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A dynamic programming approach to the approximation of nonlinear $\mathcal{L}_2$-gain

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***Abstract*— A generalization of the $\mathcal{L}_2$-gain inequality based on nonlinear gains is considered. Using optimization and dynamic programming to characterize lower bounds for the minimal gain function for which this nonlinear $\mathcal{L}_2$-gain inequality holds, a technique for computation of nonlinear $\mathcal{L}_2$-gain bounds is proposed. Some simple illustrative examples are explored.**

***Index Terms*— $\mathcal{L}_2$-gain analysis, nonlinear systems, optimization, dynamic programming.**

I. INTRODUCTION

$\mathcal{L}_2$-gain is a generalized measurement of input / output energy gain for dynamical systems, motivated by the fundamental nature of energy transport in physical systems. As a system norm, $\mathcal{L}_2$-gain has been studied extensively in both linear and nonlinear contexts, ranging from the fundamental connections to stability and generalized notions of dissipation [15], [9], frequency domain interpretations in linear systems theory [17], through to its role as a design objective in measurement feedback control design for linear [6], [18] and nonlinear systems [13], [14], [8]. Application of such analysis and design tools has thus dictated the relevance of techniques for the computation of system $\mathcal{L}_2$-gain. These techniques have largely been based on the bounded real lemma (see for example, [6], [8], for linear and nonlinear systems), which codifies a concordance between the $\mathcal{L}_2$-gain inequality and existence of solutions of an algebraic Riccati equation or Hamilton-Jacobi-Bellman equation respectively. The development of bisection style algorithms for $\mathcal{L}_2$-gain has thus become standard (e.g. [16]).

Recent developments of a variety of input-to-state style stability (ISS) properties has led to a significant expansion in the notion of system gain, away from the explicit energy interpretations of the past. This has been driven by the need to provide analysis and design tools that encompass larger classes of systems and stability properties. The fact that many nonlinear systems do not possess finite $\mathcal{L}_2$-gain, yet retain asymptotic stability, is a case in point. However, such developments do not detract from the relevance of energy gain as a fundamental property of particular importance in physical systems. Moreover, [7] recognizes that energy gain remains fundamental in the context of ISS, albeit through a state transformation. The type of energy gain thus inferred is a natural generalization of the conventional definition, requiring that output energy be related to input energy through a nonlinear gain function rather than a single gain parameter. As a generalization, this *nonlinear $\mathcal{L}_2$-gain* property naturally encompasses larger classes of systems, whilst providing the possibility of tighter gain bounds for those systems with finite "conventional" *linear $\mathcal{L}_2$-gain*. From a design point of view, both of these advantages are important, with the latter being particularly relevant in small gain design [11].

In this paper, the concept of nonlinear $\mathcal{L}_2$-gain for nonlinear systems is examined from the point of view of computation. Dynamic programming techniques [2], [1] are used in a similar way to that developed for ISS gain / transient bound computations for discrete time nonlinear systems [10]. In particular, an optimization problem that characterizes the nonlinear $\mathcal{L}_2$-gain property is proposed and studied. Application of dynamic programming, and approximate solution of the resulting dynamic programming equation, yields approximations for the nonlinear $\mathcal{L}_2$-gain (function). This approach is comparable to that in [5] in the linear gain case. A number of simple examples are considered.

Throughout this paper, consideration is restricted to continuous time nonlinear dynamical systems of the form

$$\dot{x}(s) \;=\; f(x(s), w(s))\,, \qquad z(s) \;=\; h(x(s))\,, \qquad (1)$$

where $x(s) \in \mathbf{R}^n$ is the state, $w(s) \in \mathbf{R}^m$ is the input, and $z(s) \in \mathbf{R}^p$ is the output, all at time $s \geq t \geq 0$. With a given initial state $x(t) = x \in \mathbf{R}^n$, it is assumed that a solution of (1) exists and is unique. In other notation used, a function $\gamma : \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0}$ is of class $\bar{\mathcal{K}}$ if it is continuous, non-decreasing, radially unbounded, and satisfies $\gamma(0) = 0$. There, $\mathbf{R}_{\geq 0}$ denotes the non-negative reals.

II. NOTIONS OF $\mathcal{L}_2$-GAIN

A. Conventional (linear) $\mathcal{L}_2$-gain

Conventional $\mathcal{L}_2$-gain analysis and control design [14], [8] is concerned with systems of the form of (1) that satisfy a well-known input / output gain inequality. In particular, system (1) has *linear $\mathcal{L}_2$-gain* $\leq \gamma_\ell \in \mathbf{R}_{\geq 0}$ if there exists a $\beta \in \bar{\mathcal{K}}$ such that

$$\|z\|^2_{\mathcal{L}_2[0,T]} \;\leq\; \gamma_\ell^2 \|w\|^2_{\mathcal{L}_2[0,T]} + \beta(|x_\circ|) \qquad (2)$$

for all $w \in \mathcal{L}_2[0,T]$, $T \geq 0$, $x_\circ \in \mathbf{R}^n$. Here, γ_ℓ is referred to as a *linear* $\mathcal{L}_2$-gain bound for system (1), whilst (2) is referred to as a *linear $\mathcal{L}_2$-gain inequality*. $\|\cdot\|_{\mathcal{L}_2[0,T]}$ denotes the norm defined via

$$\|z\|_{\mathcal{L}_2[0,T]} \;:=\; \sqrt{\int_0^T |z(s)|^2\, ds}\,.$$

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Should inequality (2) hold for system (1), $\gamma_\ell \in \mathbf{R}_{\geq 0}$ is then an upper bound for the conventional induced $\mathcal{L}_2$-norm γ_ℓ^* of system (1) [13]. This induced norm is referred to here as the *linear* $\mathcal{L}_2$*-gain* of system (1), and is given by

$$\gamma_\ell^* \ := \ \inf \left\{ \gamma_\ell \ \middle| \ \exists \ \beta \in \bar{\mathcal{K}} \text{ such that (2) holds} \right\}. \qquad (3)$$

By definition, the linear $\mathcal{L}_2$-gain γ_ℓ^* captures the energy gain of system (1), from input w to output z. The fact that this quantity, as defined in (3), is one dimensional and independent of input amplitude, is an artifact of the historical development of nonlinear analysis tools based on linear ideas. Indeed, the linear $\mathcal{L}_2$-gain γ_ℓ^* of system (1) has historically often been referred to as the $\mathcal{H}_\infty$-norm of (1), although this terminology should strictly be reserved for the study of linear dynamical systems (see for example [6] for a frequency domain definition of the $\mathcal{H}_\infty$-norm).

Semantics aside, it is clear that $\gamma_\ell^* \leq \gamma_\ell < \infty$ if and only if system (1) has linear $\mathcal{L}_2$-gain $\leq \gamma_\ell$. That is, if and only if (2) holds. Meanwhile, satisfaction of (2) for linear $\mathcal{L}_2$-gain bound γ_ℓ is equivalent to existence of a locally bounded, non-negative storage function [15] with supply rate $\gamma_\ell^2 |w|^2 - |z|^2$ [14]. One such storage function, called the *available storage* (the minimal such function), can be characterized in terms of the unique stabilizing viscosity solution of the corresponding Hamilton-Jacobi-Bellman (HJB) PDE, given γ_ℓ [4], [1]. That is, satisfaction of (2) is equivalent to existence of a non-negative, locally bounded solution of a HJB PDE, both of which are parameterized by the same candidate linear $\mathcal{L}_2$-gain bound $\gamma_\ell \in \mathbf{R}_{\geq 0}$. As $\leq$ defines a linear (simple) ordering on any closed interval of such gain bounds, it follows that the linear $\mathcal{L}_2$-gain (3) of system (1) can be computed via a bisection method based on existence of a solution to the HJB PDE in question [16].

B. Nonlinear $\mathcal{L}_2$-gain

System (1) has *nonlinear* $\mathcal{L}_2$-gain with gain bound $\gamma \in \bar{\mathcal{K}}$ and transient bound $\beta \in \bar{\mathcal{K}}$ if

$$\|z\|^2_{\mathcal{L}_2[0,T]} \ \leq \ \gamma\left(\|w\|^2_{\mathcal{L}_2[0,T]} \right) + \beta(|x_\circ|) \qquad (4)$$

for all $w \in \mathcal{L}_2[0,T]$, $x(0) = x_\circ \in \mathbf{R}^n$, and $T \geq 0$. Here, (4) is referred to as the *nonlinear* $\mathcal{L}_2$*-gain inequality*. Linear gain inequality (2) may be recovered from (4) by selecting a linear gain function $\gamma(s) = \left(\gamma_\ell^2\right) s \in \bar{\mathcal{K}}$, $\gamma_\ell \in \mathbf{R}_{\geq 0}$. The following proposition is immediate.

Proposition 2.1: Suppose that system (1) has linear $\mathcal{L}_2$-gain $\leq \gamma_\ell$ with transient bound $\beta \in \bar{\mathcal{K}}$. Then, system (1) has nonlinear $\mathcal{L}_2$-gain with gain bound $\gamma \in \bar{\mathcal{K}}$ and the same transient bound β, where $\gamma(s) \ \leq \ \left(\gamma_\ell^2\right) s$ for all $s \geq 0$.

Whilst obvious, the above observation confirms that any linear gain bound may be replaced with, and possibly tightened, using a nonlinear gain bound. This observation coupled with the utility of small gain results provides significant motivation for nonlinear $\mathcal{L}_2$-gain analysis based on inequality (4).

III. Dynamic programming

A. Finite horizon nonlinear $\mathcal{L}_2$-gain bound

By inspection of the nonlinear $\mathcal{L}_2$-gain inequality (4), it is clear that

$$\hat{V}_T(t,x,\xi) \ \leq \ \gamma(\xi) + \beta(|x|), \qquad (5)$$

for all $T \geq t \geq 0$, $\xi \in \mathbf{R}_{\geq 0}$, $x = x(t) \in \mathbf{R}^n$, where

$$\hat{V}_T(t,x,\xi) \ = \sup_{\|w\|^2_{\mathcal{L}_2[t,T]} \leq \xi} \left\{ \|z\|^2_{\mathcal{L}_2[t,T]} \ \middle| \ \begin{array}{c} \text{(1) holds,} \\ x(t) = x \end{array} \right\}. \qquad (6)$$

As $\beta \in \bar{\mathcal{K}}$ by definition (so that $\beta(0) = 0$), evaluation at $x = 0$ yields a lower bound for all admissible gain bounds $\gamma \in \bar{\mathcal{K}}$ for which the nonlinear $\mathcal{L}_2$-gain inequality (4) holds. However, as the optimization in (6) involves an $\mathcal{L}_2$ constraint, the associated value is difficult to compute as is. Consequently, it is useful to introduce an auxiliary state equation whose state keeps track of the energy used by the input, so that the $\mathcal{L}_2$ constraint in (6) can be replaced by a state constraint in the augmented system. To this end, consider the augmented system

$$\left[\begin{array}{c} \dot{x}(t) \\ \dot{\xi}(t) \end{array} \right] \ = \ \left[\begin{array}{c} f(x(t), w(t)) \\ -|w(t)|^2 \end{array} \right], \quad z(t) \ = \ h(x(t)), \qquad (7)$$

where $\xi(t) \in \mathbf{R}$. Then, the maximum output energy obtainable from an input with bounded $\mathcal{L}_2$-norm ξ is captured by the finite horizon value function $V_T : [0,T] \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$, $T \in [0,\infty)$,

$$V_T(t,x,\xi) \ = \sup_{w \in \mathcal{L}_2[t,T]} \left\{ \|z\|^2_{\mathcal{L}_2[t,T]} + \Psi(\xi(T)) \ \middle| \ \begin{array}{c} \text{(7) holds,} \\ x(t) = x, \\ \xi(t) = \xi \end{array} \right\} \qquad (8)$$

with terminal cost $\Psi : \mathbf{R} \to \mathbf{R} \cup \{-\infty\}$ given by

$$\Psi(\xi) \ := \ \left\{ \begin{array}{cc} 0 & \xi \geq 0, \\ -\infty & \xi < 0. \end{array} \right.$$

Lemma 3.1: Value functions (6) and (8) are equivalent.

Proof: Fix $t \in [0,\infty)$, $x \in \mathbf{R}^n$ and $\xi \in \mathbf{R}_{\geq 0}$. By integration of the augmented dynamics of (7) for any $T \geq t$,

$$\xi(T) \ = \ \xi - \|w\|_{\mathcal{L}_2[t,T]}, \quad \xi(t) = \xi \in \mathbf{R}_{\geq 0}, \qquad (9)$$

so that the following equivalence in constraints is obtained:

$$\|w\|^2_{\mathcal{L}_2[t,T]} \ \leq \ \xi \iff \xi(T) \ \geq \ 0 \iff \Psi(\xi(T)) \ = \ 0.$$

Hence, by inspection of (6),

$$\begin{aligned}
\hat{V}_T(t,x,\xi) &= \sup_{\substack{\|w\|^2_{\mathcal{L}_2[t,T]} \leq \xi \\ (w \in \mathcal{L}_2[t,T])}} \left\{ \|z\|^2_{\mathcal{L}_2[t,T]} \ \middle| \ \begin{array}{c} \text{(1) holds,} \\ x(t) = x \end{array} \right\} \\
&= \sup_{w \in \mathcal{L}_2[t,T]} \left\{ \|z\|^2_{\mathcal{L}_2[t,T]} \ \middle| \ \begin{array}{c} \text{(7) holds,} \\ x(t) = x, \\ \xi(t) = \xi, \\ \xi(T) \geq 0 \end{array} \right\} \\
&= \sup_{w \in \mathcal{L}_2[t,T]} \left\{ \|z\|^2_{\mathcal{L}_2[t,T]} + \Psi(\xi(T)) \ \middle| \ \begin{array}{c} \text{(7) holds,} \\ x(t) = x, \\ \xi(t) = \xi \end{array} \right\} \\
&=: V_T(t,x,\xi)
\end{aligned}$$

With $\xi \in \mathbf{R}_{<0}$, $V_T(t,x,\xi) = -\infty$ by definition of Ψ and the non-increasing nature of the augmented dynamics (9). For the same $\xi \in \mathbf{R}_{<0}$, defining the supremum over the empty set to be $-\infty$ yields $\hat{V}(t,x,\xi) = -\infty$, completing the proof. ■

With the value function V_T thus defined, it is possible to define the following candidate gain bound $\gamma_T^* : \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0}$,

$$\gamma_T^*(s) \;:=\; V_T(0,0,s)\,. \tag{10}$$

In order to show that the function γ_T^* is a lower bound for the nonlinear $\mathcal{L}_2$-gain of system (1) on the finite time horizon $[0,T]$ as per (5), some minor technicalities are first presented.

Lemma 3.2: Suppose system (1) has nonlinear $\mathcal{L}_2$-gain with gain bound $\gamma \in \bar{\mathcal{K}}$ and transient bound $\beta \in \bar{\mathcal{K}}$. Then, finite horizon value $V_T(t,x,\xi)$ defined by (8) satisfies the following properties for all $T \in [t,\infty)$, $t,\xi \in \mathbf{R}_{\geq 0}$, $x \in \mathbf{R}^n$:

(i) $V_T(t,x,\xi)$ satisfies the bounds

$$0 \;\leq\; V_T(t,x,\xi) \;\leq\; \gamma(\xi) + \beta(|x|)\,; \tag{11}$$

(ii) $V_T(t,x,\xi)$ is non-decreasing in ξ;

(iii) $V_T(t,x,\xi)$ is non-decreasing in T;

(iv) $\lim_{T\to\infty} V_T(t,x,\xi)$ exists.

Proof: (i) As $\xi \in \mathbf{R}_{\geq 0}$, selecting $w \equiv 0$ suboptimal in (8) implies the left-hand inequality of (11), whilst the right-hand inequality is immediate from (5), (8), and Lemma 3.1.

(ii) Fix $0 \leq \xi_\circ < \xi_1 < \infty$. Applying Lemma 3.1,

$$\begin{aligned} V_T(t,x,\xi_\circ) &= \sup_{\|w\|^2_{\mathcal{L}_2[t,T]} \leq \xi_\circ} \left\{ \|z\|^2_{\mathcal{L}_2[t,T]} \;\middle|\; \begin{array}{c} \text{(1) holds,} \\ x(t) = x \end{array} \right\} \\ &\leq \sup_{\|w\|^2_{\mathcal{L}_2[t,T]} \leq \xi_1} \left\{ \|z\|^2_{\mathcal{L}_2[t,T]} \;\middle|\; \begin{array}{c} \text{(1) holds,} \\ x(t) = x \end{array} \right\} \\ &= V_T(t,x,\xi_1)\,. \end{aligned}$$

(iii) Fix any $T_\circ \in [t,T]$. As $\xi \in \mathbf{R}_{\geq 0}$, the value is bounded by (11) (i.e. it exists). Fix any $\delta > 0$. Let $w_\circ^\delta \in \mathcal{L}_2[t,T]$ denote a δ-optimal input, such that

$$\begin{aligned} V_{T_\circ}(t,x,\xi) - \delta \;&<\; \|z\|_{\mathcal{L}_2[t,T_\circ]} + \Psi(\xi(T_\circ)) \;\left|\; \begin{array}{c} \text{(7) holds,} \\ x(t) = x, \\ \xi(t) = \xi, \\ w = w_\circ^\delta \end{array} \right. \\ &\leq\; V_{T_\circ}(t,x,\xi)\,. \end{aligned} \tag{12}$$

Define the concatenated input $w_1^\delta \in \mathcal{L}_2[t,T]$ according to

$$w_1^\delta(s) \;:=\; \left\{ \begin{array}{ll} w_\circ^\delta(s) & s \in [t,T_\circ) \\ 0 & s \in [T_\circ,T] \end{array} \right.$$

Employing the flow notation

$$\xi(T,t,\xi;w) \;:=\; \xi(T), \quad \xi(\cdot) \text{ defined by (9)}\,,$$

it is clear that $\xi(T,t,\xi;w_1^\delta) \equiv \xi(T_\circ,t,\xi;w_\circ^\delta)$, so that $\Psi(\xi(T,t,\xi;w_1^\delta)) \equiv \Psi(\xi(T_\circ,t,\xi;w_\circ^\delta))$. Furthermore, adopting a similar notation for the output $z(\cdot)$,

$$\|z(T,t,x;w_1^\delta)\|^2_{\mathcal{L}_2[t,T]} \;\geq\; \|z(T_\circ,t,x;w_\circ^\delta)\|^2_{\mathcal{L}_2[t,T_\circ]}$$

So, selecting $w_1^\delta \in \mathcal{L}_2[t,T]$ as a suboptimal input in the definition (8) of $V_T(t,x,\xi)$,

$$\begin{aligned} V_T(t,x,\xi) &\geq \|z(T,t,x;w_1^\delta)\|^2_{\mathcal{L}_2[t,T]} + \Psi(\xi(T,t,\xi;w_1^\delta)) \\ &\geq \|z(T_\circ,t,x;w_\circ^\delta)\|^2_{\mathcal{L}_2[t,T_\circ]} + \Psi(\xi(T_\circ,t,\xi;w_\circ^\delta)) \\ &> V_{T_\circ}(t,x,\xi) - \delta \end{aligned}$$

where the last inequality follows from (12). As $\delta > 0$ is arbitrary, sending $\delta \downarrow 0$ yields the non-decreasing property.

(iv) Follows immediately from assertions (i) and (iii). ■

Consider again the candidate gain bound γ_T^* defined by (10).

Theorem 3.3: Suppose system (1) has nonlinear $\mathcal{L}_2$-gain with gain bound $\gamma \in \bar{\mathcal{K}}$ and transient bound $\beta \in \bar{\mathcal{K}}$. Then, for any $T \in [0,\infty)$,

$$\gamma(s) \;\geq\; \gamma_T^*(s) \tag{13}$$

for all $s \in \mathbf{R}_{\geq 0}$. Furthermore, γ_T^* is non-decreasing and satisfies $\gamma_T^*(0) = 0$.

Proof: Applying (10) and Lemma 3.2(i),

$$\gamma_T^*(s) \;=\; V_T(0,0,s) \;\leq\; \gamma(s) + \beta(0) \;=\; \gamma(s) \tag{14}$$

as $\beta(0) = 0$. With $\gamma \in \bar{\mathcal{K}}$, gain bound (14) implies that $\gamma_T^*(0) \leq \gamma(0) = 0$. Fixing $0 \leq s_\circ < s_1 < \infty$ and applying Lemma 3.2(ii),

$$\gamma_T^*(s_\circ) = V_T(0,0,s_\circ) \leq V_T(0,0,s_1) = \gamma_T^*(s_1)\,. \quad ■$$

With a view to computing the gain bound (13), a dynamic programming equation for V_T is useful.

Lemma 3.4: The finite horizon value function V given by (8) satisfies the dynamic programming equation

$$V_T(t,x,\xi) \;=\; \sup_{w \in \mathcal{L}_2[t,\tau]} \left\{ \begin{array}{l} \|z\|^2_{\mathcal{L}_2[t,\tau]} + \\ \;V_T(\tau,x(\tau),\xi(\tau)) \end{array} \;\middle|\; \begin{array}{c} \text{(7) holds,} \\ x(t) = x, \\ \xi(t) = \xi \end{array} \right\} \tag{15}$$

for all $x \in \mathbf{R}^n$, $\xi \in \mathbf{R}$, $t,\tau \in [0,T]$, subject to the final condition

$$V_T(T,x,\xi) \;=\; \Psi(\xi)\,. \tag{16}$$

Proof: Follows standard dynamic programming arguments, see for example [1]. ■

B. Infinite horizon gain bound

Define the infinite horizon value function

$$W(x,\xi) \;:=\; \limsup_{T\to\infty} V_T(0,x,\xi)\,. \tag{17}$$

and the candidate gain bound $\gamma_\infty^* : \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0}$,

$$\gamma_\infty^*(s) \;:=\; W(0,s)\,. \tag{18}$$

Lemma 3.5: Suppose system (1) has nonlinear $\mathcal{L}_2$-gain with gain bound $\gamma \in \bar{\mathcal{K}}$ and transient bound $\beta \in \bar{\mathcal{K}}$. Then, the infinite horizon value $W(x,\xi)$ defined by (17) satisfies the following properties for all $x \in \mathbf{R}^n$, $\xi \in \mathbf{R}_{\geq 0}$:

(i) $W(x,\xi)$ satisfies the same bounds (11) as the finite horizon value function $V_T(t,x,\xi)$;

(ii) $W(x,\xi)$ is non-decreasing in ξ;

(iii) $W(x,\xi)$ satisfies the dynamic programming equation

$$W(x,\xi) \;=\; \sup_{w\in\mathcal{L}_2[0,\tau]} \left\{ \begin{array}{l} \|z\|^2_{\mathcal{L}_2[0,\tau]} + \\ \quad W(x(\tau),\xi(\tau)) \end{array} \middle| \begin{array}{r} \text{(7) holds,} \\ x(0)=x, \\ \xi(0)=\xi \end{array} \right\} \tag{19}$$

for all $\tau \in [0,\infty)$, subject to the condition that

$$W(x,\xi) \;=\; -\infty \quad \forall\, \xi \in \mathbf{R}_{<0}\,. \tag{20}$$

Proof: (i), (ii) Immediate from Lemma 3.2(i), (ii) and definition (17), as bound (11) is uniform in T.

(iii) Lemma 3.2(iii), (iv) and definition (17) imply that the limsup in (17) may be replaced with a supremum over $T \geq 0$. Hence, taking the supremum over $T \geq 0$ of both sides of (15) and noting that the $\|z\|^2_{\mathcal{L}_2[t,\tau]}$ term is independent of T, (19) immediately follows by interchanging suprema. ∎

Theorem 3.6: Suppose system (1) has nonlinear $\mathcal{L}_2$-gain with gain bound $\gamma \in \bar{\mathcal{K}}$ and transient bound bound $\beta \in \bar{\mathcal{K}}$. Then,

$$\gamma(s) \;\geq\; \gamma^*_\infty(s) \;\geq\; \gamma^*_T(s) \tag{21}$$

for all $s \in \mathbf{R}_{\geq 0}$, where γ^*_T, γ^*_∞ are given by (10), (18) respectively. Furthermore, γ^*_∞ is non-decreasing and satisfies $\gamma^*_\infty(0) = 0$.

Proof: By inspection of (10), (17) and (18), the argument used in proving Lemma 3.5(iii) implies that

$$\gamma^*_\infty(s) \;\equiv\; \sup_{T\geq 0} \gamma^*_T(s)\,.$$

Hence, Theorem 3.3 immediately yields (21) and the non-decrescent and zero at zero properties. ∎

C. Other nonlinear $\mathcal{L}_2$-gain bound candidates

A pair of obvious, generally more conservative gain bound candidates is obtained if it is required that an exact input energy be used in evaluating the output energy. These candidates are defined by

$$\bar{\gamma}^*_T(s) \;:=\; \bar{V}_T(0,0,s)\,, \tag{22}$$

$$\bar{\gamma}^*_\infty(s) \;:=\; \limsup_{T\to\infty} \bar{\gamma}^*_T(s)\,, \tag{23}$$

where $\bar{V}_T$ is the value function

$$\bar{V}_T(t,x,\xi) \;:=\; \sup_{\|w\|^2_{\mathcal{L}_2[t,T]}=\xi} \left\{ \|z\|^2_{\mathcal{L}_2[t,T]} \;\middle|\; \begin{array}{r} \text{(1) holds,} \\ x(t)=x \end{array} \right\}. \tag{24}$$

Comparing (6) and (24), $\bar{V}_T(t,x,\xi) \leq \hat{V}_T(t,x,\xi)$, so that $\bar{\gamma}^*_T(s) \leq \gamma^*_T(s)$ and $\bar{\gamma}^*_\infty(s) \leq \gamma^*_\infty(s)$ for all $s \in \mathbf{R}_{\geq 0}$. While similar dynamic programming results hold for $\bar{V}_T$, it is not immediately clear that these gains satisfy the non-decrescent property of class $\bar{\mathcal{K}}$. If the nonlinear $\mathcal{L}_2$-gain inequality (4) is known to hold for a particular fixed transient bound $\beta \in \bar{\mathcal{K}}$, a further pair of finite and infinite horizon gain bounds can be defined [10] via (8) and (17) as follows:

$$\gamma^*_{T,\beta}(s) := \sup_{x\in\mathbf{R}^n} \max\left(V_T(0,x,s) - \beta(|x|),\, 0\right)\,, \tag{25}$$

$$\gamma^*_{\infty,\beta}(s) := \sup_{x\in\mathbf{R}^n} \max\left(W(x,s) - \beta(|x|),\, 0\right)\,. \tag{26}$$

Whilst gains (25) and (26) can be computed (in-principle) if V_T, W, and β are known, the supremum over the entire state-space can render such computations infeasible.

D. Numerical method

An implicit Markov chain approximation method similar to that detailed in [3], [12] is used to compute approximate solutions of the dynamic programming equation (15) and its analogue defined by (24). These solutions depend on the existence of a differential equation form of the dynamic programming equation [1], and are used to compute the finite horizon gain bounds γ^*_T and $\bar{\gamma}^*_T$, given respectively by (10) and (22). The infinite horizon value function W (17) and associated infinite horizon gain bound γ^*_∞ (18) are approximated by selecting T large in the aforementioned finite horizon computations. This approach to the computation of the various gain bounds is applied in the examples to follow. Our experience suggests that the computation of gain bound $\bar{\gamma}^*_T$ (22) provides more conservative results, but tends to be numerically more robust than the other gain bounds presented.

IV. Examples

A. Scalar linear system

Consider system (1) in which

$$f(x,w) \;=\; -2x + w\,, \qquad h(x) \;=\; x\,, \tag{27}$$

where $x, w \in \mathbf{R}$. As this system is linear, the nonlinear $\mathcal{L}_2$-gain is known apriori to be given by the linear gain function

$$\gamma^*(s) \;=\; \|\Sigma\|^2_\infty\, s \;=\; \left(\frac{1}{4}\right) s\,, \tag{28}$$

where $\|\Sigma\|_\infty$ is the $\mathcal{H}_\infty$ norm of system (27).

1) Computation of γ^*_T and inference of γ^*_∞: ($T = 1$) Figure 1 illustrates $V_1(0,x,\xi)$, $x \in [-2,2]$, $\xi \in [0,1]$. Figure 2 illustrates that the nonlinear $\mathcal{L}_2$-gain bound obtained matches very closely the linear gain expected from $\mathcal{H}_\infty$ analysis. In view of the sandwich inequality (21), the infinite horizon gain bound γ^*_∞ must also be similarly matched to the linear $\mathcal{L}_2$-gain function (28).

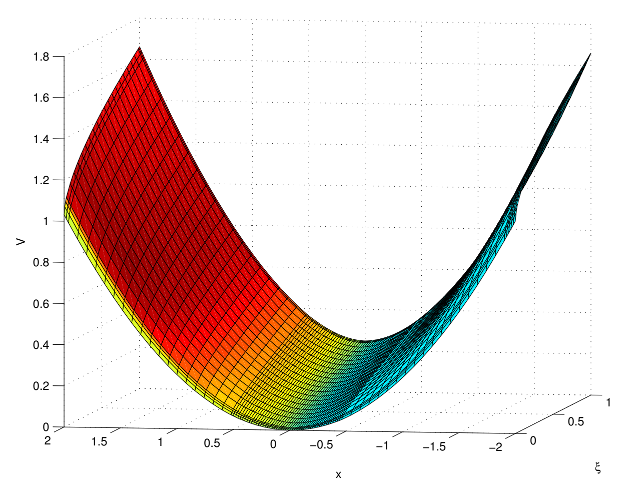

Fig. 1. Value function $V_1(0,x,\xi)$, $x \in [-2,2]$, $\xi \in [0,1]$, system (27).

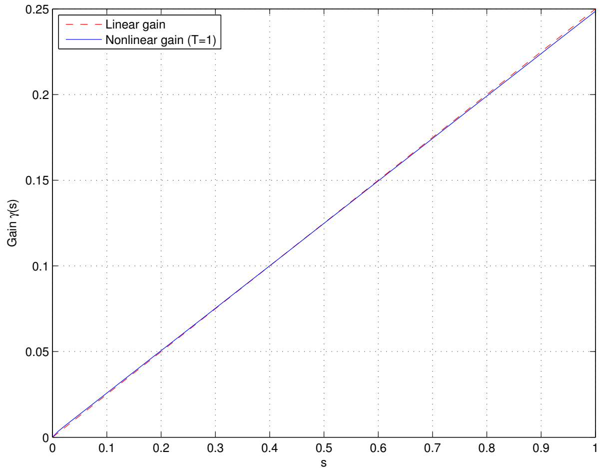

Fig. 2. Gain bound γ_1^* (dash-dot), and actual gain (dashed), system (27)

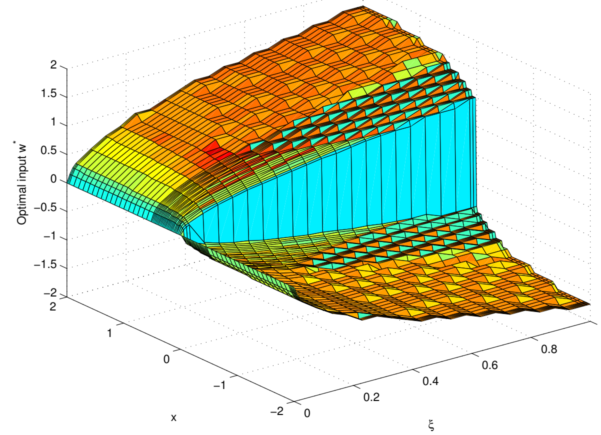

Fig. 4. Optimal input $\bar{w}_1^*(x,\xi)$, $x \in [-2,2]$, $\xi \in [0,1]$, system (27).

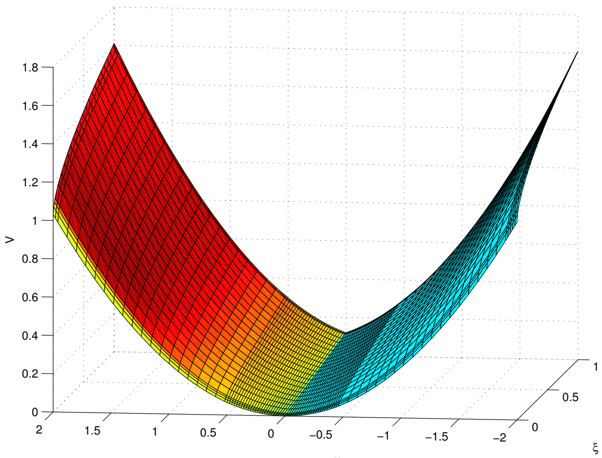

Fig. 3. Value function $\bar{V}_1(0,x,\xi)$, $x \in [-2,2]$, $\xi \in [0,1]$, system (27).

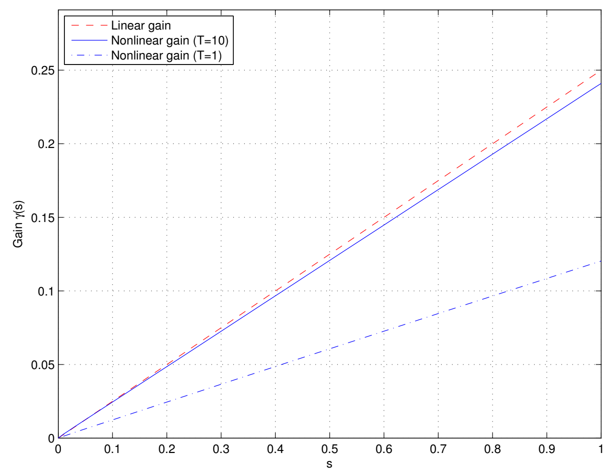

Fig. 5. Gain bound $\bar{\gamma}_{10}^*$ (solid line), gain bound $\bar{\gamma}_1^*$ (dash-dot), and actual gain (dashed), system (27)

2) Computation of $\bar{\gamma}_T^*$: ($T = 1$) Figure 3 illustrates $\bar{V}_1(0,x,\xi)$, $x \in [-2,2]$, $\xi \in [0,1]$. This value function can be observed to be non-smooth on the manifold $x = 0$. This is confirmed by Figure 4, which illustrates the state feedback characterization $\bar{w}_1^*(x,\xi)$ of the optimal input defined by (24). This state feedback clearly exhibits a step discontinuity at $x = 0$, indicating a corresponding change in the gradient of the value function. The computed gain bound $\bar{\gamma}_1^*$, illustrated by the (lowest) dash-dot line of Figure 5, is clearly conservative.

($T = 10$) In increasing the time horizon T over which the input w may be used to perturb the state of system (1), it is expected that the value function should appear smoother for each fixed input energy $\xi \in \mathbf{R}_{\geq 0}$. This follows intuitively due to the fact that energy ξ may be used more sparingly over the longer time horizon, giving rise to smaller changes in gradient of the value function. This is indeed observed to be case, with the computed value $\bar{V}_{10}(0,x,\xi)$ being very similar to that of Figure 1. The gain bound $\bar{\gamma}_{10}^*$ obtained from the computation of $\bar{V}_{10}(0,x,\xi)$ is illustrated by the solid line in Figure 5, along with $\bar{\gamma}_1^*$ (dash-dot, lowest curve) and the expected (linear) gain (28) (dashed, highest curve). Clearly, the ordering (21) is shown to be preserved.

B. Scalar nonlinear system

Suppose a cubic nonlinearity is added to the linear system (27), yielding a nonlinear system of the form of (1) with

$$f(x,w) \;=\; -2x - 8x^3 + w\,, \qquad h(x) \;=\; x\,, \qquad (29)$$

where $x, w \in \mathbf{R}$. In the absence of inputs, inclusion of this nonlinearity increases the rate of convergence of the state to the origin. Consequently, more energy is required to excite the system dynamics, and hence the output. This implies a reduction in the nonlinear $\mathcal{L}_2$-gain. In particular, it is expected that

$$\gamma^*(s) \;<\; \left(\frac{1}{4}\right) s\,, \quad s > 0\,. \qquad (30)$$

Here, computations are restricted to the gain bound $\bar{\gamma}_T^*$ of (22). ($T = 10$) Figure 6 illustrates the computed finite horizon value function $\bar{V}_{10}(0,x,\xi)$, $x \in [-2,2]$, $\xi \in [0,1]$, which is clearly non-smooth at $x = 0$. The optimal trajectory can be seen in Figure 7, which also illustrates contours of $\bar{V}_{10}(0,x,\xi)$ and the drift vector field of the augmented dynamics defined by (7), (29). The computed gain bound $\bar{\gamma}_{10}^*$ is compared with the expected gain bound of (30) in Figure 8.

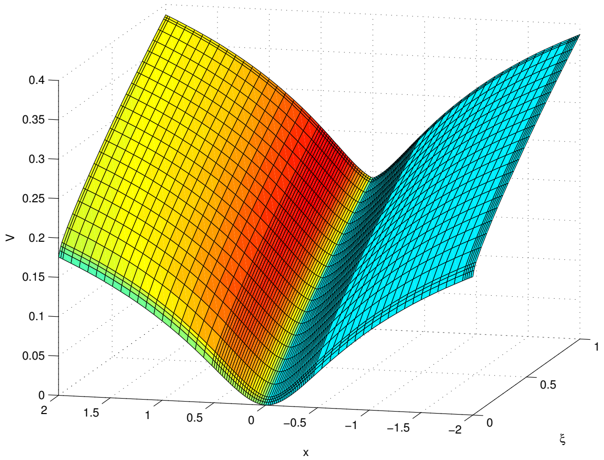

Fig. 6. Value function $\bar{V}_{10}(0, x, \xi)$, $x \in [-2, 2]$, $\xi \in [0, 1]$, system (29).

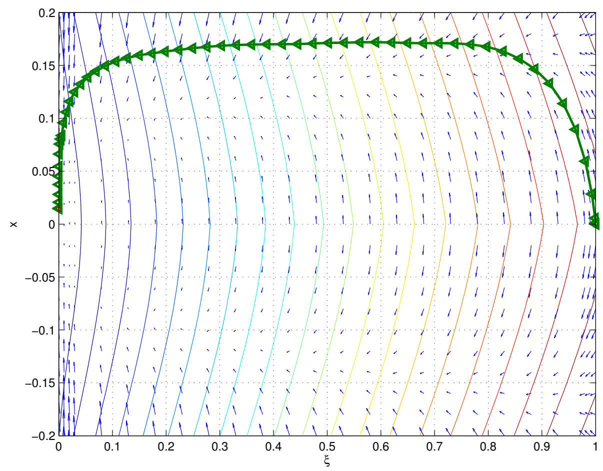

Fig. 7. Portrait $(\xi(t), x(t))$, $t \in [0, 10]$, along with contours of $\bar{V}_{10}(0, x, \xi)$, system (29)

C. A system with infinite nonlinear $\mathcal{L}_2$-gain

Consider a scalar nonlinear system of the form of (1) with

$$f(x, w) = -x^3 + w\,, \qquad h(x) = x\,, \tag{31}$$

where $x, w \in \mathbf{R}$. It is straightforward to show that with initial condition $x(t) \in \mathbf{R}$ fixed and input $w \equiv 0$,

$$x(\tau) = \frac{x(t)}{\sqrt{1 + 2(\tau - t) x(t)^2}}\,, \quad \tau \in [t, T]\,.$$

An explicit lower bound for all possible transient bounds $\beta : \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0}$ in inequality (4) follows,

$$\begin{aligned}\beta(s) &\geq \sup_{T \geq 0} \sup_{|x| \leq s} V_T(0, x, 0) \geq \sup_{T \geq 0} \|z\|^2_{\mathcal{L}_2[0,T]} \\ &= \frac{1}{2} \sup_{T \geq 0} \log\left(1 + 2Ts^2\right) = \begin{cases} 0 & s = 0 \\ \infty & s > 0 \end{cases}\end{aligned}$$

That is, no $\beta \in \bar{\mathcal{K}}$ exists such that the (4) holds, so that system (31) cannot satisfy the nonlinear $\mathcal{L}_2$-gain property.

V. Conclusions

Lower bounds on a nonlinear generalization of $\mathcal{L}_2$-gain were characterized in terms of the value of a number of optimization problems. Dynamic programming and a well-known numerical method were used to compute some of these values, thereby yielding a number of lower bounds for the nonlinear $\mathcal{L}_2$-gain of some simple dynamical systems.

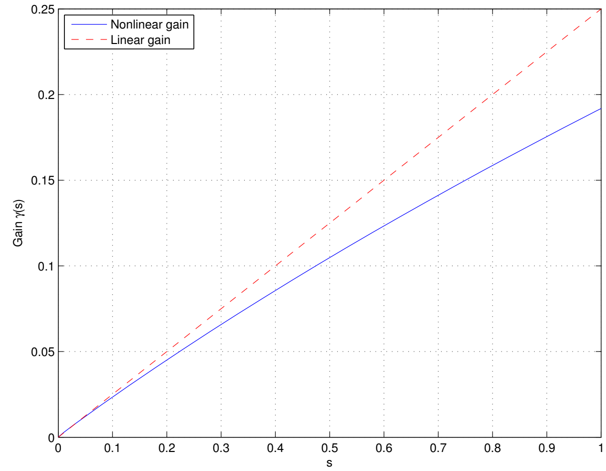

Fig. 8. Nonlinear $\mathcal{L}_2$-gain bound $\bar{\gamma}^*$ (solid line) and linear $\mathcal{L}_2$-gain (dashed), system (29)

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