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Robust Output Nash Strategies Based on Hierarchical Sliding Mode Observer in a Two-Player Differential Game

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Abstract— **This paper tackles the problem of a two-player differential game affected by matched uncertainties with only the output measurement available for each player. We suggest a state estimation based in the so called algebraic hierarchical observer for each player in order to design the Nash equilibrium strategies based on such estimation. At the same time, the use of an output integral sliding mode term (also based on the estimation processes) for the Nash strategies robustification for both players ensures the compensation of the *matched uncertainties*. A simulation example shows the feasibility of this approach in a magnetic levitator problem.**

I. INTRODUCTION

Preliminaries. Differential Game Theory deals with the dynamic optimization behavior of multiple decision makers when none of them can control the decisions made by others and the outcome for each participant is affected by the consequences of these decisions. During the last decades, the interest in the application of some modern concepts in differential games has significantly increased. This is specially seen in the kind of games affected by uncertainties (see [1], [2], [3], [4]). A common focus in recent publications has been the analysis of different uncertainty effects in players behavior. In [2], [1], [4]o LQ games with uncertainty scenarios have been considered using the $\mathscr{H}_\infty$ approach leading to the min-max formulation. Two different papers, [2] as well as [1], deal with a two-person uncertain LQ differential game with uncertainties which may "*play against*" the players. In [3], the authors propose a type of Robust Nash equilibrium concept where the game uncertainty is represented by a malevolent input, which is subjected to a cost penalty or a direct bound. Then, $\mathscr{H}_\infty$ theory is used once again to design robust strategies for all players. A second key point presented in most applications deals with *inaccurate systems,* where only a part, or a combination of, the state space coordinates is known. Games in which the players have access only to output measurements have been of interest since the late 60's in the works of [5], [6] and [7]. In [6], several problems of inaccurate state information, with white noise corrupting the output, in differential games are presented using quadratic cost functionals. However, it is still not clear how the estimation errors affect the functionals for each player. In [7], a partially observed system is considered, but the disturbances are assumed to be quadratically integrable on an infinite horizon, which implies that they tend to zero.

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Methodology. In recent years, robust observers based on Sliding Modes have been successfully developed [8], [9], [10], [11], [12] and [13]. This kind of observers is widely used because of their attractive features: (a) their insensitivity, which is a characteristic stronger than robustness, with respect to unknown inputs, (b) the possibility to use the values of the equivalent output injection for the unknown inputs compensation. Another special sliding mode technique, namely integral sliding mode (ISM) [14], has also been widely used in processes that require compensation of arising uncertainty effects. The main properties of ISM are: one, the ISM does not have a reaching phase; and two, resulting from the first one, it ensures insensitivity of the desired trajectory with respect to matched uncertainties starting from the initial moment. Such useful tools have been scarcely used in differential games [15].

Contribution. We design robust output Nash strategies for a *two person* nonzero-sum differential game affected by the presence of unknown inputs (or external matched *non vanishing* perturbations). These uncertainties influence player dynamics and are not available (measurable), neither *a priory* nor on-line. The only way to obtain information regarding the state is through an estimation process.

An output integral sliding mode control is designed for each player such that it compensates the unknown inputs allowing the design of a nominal game observer. The estimated state is used in the standard Nash control strategy. The observation error is made arbitrarily small adjusting the observer's filter parameters.

Paper Structure. In Section II, the model is presented and the control challenge is formulated. Section III is devoted to Nash strategies design for the nominal game. In Section IV, an output integral sliding mode controller rejecting the matched uncertainty is proposed. The hierarchical observer is described in Section V. The Robust Nash controller and an estimation of the closed loop error during implementation are presented in section VI. Performance issues of the robust output Nash controller are illustrated in a magnetic bearing simulation study in Section VII.

II. GAME MODEL DESCRIPTION AND BASIC ASSUMPTIONS

Let us consider an uncertain LQ differential game (LQDG) where the players' dynamic is represented by linear ordinary differential equations

$$\begin{aligned} \dot{x}(t) &= Ax(t)+\sum_{i=1}^{2} B^i u^i(t)+\sum_{i=1}^{2}\zeta^i(t) \quad (1)\\ y^1(t) &= C^1 x(t), \qquad y^2(t)=C^2x(t),\\ x(0) &= x_0, t\in[0,T], \end{aligned}$$

Here, i denotes the number of players $\left(i=\overline{1,2}\right)$, $A\in\Re^{n\times n}$ and $B^i\in\Re^{n\times m_i}$ are known **constant** system matrices and $\zeta^i(t)\in\Re$ is **an unknown input.** In addition, $x(t)\in\Re^n$ is the game state vector, with $u^i(t)\in\Re^{m_i}$ being the control strategies of each i-player and $y^i(t)\in\Re^p$ is the output of the game for each player which can be measured at each time. Finally, $C^i\in\Re^{p^i\times n}$ is the output matrix for player i.

The following assumptions will be considered through this paper.

A.1 The pairs $\left(A,B^i\right)$ are controllable and $\left(A,C^i\right)$ are observable.

A.2 The uncertainties $\zeta^i(t)$ $(i=1,2)$ are two smooth unknown disturbances which satisfy the next matching condition:

$$\Gamma^i := \left\{\zeta^i(t) \mid \zeta^i(t)=B^i\gamma^i(t), \left\|\gamma^i(t)\right\|\le q^i\|y\|\right\} \quad (2)$$

A.3 $\operatorname{rank}\left(C^iB^i\right)=m^i$.

A.4 The initial condition x_0 is bounded, i.e., there exists an η_0, such that $\|x_0\|^2\le\eta_0$.

The first equality in (2) means that $\zeta^i\in span\ B^i$, i.e. the $i-$player is able to exert a force on the perturbation. Assume also that $spanB^1\neq spanB^2$.

Control challenge. For the Robust Nash control design we propose the following two part strategy:

$$u^i(t)=u_0^i(t)+u_1^i(t);\qquad i=1,2 \quad (3)$$

where the control $u_0^i(t)$ is the Nash feedback strategy designed for the nominal game (i.e. $\zeta^i=0$) and control $u_1^i(t)$ is an integral sliding mode compensator for the unknown inputs ζ^i.

III. NASH CONTROL STRATEGY FOR THE NOMINAL SYSTEM

Consider the nominal game

$$\dot{x}_0(t)=Ax_0(t)+B^1u_0^1(t)+B^2u_0^2(t);\quad x_0(0)=x_0 \quad (4)$$

with a quadratic cost functional as an individual aim performance

$$J_T^i(u_0^i,u_0^{\hat{\imath}}) = \int_0^\infty (x^TQ^i(t)x+u_0^{iT}R^{ji}(t)u_0^i+u_0^{iT}R^{ij}(t)u_0^i)dt \quad j\neq i \quad (5)$$

The *performance index* $J_T^i(u_0^i,u_0^{\hat{\imath}})$ (5) of each i-player for *infinite time horizon* nominal game is given in the standard form, where u_0^i is the strategy for i-player and $u_0^{\hat{\imath}}$ are the strategies for the rest of the players ($\hat{\imath}$ is the player counteracting to the player with index i). We will assume also that

$$\begin{aligned} Q^i(t)=Q^{i\intercal}(t)\ge 0,\ R^{ji}(t)=R^{ji\intercal}(t)>0,\\ R^{ij}(t)=R^{ij\intercal}(t)\ge 0\ \ (j\neq i) \end{aligned} \quad (6)$$

The game solutions are understood in the Filippov sense, [16], in order to provide the possibility of discontinuous signals in the observer design. Note that Filippov solutions coincide with the usual solutions, when the right hand side is continuous.

In the case when there are no unknown inputs in the game and the complete state information is available, from the limiting solution of the finite time problem [17], the next coupled algebraic equations appear [18]:

$$\begin{aligned} -\left(A-S^2P^2\right)^\intercal P^1 &- P^1\left(A-S^2P^2\right)\\ &+ P^1S^1P^1-Q^1-P^2S^{21}P^2=0 \quad (7)\\ -\left(A-S^1P^1\right)^\intercal P^2 &- P^2\left(A-S^1P^1\right)\\ &+ P^2S^2P^2-Q^2-P^1S^{12}P^1=0 \quad (8) \end{aligned}$$

with

$$\begin{aligned} S^i &= B^i\left(R^{ji}\right)^{-1}B^{i\intercal}\\ S^{ij} &= B^i\left(R^{ji}\right)^{-1}R^{ji}\left(R^{ji}\right)^{-1}B^{i\intercal} \quad \text{for} \quad j\neq i \end{aligned}$$

The following result is well established (see [19]): for a 2-player LQDG described by (1) with (5): let P^i $(i=1,2)$ be a symmetric stabilizing solution of (7)-(8). Taking $F^{i*}:=\left(R^{ji}\right)^{-1}B^{i\intercal}P^i$ for $i=1,2$, then $\left(F^{1*},F^{2*}\right)$ is a feedback Nash equilibrium. The limiting stationary (Nash) strategies are:

$$u_0^{i*}(t)=-R^{ji^{-1}}B^{i\intercal}P^ix \quad (9)$$

Before presenting the robust Nash strategies, let us introduce the integral sliding modes compensator and the hierarchical observer.

IV. OUTPUT INTEGRAL SLIDING MODES COMPENSATOR

Define for each player the next output based sliding function $s^i\left(y^i\right)=G^iy^i+\sigma^i$.The gain matrix is defined as $G^i=\left(B^{\hat{\imath}\perp}\right)^\intercal C^{i\dagger}$. Calculating the time derivative:

$$\dot{s}^i\left(y^i\right)=\left(B^{\hat{\imath}\perp}\right)^\intercal\left[Ax+B^iu_0^i(t)+B^iu_1^i(t)+B^i\gamma^i(t)\right]+\dot{\sigma}^i \quad (10)$$

Note here that with the assignation of the matrix $G^i(x,t)$ to $B^{\hat{\imath}\perp}C^{i\dagger}$, where $B^{\hat{\imath}\perp}$ is an orthogonal complement of the control matrix of the opposite player such that $B^{\hat{\imath}\perp}B^{\hat{\imath}}=0$, makes all terms related with this player disappear. Now, $\dot{\sigma}^i$ is defined as:

$$\dot{\sigma}^i=\left(B^{\hat{\imath}\perp}\right)^\intercal A\hat{x}-\left(B^{\hat{\imath}\perp}\right)^\intercal B^iu_0^i(t),\qquad \sigma^i(0)=-G^iy^i(0)$$

$\hat{x}$ is the observer state vector which will be described later. The substitution of $\dot{\sigma}^i$ in (10), yields:

$$\dot{s}^i\left(y^i\right)=\left(B^{\hat{\imath}\perp}\right)^\intercal A(x-\hat{x})+\left(B^{\hat{\imath}\perp}\right)^\intercal B^iu_1^i(t)+\left(B^{\hat{\imath}\perp}\right)^\intercal B^i\gamma^i(t). \quad (11)$$

We propose the control:

$$u_1^i(t) = -f(t)\left(L^i\right)^{-1}\frac{s^i(t)}{\|s^i(t)\|}, \quad L^i := \left(B^{\hat{i}\perp}\right)^{\mathsf{T}} B^i, \tag{12}$$

the function $f(t)$ will be defined below. For the Lyapunov function $V^i(s) = 1/2\left\|s^i\right\|^2$:

$$\begin{aligned} \dot{V}^i(s) &= \left(s^i, \dot{s}^i\right) \quad (13)\\ &= \left(s^i, \left(B^{\hat{i}\perp}\right)^{\mathsf{T}} A(x-\hat{x}) - f(t)\frac{s^i(t)}{\|s^i(t)\|} + L^i\gamma^i(t)\right)\\ \le - &\left\|s^i\right\| \left(f(t) - \left\|\left(B^{\hat{i}\perp}\right)^{\mathsf{T}} A\right\|\|x-\hat{x}\| - \left\|L^i\right\|\left\|q^i\right\|\right) < 0 \end{aligned}$$

where $f(t) > \left\|\left(B^{\hat{i}\perp}\right)^{\mathsf{T}} A\right\|\|x-\hat{x}\| - \left\|L^i\right\|\left\|q^i\right\|$.

Thus, with adequate selection for the constants in (12), the manifold $s^i(x,t)$ is attractive since the initial time. So, from (10), we have

$$\begin{aligned} \frac{1}{2}\left\|s^i\right\|^2 = V^i\left(s^i(x(t),t)\right) &\le V^i\left(s^i(x(0),0)\right)\\ &\le \frac{1}{2}\left\|s^i(x(0),0)\right\|^2 = 0 \end{aligned}$$

which implies that for all $t \ge 0$, $\dot{s}^i(t) = 0$, which leads to $s^i(t) = 0$. This means that from the beginning of the game, the ISM strategy for each player completely compensates the matched uncertainty. The equivalent control which maintains the trajectories on the sliding surface is

$$u_{1eq}^i(t) = \left(\left(B^{\hat{i}\perp}\right)^{\mathsf{T}} B^i\right)^{-1}\left(B^{\hat{i}\perp}\right)^{\mathsf{T}} A(x-\hat{x}) + \gamma^i(t)$$

Substitution of the equivalent control in (1), yields the sliding mode equivalent dynamic

$$\begin{aligned} \dot{x}(t) &= \bar{A}x(t) + \sum_{i=1}^{2}\left(L^i\right)^{-1}\left(B^{\hat{i}\perp}\right)^{\mathsf{T}} A\hat{x} + \sum_{=1}^{2} B^i u_0^i(t) \quad (14)\\ y^1(t) &= C^1 x(t), \qquad y^2(t) = C^2 x(t), \end{aligned}$$

where $\bar{A} := A - \sum_{i=1}^{2}\left(L^i\right)^{-1}\left(B^{\hat{i}\perp}\right)^{\mathsf{T}} A$.

Remark. In [11], it has been proven that when the number of outputs is less than or equal to the number of inputs, the matrix $\bar{A}$ in (14) always belongs to the null space of the matrix C^i and, consequently, the pair $\left(\bar{A}, C^i\right)$ is not observable. This means that in the case when $p^i \le m^i$, the ISM control using only output information should not be realized.

V. OBSERVER DESIGN

The principal idea in the design of the hierarchical observer is the recovery of the elements $Cx(t)$, $CAx(t)$ and so on, until we get $C^i\bar{A}^k x$, with $k = \overline{1, \ell-1}$. Constructing the $Hx(t)$ vector with

$$H = \begin{bmatrix} C & C\bar{A} & \ldots & C\bar{A}^{\ell-1} \end{bmatrix}^T, \quad H \in \Re^{p\ell \times n}$$

where ℓ is the observability index, i.e., the least positive integer such that such that *rank* $H = n$.

Before designing the observer, it is necessary to find an error bound. Design the following dynamic system

$$\begin{aligned} \dot{\tilde{x}}(t) &= \bar{A}\tilde{x}(t) + \sum_{i=1}^{2}\left(L^i\right)^{-1}\left(B^{\hat{i}\perp}\right)^{\mathsf{T}} A\hat{x}\\ &+ \sum_{i=1}^{2}\left(B^i u_0^i(t) + K^i\left(y^i - C^i\tilde{x}\right)\right), \end{aligned}$$

defining the error $r(t) = x - \tilde{x}$, we have

$$\dot{r}(t) = \left(\bar{A} - K^i C^i\right) r(t) = \hat{A} r(t),$$

with $\hat{A}$ Hurwitz, $r(t)$ is bounded. Now, let us recover the $C^i\bar{A}^k x$ vectors with $k = \overline{1, \ell-1}$. To recover $C^i\bar{A}x$ vector, let us design the next auxiliary system

$$\begin{aligned} \dot{x}_a^{(1)}(t) &= \bar{A}\tilde{x} + \sum_{i=1}^{2}\left(L^i\right)^{-1}\left(B^{\hat{i}\perp}\right)^{\mathsf{T}} A\hat{x}\\ &+ \sum_{i=1}^{2} B^i u_0^i(t) + T(CT)^{-1} v^{(1)}(t), \end{aligned} \tag{15}$$

where $x_a^{(1)}(0)$ satisfies $C^i x_a^{(1)}(0) = y^i(0)$. For the variable

$$s^{(1)}\left(y^i(t), x_a^{(1)}(t)\right) = C^i x(t) - C^i x_a^{(1)}(t), \tag{16}$$

the time derivative is

$$\dot{s}^{(1)}(t) = C^i\bar{A}\left(x(t) - \tilde{x}(t)\right) - v^{(1)}(t), \tag{17}$$

where

$$v^{(1)}(t) = M_1^i \frac{s^{(1)}(t)}{\left\|s^{(1)}(t)\right\|}.$$

Here, the scalar gain M_1^i should satisfy the condition $\left\|C\bar{A}\right\|\|x-\hat{x}\| < M_1^i$ to reach the sliding mode regime. Then, we get $\dot{s}^1(t) = s^1(t) = 0$ for all $t \ge 0$. Thus, $C^i x(t) = C^i x_a^{(1)}(t)$ and the equivalent output injection is

$$v_{eq}^{(1)}(t) = C^i\bar{A}x(t) - C^i\bar{A}\tilde{x}(t) \qquad \forall t \ge 0,$$

finally the recovery of $C^i\bar{A}x(t)$ is made by

$$C^i\bar{A}x(t) = C^i\bar{A}\tilde{x}(t) + v_{eq}^{(1)}(t) \qquad \forall t \ge 0, \tag{18}$$

Now, to recover $C^i\bar{A}^2 x(t)$ the next auxiliary system is designed:

$$\begin{aligned} \dot{x}_a^{(2)}(t) &= \bar{A}^2\tilde{x} + \bar{A}\left(\sum_{i=1}^{2}\left(L^i\right)^{-1}\left(B^{\hat{i}\perp}\right)^{\mathsf{T}} A\hat{x} + \sum_{i=1}^{2} B^i u_0^i(t)\right)\\ &+ T\left(C^i T\right)^{-1} v^{(2)}(t), \end{aligned}$$

for the variable

$$s^{(2)}\left(v_{eq}^{(1)}(t), x_a^{(2)}(t)\right) = C^i\bar{A}x(t) - C^i x_a^{(2)}(t), \tag{19}$$

the time derivative is

$$\dot{s}^{(2)}(t) = C^i\bar{A}^2\left(x(t) - \tilde{x}(t)\right) - v^{(2)}(t).$$

The output injection for the second level is

$$v^{(2)}(t) = M_2^i \frac{s^{(2)}(t)}{\left\|s^{(2)}(t)\right\|},$$

with $\left\|C\bar{A}^2\right\| \|x-\hat{x}\| < M_2^i$, to obtain the sliding mode regime $\dot{s}^2(t) = s^2(t) = 0$ for all $t \geq 0$. The equivalent control is

$$v_{eq}^{(2)}(t) = C^i\bar{A}^2x(t) - C^i\bar{A}^2\tilde{x}(t) \qquad \forall t \geq 0,$$

and

$$C^i\bar{A}^2x(t) = C^i\bar{A}^2\tilde{x}(t) + v_{eq}^{(2)}(t) \qquad \forall t \geq 0. \tag{20}$$

We can repeat this procedure to recover the $C^i\bar{A}^kx(t)$ vectors for $k = \overline{1,\ell-1}$. The general formula for the auxiliary dynamics is

$$\begin{aligned}\dot{x}_a^{(k)}(t) &= \bar{A}^k\tilde{x} + \bar{A}^{k-1}\left(\sum_{i=1}^{2}\left(L^i\right)^{-1}\left(B^{i\perp}\right)^{\mathsf{T}}A\hat{x} + \sum_{i=1}^{2}B^iu_0^i(t)\right)\\ &+ T\left(C^iT\right)^{-1}v^{(k)}(t),\end{aligned}$$

and

$$v^{(k)}(t) = M_k^i\frac{s^{(k)}(t)}{\left\|s^{(k)}(t)\right\|},$$

with $\left\|C\bar{A}^k\right\| \|x-\hat{x}\| < M_k^i$. The general sliding surface

$$s^{(k)}\left(v_{eq}^{(k-1)}(t)\ ,x_a^{(k)}(t)\right) = \begin{cases} y^i(t) - C^ix_a^{(1)}(t) & \text{for} \quad k=1 \\ v_{eq}^{(k-1)}(t) + C^i\bar{A}^{k-1}\tilde{x}(t) - C^ix_a^{(k)}(t) & \text{for} \quad k>1 \end{cases}$$

and $s^{(k)}(0)$ should satisfy

$$s^{(k)}(0) = \begin{cases} C^iy^i(0) - C^ix_a^{(1)}(0) & \text{for} \quad k=1 \\ v_{eq}^{(k-1)}(0) + C^i\bar{A}^{k-1}\tilde{x}(0) - C^ix_a^{(k)}(0) & \text{for} \quad k>1 \end{cases}$$

Equations (18) and (20) can be rewritten in matrix form

$$\underbrace{\begin{bmatrix} C \\ C\overline{A} \\ \vdots \\ C\overline{A}^k \end{bmatrix}}_{H} x(t) = \underbrace{\begin{bmatrix} C \\ C\overline{A} \\ \vdots \\ C\overline{A}^k \end{bmatrix}}_{H} \tilde{x}(t) + \underbrace{\begin{bmatrix} Cx_a^{(1)} - C\overline{x} \\ v_{eq}^{(1)} \\ \vdots \\ v_{eq}^{(k)} \end{bmatrix}}_{v_{eq}} \tag{21}$$

As mentioned earlier, *rank* $H = n$; therefore we can premultiply (21) by H^+ and recover the $\hat{x}(t)$ state

$$\hat{x}(t) := \overline{x}(t) + H^+v_{eq} \tag{22}$$

Observer Realization To carry out the observer in the form 22, the surface $s^{(k)}$ must be realizable. To guarantee this, the equivalent output injection $v^{(k)}$ must be available. However, the non idealities in the implementation of $v^{(k)}$ cause the so-called chattering effect. Therefore, $v^{(k)}$ can not be directly measured. However, we can apply a first order filter

$$\tau\dot{v}_{av}^{(k)} + v_{av}^{(k)} = v_{eq}^{(k)}, \qquad v_{av}(0) = 0 \tag{23}$$

For a very small $\tau > 0$, the filter output approaches to the equivalent control $v_{eq}^{(k)}$, i.e., $\lim\limits_{\substack{\tau\to 0\\ \delta/\tau\to 0}} v_{av}^{(k)} = v_{eq}^{(k)}$ (see [20]), where δ is the sampling time. We can select $\tau = \delta^{\eta}$, where $0 < \eta < 1$. Finally, to realize the observer, select a very small sampling interval δ and substitute $v_{eq}^{(k)}$ by $v_{av}^{(k)}$:

$$\hat{x}(t) := \tilde{x}(t) + H^+v_{av} \tag{24}$$

$$v_{av} = \left[\begin{array}{cccc} Cx_a^{(1)} - C\tilde{x} & v_{av}^{(1)} & \ldots & v_{av}^{(k)} \end{array}\right]^T. \tag{25}$$

VI. ROBUST OUTPUT NASH STRATEGY

Before presenting the new Robust Nash strategies we have the next proposition:

Proposition 1: Due to the inaccurate state information, it seems natural to use a *current state estimate* $\hat{x}(t)$ (if it is available) instead of $x(t)$ in the feedback equilibrium control laws (9), that is,

$$\hat{u}_0^{i*}(\hat{x}) = -R^{ji^{-1}}B^{i\mathsf{T}}P^i\hat{x}, \tag{26}$$

where $\hat{u}^{i*}$ denotes the control action based on estimations $\hat{x}$.

Thus, the proposed control law in (3) yields

$$u^i(\hat{x},t) = -R^{ji^{-1}}B^{i\mathsf{T}}P^i\hat{x} - f(t)\left(L^i\right)^{-1}\frac{s^i(t)}{\|s^i(t)\|}; \qquad i=1,2 \tag{27}$$

A. Error estimation during implementation of the closed loop control

The filter causes some errors in the state vector reconstruction. Evidently those errors directly affect the controller. Hence, we will estimate the error which appears during the realization of the closed loop control, that is, the error due to the actuators plus the error due to the observation process.

The control error is $O(\mu)$, where μ is a control execution constant which generally depends on the actuators time constants. Now, let us estimate the error order due to the observation process. As we saw, the observer design is based on the recursive use of filters of the form (23). Firstly, let us recall the following lemma regarding the error induced by this type of filters.

Lemma 2: [20] If in the differential equation

$$\tau\dot{z} + z = h(t) + H(t)\dot{s}, \tag{28}$$

where τ is a constant and z, h and s are m-dimensional vectors functions

(1) the functions $h(t)$ and $H(t)$, and their first order derivatives are bounded in magnitude by a certain number M and

(2) $\|s(t)\| \leq \xi$ (ξ is a constant positive value)

then for any pair of positive numbers Δt and υ there exists a number $d(\upsilon,\Delta t,z(0))$ such that $\|z(t)-h(t)\| \leq \upsilon$with $0 < \tau \leq d$, $\xi/\tau \leq d$ and $t \geq \Delta t$.

Indeed, $\|z(t)-h(t)\|$ satisfies the following inequality

$$\begin{aligned}\|z(t)-h(t)\| \quad &\leq \quad \|z(0)-h(0)\|\exp(-t/\tau) + M(\tau+\xi)\\ &+3M\left(\frac{\xi}{\tau}\right)\end{aligned}$$

In our case, expression (28) can be related with $\tau_1\dot{v}_{av}^{(1)} + v_{av}^{(1)} = v_{eq}^1 - \dot{s}^1$, which is obtained from equations (23) and

(17). Thus, in this case $h(t)$ refers to the equivalent output injection. Furthermore, the sliding mode control error directly affects the performance of the first sliding mode in the observation process. It is also known that the sampling step δ induces an error of order $O(\delta)$ in the variable $\dot{s}^{(1)}$ during the sliding motion. Hence, it is reasonable to accept that the error in the sliding variable $\dot{s}^{(1)}$ is of order $O(\mu)+O(\delta)$. By defining $\Delta := \mu+\delta$, we have that the constant ξ in the lemma (2) is $\xi = O(\Delta) = O(\mu)+O(\delta)$. Therefore, choosing $\tau = O\left(\Delta^{1/2}\right)$, the error in the first step of the observation scheme is of order $O\left(\Delta^{1/2}\right)$, that is $v_{\text{av}}^{(1)} - v_{\text{eq}}^{(1)} = O\left(\Delta^{1/2}\right)$. As it was mentioned before, we must substitute $v_{\text{eq}}^{(1)}$ by $v_{\text{av}}^{(1)}$ into the variable $s^{(2)}$ in (19). Thus, we can consider that during the sliding motion, $s^{(2)}$ will be bounded by a constant of order $O\left(\Delta^{1/2}\right)$, and consequently, using a filter constant $\tau_2 = O\left(\Delta^{1/4}\right)$, the error induced for the second filter will be $v_{\text{av}}^{(2)} - v_{\text{eq}}^{(2)} = O\left(\Delta^{1/4}\right)$. Following a similar analysis, we obtain an error of order $O\left(\Delta^{1/2^k}\right)$ in the k-th step for the observer reconstruction. Thus, it turns out to be that the observation error is of order $O\left(\Delta^{1/2^\ell}\right)$, recalling that ℓ is the smallest integer such that the H matrix has rank n. Thus, we can say that during the realization of the control process, the closed loop control total error ε_c is

$$\varepsilon_c = O(\mu) + O\left(\Delta^{1/2^\ell}\right).$$

VII. MAGNETIC LEVITATOR EXAMPLE

Consider the magnetic bearing system depicted in Fig. 1, which is composed of a planar rotor disk and two sets of stator electromagnets: one acting in the y-direction and the other acting in the *x-direction*. This system may be decoupled into two subsystems, one for each direction, with similar equations. (see [21] for details). Here, only the linearized subsystem in the *y-direction* is considered.

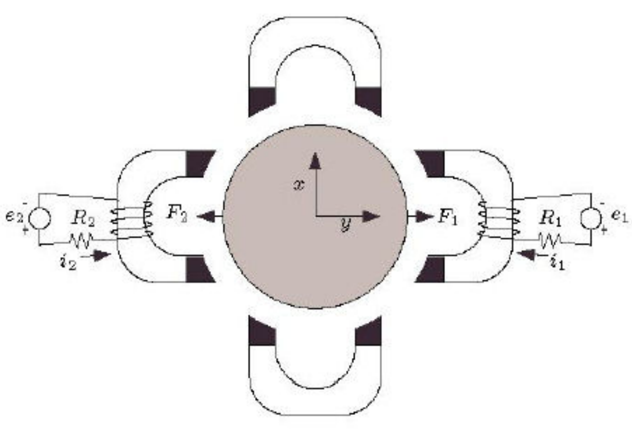

Fig. 1. Top view of a planar rotor disk magnetic bearing system [21].

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{8L_oI_o^2}{mk^2} & 0 & \frac{2L_oI_o}{mk^2} & -\frac{2L_oI_o}{mk^2} \\ 0 & -\frac{2I_o}{k} & -\frac{kR_1}{L_o} & 0 \\ 0 & \frac{2I_o}{k} & 0 & -\frac{kR_2}{L_o} \end{bmatrix}}_{A} x$$

$$+ \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{k}{L_o} \\ 0 \end{bmatrix}}_{B^1} \left(u^1 + \zeta^1(t)\right) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k}{L_o} \end{bmatrix}}_{B^2} \left(u^2 + \zeta^2(t)\right)$$

where $k = 2g_o + a$, g_o is the air gap when the rotor is in the position $y = 0$; a is a positive constant introduced to model the fact that the permeability of electromagnets is finite; $L_o > 0$ is a constant which depends on the system construction; I_o is the premagnetization constant, m is the mass of the rotor and R_1, R_2 are the resistances in the first set of stator electromagnets.The state variables $x = \begin{bmatrix} y & \dot{y} & i_1 - I_o & i_2 - I_o \end{bmatrix}^T$ and the control inputs $u^1 = e_1 - I_oR_1$ and $u^2 = e_2 - I_oR_2$.

Considering $m = 2kg$, $L_0 = 0.3mH$, $I_0 = 60mA$, $R_{1...4} = 1\Omega$ and $k = 0.002m$. With

$$C^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad C^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the controller parameters $R^{11} = diag\left(\begin{bmatrix} 1 & 1 \end{bmatrix}\right)$; $R^{22} = diag\left(\begin{bmatrix} 1 & 1 \end{bmatrix}\right)$; $Q^1 = Q^2 = 50I$, $R^{12} = R^{21} = 1$. It can be verified that for this system the triplet (A, B^i, C^i) does not have invariant zeros. The initial conditions are $x(0) = \begin{bmatrix} 0.0005 & 0 & 0.06 & 0.06 \end{bmatrix}^T$; so $y(0) = \begin{bmatrix} 0.0005 & 0.06 & 0.06 \end{bmatrix}^T$ .The pair $\left(\overline{A}, C^1\right)$ is observable with

$$\overline{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 530 & 0 & 0.2 & -0.2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, K = \begin{bmatrix} 25 & 0 & 0 \\ 686 & 0.2 & -0.2 \\ 0 & 10.2 & -0.4 \\ 0 & -0.4 & 10.8 \end{bmatrix}$$

The gain K guarantees that the $\hat{A} = A - KC^1$ matrix is Hurwitz. Applying the Lyapunov iterations algorithm [22] we find $F^1 = \begin{pmatrix} 20949 & 901 & 10 & 3 \end{pmatrix}$ and $F^2 = \begin{pmatrix} -20949 & -901 & -3 & 10 \end{pmatrix}$. The uncertainties are $\zeta^1(t) = 2sin(4t) + 2cos(2t) + 1$ and $\zeta^2(t) = 3cos(5t)$. The output ISM gains are $G^1 = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$, $G^2 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$, $M_1^1 = -10$, $M_1^2 = -10$. The simulation integration time was $10\mu s$, i.e. $\delta = \mu = 10\mu s$; $\Delta = 20\mu s$, and the filter constant was chosen as $\tau = \Delta^{1/2}$.

VIII. CONCLUSIONS

A two player differential game affected by matched uncertainties and with only the partial state measurable by all players was presented. We proposed an algebraic hierarchical observer to design the Nash equilibrium strategies based on such estimation. At the same time, the use of an output integral sliding mode term (also based on the estimation process) for the robustification of the Nash strategies was

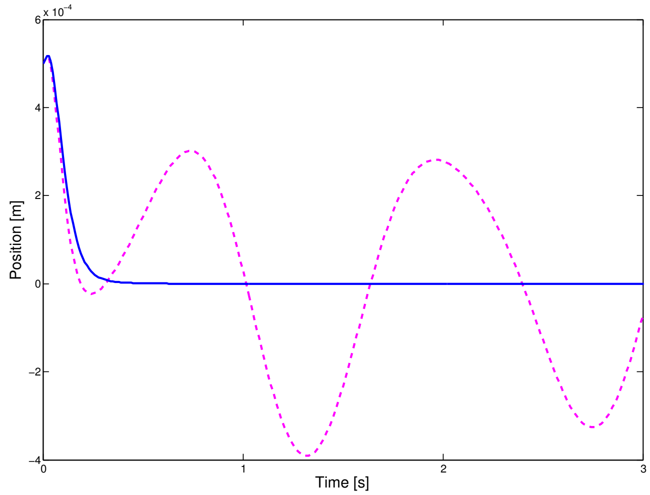

Fig. 2. Position of rotor for the perturbed system without compensation (dotted-line) and using Robust Nash strategy (solid-line).

proposed, ensuring the compensation of matched uncertainties. A simulation example showed the feasibility of this approach.

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