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Optimal Controller for Stochastic Nonlinear Polynomial Systems

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Abstract— **This paper presents the optimal quadratic-Gaussian controller for stochastic nonlinear polynomial systems with linear control input and a quadratic criterion over linear observations. The optimal closed-form controller equations are obtained using the separation principle, whose applicability to the considered problem is substantiated. As an intermediate result, the paper gives a closed-form solution of the optimal regulator (control) problem for nonlinear polynomial systems with linear control input and a quadratic criterion. Performance of the obtained optimal controller is verified in the illustrative example against the conventional LQG controller that is optimal for linearized systems. Simulation graphs demonstrating overall performance and computational accuracy of the designed optimal controller are included.**

I. INTRODUCTION

Although the optimal LQG controller problem for linear systems was solved in 1960s, based on the solutions to the optimal filtering [1] and optimal regulator [2], [3] problems, the optimal controller for nonlinear systems has to be determined using the nonlinear filtering theory (see [4], [5], [6]) and the general principles of maximum [3] or dynamic programming [7], which do not provide an explicit form for the optimal control in most cases. However, taking into account that the optimal filtering and control problems can be explicitly solved in a closed form in the linear case, and the optimal controller can be then obtained using the separation principle [2], [3], this paper exploits the same approach for designing the optimal controller for polynomial systems with linear control input over linear observations. The designed optimal solution is based on the recently obtained optimal filter and regulator for polynomial systems states. Thus, this paper continues a long tradition of the optimal control design for nonlinear systems (see, for example, [8]–[13]) and not so long research on the optimal closed-form filter design for nonlinear ([14]–[19]), and in particular, polynomial ([20]–[23]) systems. Nevertheless, to the best of authors' knowledge, the optimal closed-form controller design for polynomial systems has not been yet considered in the literature, due to the absence of closed-form solutions to the optimal filtering and control problems for polynomial system states.

This paper presents solution to the optimal quadratic-Gaussian controller problem for stochastic nonlinear polynomial systems with linear control input and a quadratic criterion over linear observations. First, the separation principle is substantiated for nonlinear polynomial systems with a quadratic criterion over linear observations. Then, the paper gives a closed-form solution of the optimal regulator (control) problem for polynomial systems with linear control input and a quadratic criterion. The obtained solution consists of a linear feedback control law and two differential equations, linear and Riccati ones, for forming the optimal control gain matrix. This result is proven in Appendix. Finally, based on that closed-form optimal control problem solution, the optimal filter for polynomial system states over linear observations [23], and the separation principle, the paper presents the optimal solution to the original quadratic-Gaussian controller problem, which has essentially the same structure as the solved optimal regulator (control) problem plus the variance equation for forming the optimal filter gain matrix. All four differential equations included in the optimal controller are interconnected.

Performance of the designed optimal controller for stochastic nonlinear polynomial systems with linear control input and a quadratic criterion over linear observations is verified in the illustrative example against the conventional LQG controller that is optimal for a linearized system. The simulation results show a big advantage in favor of the designed optimal controller for nonlinear polynomial systems: the terminal values of the cost function are 10^5 times less for the designed optimal controller than for the best controller available for a linearized system.

II. OPTIMAL CONTROLLER PROBLEM

A. Problem statement

Let (Ω, F, P) be a complete probability space with an increasing right-continuous family of σ-algebras $F_t, t \geq t_0$, and let $(W_1(t), F_t, t \geq t_0)$ and $(W_2(t), F_t, t \geq t_0)$ be independent Wiener processes. The F_t-measurable random process $(x(t), y(t))$ is described by a nonlinear differential equation with a polynomial drift term for the system state

$$dx(t) = f(x,t)dt + B(t)u(t)dt + b(t)dW_1(t), \quad x(t_0) = x_0, \tag{1}$$

and a linear differential equation for the observation process

$$dy(t) = (A_0(t) + A(t)x(t))dt + G(t)dW_2(t). \tag{2}$$

Here, $x(t) \in R^n$ is the state vector, $u(t) \in R^l$ is the control input, and $y(t) \in R^m$ is the linear observation vector, $m \leq n$. The initial condition $x_0 \in R^n$ is a Gaussian vector such that x_0, $W_1(t) \in R^p$, and $W_2(t) \in R^q$ are independent. The observation matrix $A(t) \in R^{m\times n}$ is not supposed to be invertible or even square. It is assumed that $G(t)G^T(t)$ is a positive definite

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matrix, therefore, $m \le q$. All coefficients in (1)–(2) are deterministic functions of appropriate dimensions.

The nonlinear function $f(x,t)$ is considered polynomial of n variables, components of the state vector $x(t) \in R^n$, with time-dependent coefficients. Since $x(t) \in R^n$ is a vector, this requires a special definition of the polynomial for $n > 1$. In accordance with [22], a p-degree polynomial of a vector $x(t) \in R^n$ is regarded as a p-linear form of n components of $x(t)$

$$f(x,t) = a_0(t) + a_1(t)x + a_2(t)xx^T + \ldots + a_p(t)x \ldots_{p\ times} \ldots x, \tag{3}$$

where a_0 is a vector of dimension n, a_1 is a matrix of dimension $n \times n$, a_2 is a 3D tensor of dimension $n \times n \times n$, a_p is an $(p+1)$D tensor of dimension $n \times \ldots_{(p+1)\ times} \ldots \times n$, and $x \times \ldots_{p\ times} \ldots \times x$ is a pD tensor of dimension $n \times \ldots_{p\ times} \ldots \times n$ obtained by p times spatial multiplication of the vector $x(t)$ by itself. Such a polynomial can also be expressed in the summation form

$$f_k(x,t) = a_{0\ k}(t) + \sum_i a_{1\ ki}(t)x_i(t) + \sum_{ij} a_{2\ kij}(t)x_i(t)x_j(t) + \ldots$$

$$+ \sum_{i_1 \ldots i_p} a_{p\ ki_1 \ldots i_p}(t)x_{i_1}(t) \ldots x_{i_p}(t), \quad k,i,j,i_1 \ldots\ i_p = 1,\ldots,n.$$

The quadratic cost function J to be minimized is defined as follows

$$J = \frac{1}{2}E[x^T(T)\Phi x(T) + \int_{t_0}^{T} u^T(s)R(s)u(s)ds + \int_{t_0}^{T} x^T(s)L(s)x(s)ds], \tag{4}$$

where R is positive definite and Φ, L are nonnegative definite symmetric matrices, $T > t_0$ is a certain time moment, the symbol $E[f(x)]$ means the expectation (mean) of a function f of a random variable x, and a^T denotes transpose to a vector (matrix) a.

The optimal controller problem is to find the control $u^*(t)$, $t \in [t_0,T]$, that minimizes the criterion J along with the unobserved trajectory $x^*(t)$, $t \in [t_0,T]$, generated upon substituting $u^*(t)$ into the state equation (1).

B. Separation principle

It can be observed that the separation principle [2], [3] remains valid for polynomial stochastic systems. Indeed, let us replace the unmeasured polynomial state $x(t)$, satisfying (1), with its optimal estimate $m(t)$ over linear observations $y(t)$ (2), which is obtained using the following optimal filter for polynomial states over linear observations (see [23] for the corresponding filtering problem statement and solution)

$$dm(t) = E(f(x,t) \mid F_t^Y)dt + B(t)u(t)dt + \tag{5}$$

$$P(t)A^T(t)(G(t)G^T(t))^{-1}(dy(t) - (A_0(t) + A(t)m(t))dt).$$

$$m(t_0) = E(x(t_0) \mid F_t^Y),$$

$$dP(t) = (E((x(t)-m(t))(f(x,t))^T \mid F_t^Y) + \tag{6}$$

$$E(f(x,t)(x(t)-m(t))^T) \mid F_t^Y) +$$

$$b(t)b^T(t) - P(t)A^T(t)(G(t)G^T(t))^{-1}A(t)P(t))dt,$$

$$P(t_0) = E((z(t_0)-m(t_0))(z(t_0)-m(t_0))^T \mid F_t^Y),$$

where $P(t)$ is the conditional variance of the estimation error $x(t)-m(t)$ with respect to the observations $Y(t)$.

Recall that $m(t)$ is the optimal estimate for the state vector $x(t)$, based on the observation process $Y(t) = \{y(s), t_0 \le s \le t\}$, that minimizes the Euclidean 2-norm

$$H = E[(x(t)-m(t))^T(x(t)-m(t)) \mid F_t^Y]$$

at every time moment t. Here, $E[\xi(t) \mid F_t^Y]$ means the conditional expectation of a stochastic process $\xi(t) = (x(t)-m(t))^T(x(t)-m(t))$ with respect to the σ - algebra F_t^Y generated by the observation process $Y(t)$ in the interval $[t_0,t]$. As known [24], this optimal estimate is given by the conditional expectation $m(t) = E(x(t) \mid F_t^Y)$ of the system state $x(t)$ with respect to the σ - algebra F_t^Y generated by the observation process $Y(t)$ in the interval $[t_0,t]$. As usual, the matrix function $P(t) = E[(x(t)-m(t))(x(t)-m(t))^T \mid F_t^Y]$ is the estimation error variance.

Remark 1. The equations (5) and (6) do not form a closed system of equations due to the presence of polynomial terms depending on x, such as $E(f(x,t) \mid F_t^Y)$, and $E((x(t)-m(t))f^T(x,t)) \mid F_t^Y)$, which are not expressed yet as functions of the system variables, $m(t)$ and $P(t)$. However, as shown in [20]-[23], the closed system of the filtering equations can be obtained for any polynomial state (1) over linear observations (2), using the technique of representing superior moments of the conditionally Gaussian random variable $x(t)-m(t)$ as functions of only two its lower conditional moments, $m(t)$ and $P(t)$ (see [20]-[23] for more details of this technique). Apparently, the polynomial dependence of $f(x,t)$ and $(x(t)-m(t))f^T(x,t)$ on x is the key point making this representation possible.

It is readily verified (see [2]) that the optimal control problem for the system state (1) and cost function (4) is equivalent to the optimal control problem for the estimate (5) and the cost function J represented as

$$J = E\{\frac{1}{2}m^T(T)\Phi m(T) + \frac{1}{2}\int_{t_0}^{T} u^T(s)R(s)u(s)ds + \frac{1}{2}\int_{t_0}^{T} m^T(s)L(s)m(s)ds + \frac{1}{2}\int_{t_0}^{T} tr[P(s)L(s)]ds + \frac{1}{2}tr[P(T)\Phi]\}, \tag{7}$$

where $tr[A]$ denotes trace of a matrix A. Since the latter part of J does not directly depend on control $u(t)$ or state $x(t)$, the reduced effective cost function M to be minimized takes the form

$$M = E\{\frac{1}{2}m^T(T)\Phi m(T) + \frac{1}{2}\int_{t_0}^{T} u^T(s)R(s)u(s)ds + \frac{1}{2}\int_{t_0}^{T} m(s)L(s)m(s)ds\}. \tag{8}$$

Thus, the solution for the optimal control problem specified by (1),(4) can be found solving the optimal control problem given by (5),(8). Finally, the minimal value of the criterion J should be determined using (7). This conclusion presents the separation principle for polynomial systems with a quadratic cost function.

C. Optimal control problem solution: Measured state

To handle the optimal control problem given by (5),(8), let us first give the solution to the general optimal control problem for a polynomial system with linear control input and a quadratic cost function.

Consider a polynomial system with linear control input

$$dx(t) = f(x,t)dt + B(t)u(t)dt + b(t)dW_1(t), \quad x(t_0) = x_0, \tag{9}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^l$ is the control input, the polynomial drift function $f(x,t)$ is defined by (3), and the assumptions made for the system (1) hold. The quadratic cost function J to be minimized is defined by (4).

The optimal control problem is to find the control $u^*(t)$, $t \in [t_0,T]$, that minimizes the criterion J along with the trajectory $x^*(t)$, $t \in [t_0,T]$, generated upon substituting $u^*(t)$ into the state equation (1). The solution to the stated optimal control problem is given by the following theorem.

Theorem 1. The optimal regulator for the polynomial system (9) with linear control input with respect to the quadratic criterion (4) is given by the control law

$$u^*(t) = R^{-1}(t)B^T(t)[Q(t)x(t) + p(t)], \tag{10}$$

where the matrix function $Q(t)$ is the solution of the Riccati equation

$$\dot{Q}(t) = L(t) - [a_1(t) + 2a_2(t)x(t) + 3a_3(t)x(t)x^T(t) + \ldots \tag{11}$$
$$+pa_p(t)x(t)\ldots_{p-1 \ times}\ldots x(t)]^T Q(t)-$$
$$Q(t)[a_1(t) + a_2(t)x(t) + a_3(t)x(t)x^T(t) + \ldots$$
$$+a_p(t)x(t)\ldots_{p-1 \ times}\ldots x(t)] - Q(t)B(t)R^{-1}(t)B^T(t)Q(t),$$

with the terminal condition $Q(T) = -\Phi$, and the vector function $p(t)$ is the solution of the linear equation

$$\dot{p}(t) = -Q(t)a_0(t) - [a_1(t) + 2a_2(t)x(t)+ \tag{12}$$
$$3a_3(t)x(t)x^T(t) + \ldots + pa_p(t)x(t)\ldots_{p-1 \ times}\ldots x(t)]^T p(t)-$$
$$Q(t)B(t)R^{-1}(t)B^T(t)p(t),$$

with the terminal condition $p(T) = 0$. The optimally controlled state of the polynomial system (9) is governed by the equation

$$dx(t) = f(x,t)dt + B(t)R^{-1}(t)B^T(t)[Q(t)x(t)+ \tag{13}$$
$$p(t)]dt + b(t)dW_1(t), \quad x(t_0) = x_0.$$

Proof of the theorem is given in Appendix.

D. Optimal controller problem solution: Unmeasured state

Based on the result of Theorem 1 and the preceding derivations substantiating separation of the filtering and control problems, the solution to the original optimal controller problem (1)–(4) is given as follows. The corresponding optimal control law takes the form

$$u^*(t) = R^{-1}(t)B^T(t)[Q(t)m(t) + p(t)], \tag{14}$$

where the matrix function $Q(t)$ is the solution of the Riccati equation

$$\dot{Q}(t) = L(t) - [c_1(t) + 2c_2(t)m(t) + 3c_3(t)m(t)m^T(t) + \ldots \tag{15}$$
$$+pc_p(t)m(t)\ldots_{p-1 \ times}\ldots m(t)]^T Q(t)-$$
$$Q(t)[c_1(t) + c_2(t)m(t) + c_3(t)m(t)m^T(t) + \ldots$$
$$+c_p(t)m(t)\ldots_{p-1 \ times}\ldots m(t)] - Q(t)B(t)R^{-1}(t)B^T(t)Q(t),$$

with the terminal condition $Q(T) = -\Phi$, and the vector function $p(t)$ is the solution of the linear equation

$$\dot{p}(t) = -Q(t)c_0(t) - [c_1(t) + 2c_2(t)m(t)+ \tag{16}$$
$$3c_3(t)m(t)m^T(t) + \ldots + pc_p(t)m(t)\ldots_{p-1 \ times}\ldots m(t)]^T p(t)-$$
$$Q(t)B(t)R^{-1}(t)B^T(t)p(t),$$

with the terminal condition $p(T) = 0$, where $c_0(t), c_1(t), \ldots, c_p(t)$ are the coefficients in the representation of the term $E(f(x,t) \mid F_t^Y)$ in the right-hand side of (5) as a polynomial of m, that is,

$$E(f(x,t) \mid F_t^Y) = c_0(t) + c_1(t)m+$$
$$c_2(t)mm^T + \ldots + c_p(t)m\ldots_{p \ times}\ldots m.$$

Upon substituting the optimal control (14) into the equation (5), the following optimally controlled state estimate equation is obtained

$$dm(t) = E(f(x,t) \mid F_t^Y)dt+ \tag{17}$$
$$B(t)R^{-1}(t)B^T(t)[Q(t)m(t) + p(t)]dt+$$
$$P(t)A^T(t)(B(t)B^T(t))^{-1}(dy(t) - (A_0(t) + A(t)m(t))dt).$$

with the initial condition $m(t_0) = E(x(t_0) \mid F_t^Y)$.

Thus, the optimally controlled state estimate equation (17), the gain matrix constituent equations (15) and (16), the optimal control law (14), and the variance equation (6) give the complete solution to the optimal controller problem for polynomial systems with linear control input and a quadratic cost function. This solution is not yet written in a closed form due to non-closeness of the filtering equations (5),(6) in the general situation; however, as noted in Remark 1, the closed-form solution can be obtained for any specific form of the polynomial drift $f(x,t)$ in the equation (1). In the next subsection, the closed-form optimal solution is obtained for the particular case of a second degree polynomial function $f(x,t)$.

1) Optimal controller problem solution for second degree polynomial state: Let the function

$$f(x,t) = a_0(t) + a_1(t)x + a_2(t)xx^T \quad (18)$$

be a second degree polynomial, where x is an n-dimensional vector, $a_0(t)$ is an n-dimensional vector, $a_1(t)$ is a $n \times n$-dimensional matrix, and $a_2(t)$ is a 3D tensor of dimension $n \times n \times n$. In this case, the representations for $E(f(x,t) \mid F_t^Y)$ and $E((x(t)-m(t))(f(x,t))^T \mid F_t^Y)$ as functions of $m(t)$ and $P(t)$ are derived as follows (see also the results in [20]-[23])

$$E(f(x,t) \mid F_t^Y) = a_0(t) + a_1(t)m(t) + \quad (19)$$
$$a_2(t)m(t)m^T(t) + a_2(t)P(t),$$
$$E(f(x,t)(x(t)-m(t))^T) \mid F_t^Y) +$$
$$E((x(t)-m(t))(f(x,t))^T \mid F_t^Y) = \quad (20)$$
$$a_1(t)P(t) + P(t)a_1^T(t) + 2a_2(t)m(t)P(t) + 2(a_2(t)m(t)P(t))^T.$$

Substituting the expression (19) in (5) and the expression (20) in (6), the filtering equations for the optimal estimate $m(t)$ and the error variance $P(t)$ are obtained

$$dm(t) = (a_0(t) + a_1(t)m(t) + a_2(t)m(t)m^T(t) + \quad (21)$$
$$a_2(t)P(t))dt + B(t)u(t)dt +$$
$$P(t)A^T(t)(G(t)G^T(t))^{-1}[dy(t) - (A_0(t) + A(t)m(t))dt],$$
$$m(t_0) = E(x(t_0) \mid F_t^Y)),$$
$$dP(t) = (a_1(t)P(t) + P(t)a_1^T(t) + \quad (22)$$
$$2a_2(t)m(t)P(t) + 2(a_2(t)m(t)P(t))^T +$$
$$b(t)b^T(t))dt - P(t)A^T(t)(G(t)G^T(t))^{-1}A(t)P(t)dt.$$
$$P(t_0) = E((x(t_0) - m(t_0))(x(t_0) - m(t_0))^T \mid F_t^Y)).$$

Taking into account the representation (19): $c_0(t) = a_0(t) + a_2(t)P(t)$, $c_1(t) = a_1(t)$, $c_2(t) = a_2(t)$, the equations (15) and (16) take the following particular forms in the case of a second degree polynomial function (18)

$$\dot{Q}(t) = L(t) - [a_1(t) + 2a_2(t)m(t)]^T Q(t) - \quad (23)$$
$$Q(t)[a_1(t) + a_2(t)m(t)] - Q(t)B(t)R^{-1}(t)B^T(t)Q(t),$$

with the terminal condition $Q(T) = -\Phi$, and the vector function $p(t)$ is the solution of the linear equation

$$\dot{p}(t) = -Q(t)(a_0(t) + a_2(t)P(t)) - \quad (24)$$
$$[a_1(t) + 2a_2(t)m(t)]^T p(t) - Q(t)B(t)R^{-1}(t)B^T(t)p(t),$$

with the terminal condition $p(T) = 0$.

The optimally controlled state estimate equation (17) takes the the following particular form

$$dm(t) = (a_0(t) + a_1(t)m(t) + a_2(t)m(t)m^T(t) + a_2(t)P(t))dt +$$
$$B(t)R^{-1}(t)B^T(t)[Q(t)m(t) + p(t)]dt + \quad (25)$$
$$P(t)A^T(t)(G(t)G^T(t))^{-1}(dy(t) - (A_0(t) + A(t)m(t))dt).$$

with the initial condition $m(t_0) = E(x(t_0) \mid F_t^Y)$.

Thus, the optimally controlled state estimate equation (25), the gain matrix constituent equations (23) and (24), the optimal control law (14), and the variance equation (22) give the complete closed-form solution to the optimal controller problem for second degree polynomial systems with linear control input and a quadratic cost function. In the next section, performance of the designed closed-form optimal controller for second degree polynomial systems is verified in an example.

III. EXAMPLE

This section presents an example of designing the optimal controller for a second degree polynomial system (1) over linear observations (2) with a quadratic criterion (4), using the scheme (21)–(25), and comparing it to the best linear controller available for a linearized system.

Consider a scalar quadratic polynomial state equation

$$\dot{x}(t) = 0.1x^2(t) + u(t), \quad x(0) = x_0, \quad (26)$$

and linear observations

$$y(t) = x(t) + \psi(t), \quad (27)$$

where $\psi(t)$ is a white Gaussian noise, which is the weak mean square derivative of a standard Wiener process (see [24]), and x_0 is a Gaussian random variable. The equations (26) and (27) present the conventional form for the equations (1) and (2), which is actually used in practice [25].

The controller problem is to find the control $u(t)$, $t \in [0,T]$, $T = 5$, that minimizes the criterion

$$J = \frac{1}{2}E[\int_0^T u^2(t)dt + \int_0^T x^2(t)dt]. \quad (28)$$

In other words, the control problem is to minimize the overall energy of the state x using the minimal overall energy of control u.

Let us first construct the controller where the control law $u(t)$ and the matrices $P(t)$ and $Q(t)$ are calculated in the same manner as for the optimal linear controller for the linearized system (26)

$$\dot{x}(t) = 0.2m(t)x(t) - 0.1m^2(t) + u(t), \quad x(0) = x_0, \quad (29)$$

which yields $u(t) = R^{-1}(t)B^T(t)Q(t)m(t)$ (see [2] for reference). Since $B(t) = 1$ in (26) and $R(t) = 1$ in (28), the control law is actually equal to

$$u(t) = Q(t)m(t); \quad (30)$$

where $m(t)$ satisfies the equation

$$\dot{m}(t) = a(t)m(t) + B(t)u(t) + P(t)A^T(t) \times$$
$$G(t)G^T(t))^{-1}(y(t) - (A_0(t) + A(t)m(t))),$$

$m(t_0) = m_0 = E(x_0 \mid F_{t_0}^Y)$; $Q(t)$ satisfies the Riccati equation

$$\dot{Q}(t) = -a^T(t)Q(t) - Q(t)a(t) +$$
$$L(t) - Q(t)B(t)R^{-1}(t)B^T(t)Q(t)),$$

with the terminal condition $Q(T) = -\Phi$; and $P(t)$ satisfies the Riccati equation

$$\dot{P}(t) = P(t)a(t) + a(t)P(t) + b(t)b^T(t) - P(t)A^T(t)(G(t)G^T(t))^{-1}A(t)P(t),$$

with the initial condition $P(t_0) = E((x_0 - m_0)(x_0 - m_0)^T \mid y(t_0))$. Since $t_0 = 0$, $a(t) = 0.2m(t)$, $B(t) = 1$, $b(t) = 0$ in (29), $A_0(t) = 0$, $A(t) = 1$, $G(t) = 1$ in (27), and $L = 1$ and $\Phi = 0$ in (28), the last equations turn to

$$\dot{m}(t) = 0.1m^2(t) + u(t) + P(t)(y(t) - m(t)),\ m(0) = m_0, \quad (31)$$

$$\dot{Q}(t) = 1 - 0.4m(t)Q(t) - (Q(t))^2,\ Q(5) = 0, \quad (32)$$

$$\dot{P}(t) = 0.4m(t)P(t) - (P(t))^2,\ P(0) = P_0. \quad (33)$$

Upon substituting the control (30) into (31), the controlled estimate equation takes the form

$$\dot{m}(t) = 0.1m^2(t) + Q(t)m(t) + P(t)(y(t) - m(t)),\ m(0) = m_0. \quad (34)$$

For numerical simulation of the system (26),(27) and the controller (30)-(34), the initial values $x(0) = 1$, $m(0) = 2$, and $P(0) = 10$ are assigned. The disturbance $\psi(t)$ in (27) is realized using the built-in MatLab white noise function.

The results of applying the controller (30)–(34) to the system (26),(27) are shown in Fig. 1, which presents the graph of control function (30) and the graph of the criterion (28) $J(t)$ in the interval $[0,5]$. The value of the criterion (28) at the final moment $T = 5$ is too large and unacceptable: $J(5) = 6.68 * 10^4$.

Let us now apply the optimal controller for second degree polynomial systems designed according to the optimal scheme (21)–(25),(14) to the system (26), (27). The control law (14) takes the form

$$u^*(t) = Q(t)m(t) + p(t), \quad (35)$$

where

$$\dot{m}(t) = 0.1m^2(t) + 0.1P(t) + u(t) + P(t)(y(t) - m(t)), \quad (36)$$

$$m(0) = m_0, \text{ and}$$

$$\dot{Q}(t) = 1 - 0.3m(t)Q(t) - (Q(t))^2,\ Q(5) = 0, \quad (37)$$

$$\dot{p}(t) = -0.1Q(t)P(t) - 0.2m(t)p(t) - Q(t)p(t),\ p(5) = 0, \quad (38)$$

$$\dot{P}(t) = 0.4m(t)P(t) - (P(t))^2,\ P(0) = P_0. \quad (39)$$

Note that the obtained system (36)–(39) can be solved using simple numerical methods, such as "shooting." This method consists in varying initial conditions of (37) and (38) until the given terminal conditions are satisfied.

Upon substituting the control (35) into (36), the optimally controlled estimate equation takes the form

$$\dot{m}(t) = 0.1m^2(t) + 0.1P(t) + Q(t)m(t) + p(t) + P(t)(y(t) - m(t)), \quad m(0) = m_0, \quad (40)$$

For numerical simulation of the system (26),(27) and the controller (35)-(40), the initial values $x(0) = 1$, $m(0) = 2$, and $P(0) = 10$ are assigned. The disturbance $\psi(t)$ in (27) is realized using the built-in MatLab white noise function.

The results of applying the controller (35)–(40) to the system (26),(27) are shown in Fig. 2, which presents the graph of control function (30) and the graph of the criterion (28) $J(t)$ in the interval $[0,5]$. The value of the criterion (28) at the final moment $T = 5$ is 10^5 times less than in the preceding case: $J(5) = 0.64$.

It can be observed that the final criterion values at $T = 5$ are definitively better for the designed optimal controller for second degree polynomial systems in comparison to the best controller available for a linearized system. This successfully verifies overall performance and computational accuracy of the designed optimal controller for polynomial systems.

IV. Appendix

Proof of Theorem 1. Necessity. Define the Hamiltonian function [3] for the optimal control problem (9),(4) as

$$H(x,u,q,t) = E\{\frac{1}{2}(u^T R(t)u + x^T L(t)x) + q^T \dot{x}(t)\} =$$

$$= E\{\frac{1}{2}(u^T R(t)u + x^T L(t)x) + q^T [f(x,t) + B(t)u]\}. \quad (41)$$

Applying the maximum principle condition $\partial H/\partial u = 0$ to this specific Hamiltonian function (41) yields

$$\partial H/\partial u = 0 \Rightarrow R(t)u(t) + B^T(t)q(t) = 0.$$

Accordingly, the optimal control law is obtained as

$$u^*(t) = -R^{-1}(t)B^T(t)q(t).$$

Let us seek $q(t)$ as an affine function of $x(t)$

$$q(t) = -Q(t)x(t) - p(t), \quad (42)$$

where $Q(t)$ is a square matrix of dimension $n \times n$, such that $Q(T)$ is a symmetric matrix, and $p(t)$ is a vector of dimension n. This yields the complete form of the optimal control

$$u^*(t) = R^{-1}(t)B^T(t)[Q(t)x(t) + p(t)]. \quad (43)$$

Note that the transversality condition [3] for $q(T)$ implies that $q(T) = -Q(T)x(T) - p(T) = \partial J/\partial x(T) = \Phi x(T)$ and, therefore,

$$Q(T) = -\Phi \text{ and } p(T) = 0. \quad (44)$$

Using the co-state equation $E\{dq(t)/dt\} = -\partial H/\partial x$, which gives

$$E\{-dq(t)/dt = L(t)x(t) + [\partial f(x,t)/\partial x]^T q(t)\}, \quad (45)$$

and substituting (42) into (45), we obtain

$$E\{\dot{Q}(t)x(t) + Q(t)d(x(t))/dt + \dot{p}(t) = L(t)x(t) - [\partial f(x,t)/\partial x]^T(Q(t)x(t) + p(t))\}. \quad (46)$$

Substituting the expression for $\dot{x}(t)$ from the state equation (9) into (46) yields

$$E\{\dot{Q}(t)x(t) + Q(t)f(x,t) + Q(t)B(t)u(t) + \dot{p}(t) = \quad (47)$$

$$L(t)x(t) - [\partial f(x,t)/\partial x]^T (Q(t)x(t) + p(t))\}.$$

Substituting now the representation (3) for $f(x,t)$ and the optimal control law (43) into (47) and taking into account the expression for $\partial f(x,t)/\partial x$

$$\partial f(x,t)/\partial x = a_1(t) + 2a_2(t)x +$$
$$3a_3(t)xx^T + \ldots + pa_p(t)x \ldots_{p-1 \text{ times}} \ldots x,$$

the following equation including $Q(t)$ and $p(t)$ is obtained upon omitting the expectation sign

$$\dot{Q}(t)x(t) + Q(t)[a_0(t) + a_1(t)x(t) +$$
$$a_2(t)x(t)x^T(t) + \ldots + a_p(t)x(t) \ldots_{p \text{ times}} \ldots x(t)] +$$
$$\dot{p}(t) + Q(t)B(t)R^{-1}(t)B^T(t)[Q(t)x(t) + p(t)] = \qquad (48)$$
$$L(t)x(t) - [a_1(t) + 2a_2(t)x(t) + 3a_3(t)x(t)x^T(t) + \ldots +$$
$$pa_p(t)x(t) \ldots_{p-1 \text{ times}} \ldots x(t)]^T (Q(t)x(t) + p(t)).$$

The equation (48) is satisfied, if $Q(t)$ and $p(t)$ are assigned as the solutions of the equations (11) and (12), respectively, with the initial conditions defined by (44). The necessity part is proved.

Sufficiency. The optimality of the optimal control law $u^*(t)$ given in Theorem 1 and by the formula (43) is proved in a standard way (see details, for example, in [26]): composing the Hamilton-Jacobi-Bellman (HJB) equation, corresponding to the Hamiltonian (41), and demonstrating that it is satisfied with the Bellman function $V(x,t) = -\frac{1}{2}x^T Qx - p^T x$, where $Q(t)$ and $p(t)$ are the solutions of the equations (11) and (12), respectively. The demonstration mostly repeats the formulas (44)–(48) in the necessity part. Finally, minimizing the right-hand side of the HJB equation over u yields the optimal control $u^*(t)$ in the form (43). The theorem is proved. ■

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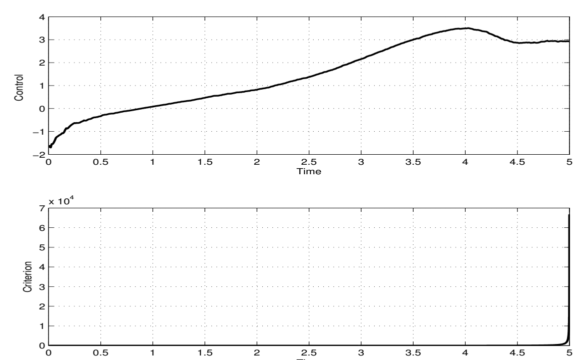

Fig. 1. Graphs of the control (30) $u(t)$ and the criterion (28) $J(t)$ corresponding to the controller (30)–(34) in the interval $[0,5]$.

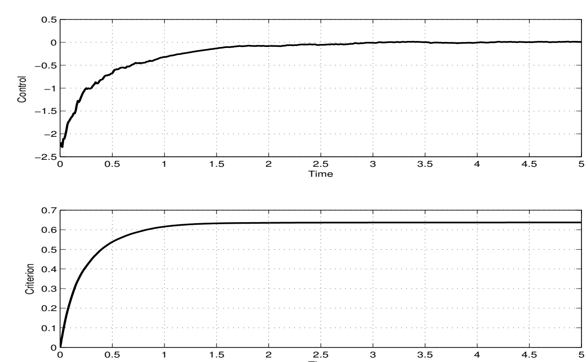

Fig. 2. Graphs of the control (35) $u(t)$ and the criterion (28) $J(t)$ corresponding to the controller (35)–(40) in the interval $[0,5]$.