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A Lyapunov-based small-gain theorem for hybrid ISS systems

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***Abstract*— A Lyapunov-based small-gain theorem is presented for hybrid systems modelled using a recently proposed framework [9]. Lyapunov small-gain theorems for continuous-time and discrete-time systems are special cases of our result. Several examples including networked control systems and reset systems are presented to illustrate our main result. Our results are general and they apply to a range of other situations.**

I. PRELIMINARIES

Small-gain theorems are ubiquitous in stability and robustness analysis and design of general control systems and they are indispensable in numerous problems, making them one of the pillars of stability theory. Small-gain theorems involving linear input-output gains are now regarded as classical and a good account of these techniques and tools can be found in [8]. In the nonlinear context, it was realized in [17] that working with linear gains is too restrictive and a small-gain result for monotone stability was proposed. Moreover, the notion of input-to-state stability (ISS) proposed by Sontag [21], [22] turned out to be very natural for formulating and stating general small-gain theorems with nonlinear gains as first illustrated in [12] for continuous-time systems. These results were shown to be extremely useful in *design* of general control systems and they have already become a part of standard texts on nonlinear control [11].

Analytic construction of Lyapunov functions is of utmost importance for nonlinear control systems because they provide a means to quantify robustness or redesign the controller to improve robustness. Small-gain theorems provide a unique opportunity for construction of Lyapunov functions by using ISS Lyapunov functions of the subsystems in the feedback loop with an appropriate small-gain condition. This approach was first used for the special case of cascades[1] of continuous-time systems [23] and the discrete-time systems [20]. Lyapunov-based small-gain theorem for general feedback connections was first reported for continuous-time systems [13] and then for discrete-time systems [15].

Recent progress in the area of hybrid control systems [9], [5] has led to a new class of hybrid models that are proving to be very general and natural from the point of view of Lyapunov stability theory [3], [4]. An appropriate extension of ISS Lyapunov functions for this class of hybrid systems was reported in [2]. These hybrid models cover a range of important classes of systems, such as networked control systems [6] and reset systems [19].

This novel hybrid systems modelling framework requires an appropriate generalization of Lyapunov small-gain theorems since the existing continuous-time and discrete-time results do not apply directly. Motivation for obtaining such results stems from their proven usefulness in the continuous-time and discrete-time settings. Moreover, there is already enough evidence that such results would be useful for various classes of hybrid systems found in the literature. Indeed, it was shown in [6] that an appropriate Lyapunov-based small-gain proof can be used to prove stability of a large class of networked control systems arising from an appropriate emulation based controller design approach. A Lyapunov-based small-gain theorem for a class of hybrid systems was considered in [16].

It is a purpose of this paper to prove a general Lyapunov-based small-gain theorem that is based on the modelling framework of [9], [5]. This result generalizes some known results, such as [16], and it allows us to deal with numerous other important cases, some of which we present for illustration.

The paper is organized as follows. In Section 2 we present background and mathematical preliminaries. Section 3 contains the main result of the paper and several special cases and examples that illustrate its utility. A summary is given in the last section.

A. Preliminaries

For locally Lipschitz functions, we use the Clarke derivative, which is defined as follows:

$$V^\circ(x;v) := \lim_{h\to 0^+,\, y\to x} \sup \frac{V(y+hv)-V(y)}{h} .$$

Consider $f:\mathbb{R}^n\to\mathbb{R}$. Then, we can define the generalized gradient of f at x:

$$\partial f(x) := \{\zeta\in\mathbb{R}^n : f^\circ(x;v)\geq\langle\zeta,v\rangle\ \forall v\in\mathbb{R}^n\} .$$

For a continuously differentiable function $f(\cdot)$, the generalized gradient $\partial f(\cdot)$ coincides with the classical notion of the gradient, which we denote as $\nabla f(\cdot)$. The following is a direct consequence of [7, Propositions 2.1.2 and 2.3.12].

Proposition 1.1: Consider two continuously differentiable functions $f_1:\mathbb{R}^n\to\mathbb{R}$ and $f_2:\mathbb{R}^n\to\mathbb{R}$. Introduce three sets $A:=\{x: f_1(x)>f_2(x)\}$; $B:=\{x: f_1(x)<f_2(x)\}$; $\Gamma := \{x : f_1(x)=f_2(x)\}$. Then, for any $v\in\mathbb{R}^n$, the

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[1]Indeed, in this case one of the gains is zero and, hence, the small-gain condition automatically holds.

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function $f(x) := \max\{f_1(x), f_2(x)\}$ satisfies

$$\begin{array}{rcl} f^\circ(x;v) & = & \langle \nabla f_1(x), v\rangle \ \forall x \in A; \\ f^\circ(x;v) & = & \langle \nabla f_2(x), v\rangle \ \forall x \in B; \\ f^\circ(x;v) & = & \max\{\langle \nabla f_1(x), v\rangle, \langle \nabla f_2(x), v\rangle\} \ \forall x \in \Gamma \end{array} \quad (1)$$

■

The following lemma was proved in [13] and it will be used in the proof of our main result:

Lemma 1.1: Let $\chi_1, \chi_2 \in \mathcal{K}_\infty$ satisfy $\chi_1 \circ \chi_2(r) < r$ for all $r > 0$. Then, there exists a $\mathcal{K}_\infty$ function ρ such that

- $\chi_1(r) < \rho(r)$ for all $r > 0$;
- $\chi_2(r) < \rho^{-1}(r)$ for all $r > 0$;
- $\rho(r)$ is C^1 on $(0, \infty)$ and $\frac{d\rho}{dr}(r) > 0$ for all $r \in (0, \infty)$.

■

Motivated by hybrid system models proposed in [9], [5] we consider hybrid systems with inputs that take the following form (see also [2]):

$$\dot{x} = f(x, u) \qquad (x, u) \in C \quad (2)$$

$$x^+ = g(x, u) \qquad (x, u) \in D\ , \quad (3)$$

where $x \in \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$, C, D are sets closed in $\mathbb{R}^n \times U$. Hence, any hybrid system is defined by a tuple (U, C, D, f, g). The solutions of the hybrid system are defined on so-called hybrid time domains. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is called a compact hybrid time domain if $E = \cup_{j=0}^{J}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_J$. E is a hybrid time domain if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain. A hybrid signal is a function defined on a hybrid time domain. A hybrid input is a hybrid signal such that u : dom $u \to U$. A hybrid arc is a function ϕ defined on a hybrid time domain dom ϕ, and such that $\phi(\cdot, j)$ is locally absolutely continuous for each j. A hybrid arc ϕ : dom $\phi \to \mathbb{R}^n$ and a hybrid signal u : dom $u \to U$ is a solution pair to the hybrid model (2), (3) if:

(S1) dom ϕ = dom u.

(S2) For all $j \in \mathbb{Z}_{\geq 0}$ and almost all $t \in \mathbb{R}_{\geq 0}$ such that $(t, j) \in$ dom ϕ we have: $(\phi(t, j), u(t, j)) \in C$, $\dot{\phi}(t, j) = f(\phi(t, j), u(t, j))$.

(S3) $(t, j) \in$ dom ϕ such that $(t, j+1) \in$ dom ϕ we have $(\phi(t, j), u(t, j)) \in D$, $\phi(t, j+1) = g(\phi(t, j), u(t, j))$.

Under general regularity conditions on (U, C, D, f, g) the hybrid system possesses solutions that may be non-unique (see [9]).

II. MAIN RESULT

Our main result is a Lyapunov small-gain theorem that applies to classes of hybrid systems (2), (3) that can be decomposed as a feedback connection of two hybrid systems. In particular, we assume that the hybrid system (2), (3) can be decomposed as follows:

$$\dot{x}_1 = f_1(x_1, x_2, u) \qquad (x, u) \in C \quad (4)$$

$$\dot{x}_2 = f_2(x_1, x_2, u) \qquad (x, u) \in C \quad (5)$$

$$x_1^+ = g_1(x_1, x_2, u) \qquad (x, u) \in D \quad (6)$$

$$x_2^+ = g_2(x_1, x_2, u) \qquad (x, u) \in D\ , \quad (7)$$

where $x := (x_1, x_2)$, $x_i \in \mathbb{R}^{n_i}$, $u \in U \subseteq \mathbb{R}^m$, $f := (f_1, f_2)$, $g := (g_1, g_2)$ and $n := n_1 + n_2$ (i.e. $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$). We regard the system (2), (3) as a feedback connection of two hybrid subsystems with states x_1 and x_2.

The following assumption is crucial for our main result and it is an appropriate generalization of assumptions typically used for continuous-time [13] and discrete-time [15] Lyapunov-based small-gain theorems:

Assumption 2.1: For $i = 1, 2$ there exist continuously differentiable functions $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ such that the following hold:

A1: There exist functions $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ and $h_i : \mathbb{R}^{n_i} \to \mathbb{R}^{p_i}$ such that for all $x_i \in X_i$

$$\psi_{i1}(|h_i(x_i)|) \leq V_i(x_i) \leq \psi_{i2}(|h_i(x_i)|)\ , \quad (8)$$

where X_i is the projection of the set $C \cup D \cup (g(D, U) \times U)$ on the subspace $\mathbb{R}^{n_i}$.

A2: There exist functions $\chi_i, \gamma_i \in \mathcal{K}_\infty$, positive definite functions α_i and positive definite functions λ_i with $\lambda_i(s) < s, \forall s > 0$ such that for all $(x, u) \in C$

$$\begin{array}{c} V_1(x_1) \geq \max\{\chi_1(V_2(x_2)), \gamma_1(|u|)\} \\ \Downarrow \\ \langle \nabla V_1(x_1), f_1(x_1, x_2, u)\rangle \leq -\alpha_1(V_1(x_1)) \end{array} \quad , \quad (9)$$

and for all $(x, u) \in D$

$$V_1(g_1(x_1, x_2, u)) \leq \max\{\lambda_1(V_1(x_1)), \chi_1(V_2(x_2)), \gamma_1(|u|)\} \quad (10)$$

Moreover, for all $(x, u) \in C$

$$\begin{array}{c} V_2(x_2) \geq \max\{\chi_2(V_1(x_1)), \gamma_2(|u|)\} \\ \Downarrow \\ \langle \nabla V_2(x_2), f_2(x_1, x_2, u)\rangle \leq -\alpha_2(V_2(x_2)) \end{array} \quad , \quad (11)$$

and for all $(x, u) \in D$

$$V_2(g_2(x_1, x_2, u)) \leq \max\{\lambda_2(V_2(x_2)), \chi_2(V_1(x_1)), \gamma_2(|u|)\} \quad (12)$$

A3: The following holds:

$$\chi_1 \circ \chi_2(s) < s \qquad \forall s > 0\ . \quad (13)$$

■

We note that in (9) and (10) we use the same function χ_1 (respectively χ_2 is the same in (11) and (12)). There are examples that show that it is not possible, in general, to exploit different functions for χ_1 in (9) and (10) (similarly, for χ_2 in (11) and (12)). Such examples are omitted due to space limitations. Moreover, note that we use different forms of ISS Lyapunov conditions on the sets C and D because this greatly simplifies the proofs.

Theorem 2.1: Suppose that Assumption 2.1 holds. Let $\rho \in \mathcal{K}_\infty$ be generated via Lemma 1.1 using χ_1, χ_2 from Assumption 2.1. Let:

$$V(x) := \max\{V_1(x_1), \rho(V_2(x_2))\}\ . \quad (14)$$

Then, there exist a positive definite function α, $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \mathcal{K}$ and a positive definite function λ, with $\lambda(s) <$

$s, \forall s > 0$, such that the following holds:
1) For all $x \in X$, where X is a projection of the set $C \cup D \cup (g(D,U) \times U)$ on $\mathbb{R}^n$, we have:

$$\psi_1(|(h_1(x_1), h_2(x_2))|) \leq V(x) \leq \psi_2(|(h_1(x_1), h_2(x_2))|) \tag{15}$$

2) For all $(x,u) \in C$ we have:

$$V(x) \geq \tilde{\gamma}_1(|u|) \implies V^\circ(x; f(x,u)) \leq -\alpha(V(x)) \tag{16}$$

3) For all $(x,u) \in D$ we have:

$$V(g(x,u)) \leq \max\{\lambda(V(x)), \tilde{\gamma}_2(|u|)\} \ . \tag{17}$$

■

Proof of Theorem 2.1: Since ρ is generated using χ_1, χ_2 via Lemma 1.1, we have

$$\chi_1(r) < \rho(r) \text{ and } \chi_2(r) < \rho^{-1}(r) \ , \quad \forall r > 0 \ . \tag{18}$$

Denote $q(r) := \frac{d\rho}{dr}(r)$ and let V be defined as in (14). The proof of item 1) is straightforward and it is omitted.

We now establish item 2). The proof is almost identical to the proof of Theorem 3.1 in [13] but it is reported for sake of completeness. Let $\tilde{\gamma}_1(s) := \max\{\rho \circ \gamma_2(s), \gamma_1(s)\}$ and $\alpha(s) := \min\{q \circ \rho^{-1}(s) \cdot \alpha_2 \circ \rho^{-1}(s), \alpha_1(s)\}$. Suppose that $V(x) \geq \tilde{\gamma}_1(|u|)$. Now we introduce three sets and investigate $V^\circ(x, f(x,u))$ on each set.

$$\begin{aligned} A &= \{(x_1,x_2) \in C : V_1(x_1) < \rho(V_2(x_2))\} \\ B &= \{(x_1,x_2) \in C : V_1(x_1) > \rho(V_2(x_2))\} \\ \Gamma &= \{(x_1,x_2) \in C : V_1(x_1) = \rho(V_2(x_2))\} \ . \end{aligned}$$

Consider first $x \in A$. In this case $V(x) = \rho(V_2(x_2))$ and we have that $V_1(x_1) < \rho(V_2(x_2))$ which implies $V_2(x_2) > \chi_2(V_1(x_1))$ using (18). Hence, (11) holds and we can write that whenever $V(x) \geq \rho \circ \gamma_2(|u|)$

$$\begin{aligned} V^\circ(x; f(x,u)) &= q(V_2(x_2))\langle \nabla V_2(x_2), f_2(x_1,x_2,u)\rangle \\ &\leq -q(V_2(x_2))\alpha_2(V_2(x_2)) \\ &= -q \circ \rho^{-1}(V(x)) \cdot \alpha_2 \circ \rho^{-1}(V(x)) \ . \end{aligned}$$

Now, consider $x \in B$. Since $V_1(x_1) > \rho(V_2(x_2))$, we have using (18) that $V_1(x_1) > \chi_1(V_2(x_2))$ and $V(x) = V_1(x_1)$. Hence, (9) holds and whenever $V(x) \geq \gamma_1(|u|)$ we can write

$$V^\circ(x; f(x,u)) = V_1^\circ(x_1, f_1(x_1,x_2,u)) \leq -\alpha_1(V(x)) \ .$$

Finally, consider $x \in \Gamma$. Then, using Proposition 1.1, we have that when $V(x) \geq \max\{\rho \circ \gamma_2(|u|), \gamma_1(|u|)\}$, then

$$\begin{aligned} V^\circ(x; f(x,u)) \leq -\min\{& q \circ \rho^{-1}(V(x)) \cdot \alpha_2 \circ \rho^{-1}(V(x)), \\ & \alpha_1(V(x))\} = -\alpha(V(x)) \ . \end{aligned}$$

Hence, (16) holds.

We now show that item 3) holds. Let

$$\lambda(s) := \max\{\lambda_1(s), \rho \circ \lambda_2 \circ \rho^{-1}(s), \chi_1 \circ \rho^{-1}(s), \rho \circ \chi_2(s)\};$$

$$\tilde{\gamma}_2(s) := \max\{\gamma_1(s), \rho \circ \gamma_2(s)\} \ .$$

Note that $\lambda(s) < s$ for all $s > 0$. Indeed, $\lambda_1(s) < s$ and $\lambda_2(s) < s$ for all $s > 0$ by assumption. The latter implies that $\rho \circ \lambda_2 \circ \rho^{-1}(s) < s$ for all $s > 0$. By construction of ρ (see (18)) we have that $\chi_1 \circ \rho^{-1}(s) < s$ and $\rho \circ \chi_2(s) < s$ for all $s > 0$, which shows that $\lambda(s) < s$ for all $s > 0$.

Using the definition of V in (14) and (10), (12) we can write for all $x \in D$, $u \in U$:

$$\begin{aligned} V(g(x,u)) &= \max\{V_1(g_1(x_1,x_2,u)), \rho(V_2(g_2(x_1,x_2,u)))\} \\ &\leq \max\{\lambda_1(V_1(x_1)), \chi_1(V_2(x_2)), \gamma_1(|u|), \\ &\quad \rho \circ \lambda_2(V_2(x_2)), \rho \circ \chi_2(V_1(x_1)), \rho \circ \gamma_2(|u|)\} \\ &= \max\{\lambda_1(V_1(x_1)), \chi_1 \circ \rho^{-1} \circ \rho(V_2(x_2)), \gamma_1(|u|), \\ &\quad \rho \circ \lambda_2 \circ \rho^{-1} \circ \rho(V_2(x_2)), \rho \circ \chi_2(V_1(x_1)), \rho \circ \gamma_2(|u|)\} \\ &\leq \max\{\lambda(V(x)), \tilde{\gamma}_2(|u|)\} \ . \end{aligned}$$

Hence, (17) holds. ■

Remark 2.1: Our construction covers pure continuous-time systems (when $D = \emptyset$) and pure discrete-time systems (when $C = \emptyset$). We note that we have not presented a construction of a smooth Lyapunov function but this can be achieved in the same manner as in [13] by using an appropriate converse ISS Lyapunov theorem for hybrid systems that was provided in [2]. We omit the details for space reasons. ■

Remark 2.2: Our condition (15) is more general than those in [13], [15] since we consider ISS with respect to general sets whereas in the cited references only ISS with respect to the origin is considered (i.e. the references consider only the case when $h_i(x_i) = x_i$). While this generalization is easily achieved if we revisit results in [13], [15], it is very useful in the context of hybrid systems in situations when additional "clock" variables need to be introduced in order to constrain the hybrid time domain with the aim of ensuring that all conditions of Assumption 2.1 hold. The use of clock variables will be illustrated in the next section. ■

III. SPECIAL CASES

The purpose of this section is to show that our main result applies to various examples, such as reset systems and networked control systems. However, for networked control systems, the conclusion of Theorem 2.1 typically does not hold without introducing extra clock variables that ensure certain conditions, such as average dwell time or reverse average dwell time [10]. With clock variables satisfying appropriate dwell time conditions, it is often possible to construct the functions V_i satisfying Assumption 2.1.

The list of examples that we present is not exhaustive and our main result applies to many other cases that are not presented for space reasons. Moreover, most proofs in this section are omitted for space reasons.

A. A second order reset system

Consider a first order plant controlled by a first order reset element (FORE) (for similar models of reset systems see [19]):

$$\begin{aligned} x \in C &\implies \left\{ \begin{array}{rcl} \dot{x}_1 &=& \lambda_p x_1 + b x_2 =: f_1(x_1,x_2) \\ \dot{x}_2 &=& \lambda_r x_2 + k x_1 =: f_2(x_1,x_2) \end{array} \right. \\ x \in D &\implies \left\{ \begin{array}{rcl} x_1^+ &=& x_1 \\ x_2^+ &=& 0 \end{array} \right. \ , \end{aligned} \tag{19}$$

where $x := (x_1, x_2)$; x_1 and x_2 are respectively (scalar) plant and controller (FORE) states; $C := \{x : x_1(x_2 - \epsilon x_1) \le 0\}$; $D := \{x : x_1(x_2 - \epsilon x_1) \ge 0\}$; $\epsilon, b, k > 0$, $\lambda_p, \lambda_r < 0$ are such that the following holds:

$$\frac{2k}{|\lambda_r|} < \epsilon < \frac{|\lambda_p|}{2b} \tag{20}$$

Next we show that all conditions of Theorem 2.1 hold. First, we let $V_1(x_1) = x_1^2$ and $V_2(x_2) = x_2^2$, which shows that **A1** of Assumption 2.1 holds with $h_1 = |x_1|$ and $h_2 = |x_2|$. Now we show that **A2** holds. Hence, considering $x \in C$, we obtain using $b > 0$ and the second inequality in (20):

$$\begin{aligned} \langle \nabla V_1(x_1), f_1(x_1, x_2)\rangle &= 2x_1(\lambda_p x_1 + b x_2) \\ &\le \lambda_p x_1^2 + (\lambda_p + 2b\epsilon)x_1^2 \\ &\le \lambda_p V_1(x_1) \ , \end{aligned} \tag{21}$$

and since $\lambda_p < 0$, we conclude that (9) holds. In a similar fashion, since $k > 0$ we have for all $x \in C$:

$$\begin{aligned} \langle \nabla V_2(x_2), f_2(x_1, x_2)\rangle &= 2x_2(\lambda_r x_2 + k x_1) \\ &\le 2\lambda_r x_2^2 + 2k\epsilon x_1^2 \end{aligned}$$

which implies:

$$\begin{aligned} V_2(x_2) &\ge \frac{2k\epsilon}{|\lambda_r|} V_1(x_1) \\ &\Downarrow \\ \langle \nabla V_2(x_2), f_2(x_1, x_2)\rangle &\le \lambda_r V_2(x_2) \ , \end{aligned} \tag{22}$$

and since $\lambda_r < 0$ we conclude that (11) holds.

Now we consider jump equations on the set D. Note that $x \in D$ implies that $|x_2| \ge \epsilon |x_1|$. Hence, we can write:

$$\begin{aligned} V_1(x_1^+) &= x_1^2 \le \epsilon^{-2} x_2^2 \\ &= \max\{0 \cdot V_1(x_1), \epsilon^{-2} V_2(x_2)\} \ , \end{aligned} \tag{23}$$

and the system satisfies (10). Now consider V_2:

$$V_2(x_2^+) = 0 = \max\{0 \cdot V_2(x_2), 0 \cdot V_1(x_1)\} \ , \tag{24}$$

which shows that (12) holds and this completes the proof of the condition **A2** in Assumption 2.1 with the gains $\kappa_1(s) = \epsilon^{-2} s$ and $\kappa_2(s) = \frac{2k\epsilon}{|\lambda_r|} s$. Hence, the first inequality in (20) guarantees that **A3** holds. Therefore, all conditions of Theorem 2.1 hold and the Lyapunov function construction in the theorem applies in this case. We emphasize that the reset system (19) may have the origin asymptotically stable even when (20) is not satisfied (for example, see [19]). The condition (20) facilitates establishing stability using the small-gain analysis tool of this paper using the magnitude squared for the individual Lyapunov functions.

B. An impulsive system

Suppose that a sequence of switching times t_i are given and an impulsive system is given by[2]:

$$\begin{aligned} \dot{x}(t) &= \tilde{f}(x(t), \mu(t), u(t)) \qquad t \in [t_i, t_{i+1}] \quad (25) \\ \mu(t_i^+) &= \tilde{g}(x(t_i), \mu(t_i), u(t_i)) \ . \quad (26) \end{aligned}$$

[2]We assume that u is continuous so that the second equation makes sense. We use the notation $\mu(t_i^+) := \lim_{t \to t_i, t > t_i} \mu(t)$.

In many cases it is very natural to decompose the above system into a feedback connection of the "continuous" x subsystem and the "jump" μ subsystem (see [16]). Moreover, sometimes it is natural to assume that the x subsystem is ISS from the inputs (μ, u) to the state x and the μ subsystems is ISS from the inputs (x, u) to the state μ. More precisely, we assume:

Assumption 3.1: There exist continuously differentiable functions W_1, W_2 such that:

B1 There exist $c > 0$, $\mathcal{K}_\infty$ functions $\overline{\psi}_{1,i}, i = 1, 2$, $\overline{\kappa}_1$ and $\overline{\gamma}_1$ such that for all x, μ, u we have:

$$\overline{\psi}_{11}(|x|) \le W_1(x) \le \overline{\psi}_{12}(|x|) \tag{27}$$

$$\begin{aligned} W_1(x) &\ge \max\{\overline{\kappa}_1(W_2(\mu)), \overline{\gamma}_1(|u|)\} \\ &\Downarrow \\ \langle \nabla W_1(x), \tilde{f}(x, \mu, u)\rangle &\le -c W_1(x) \end{aligned} \tag{28}$$

B2: There exist $d > 0$, $\mathcal{K}_\infty$ functions $\overline{\psi}_{2,i}, i = 1, 2$, $\overline{\kappa}_2$ and $\overline{\gamma}_2$ such that for all x, μ, u we have:

$$\overline{\psi}_{21}(|\mu|) \le W_2(\mu) \le \overline{\psi}_{22}(|\mu|) \tag{29}$$

$$W_2(\tilde{g}(x, \mu, u)) \le \max\{e^{-d} W_2(\mu), \overline{\kappa}_2(W_1(x)), \overline{\gamma}_2(|u|)\} \tag{30}$$

B3: There exists $\sigma > 0$ such that:

$$e^{\sigma} \overline{\kappa}_1(e^{\sigma} \overline{\kappa}_2(s)) < s, \forall s > 0 \ . \tag{31}$$

■

While **B1** and **B2** are very related to conditions **A1** and **A2** in Assumption 2.1, it is not hard to see that the conclusion of Theorem 2.1 can not hold under Assumption 3.1. However, we will show next that if all of the above conditions hold and the sequence of times satisfies

$$\underline{\epsilon} \le t_{i+1} - t_i \le \overline{\epsilon} \tag{32}$$

for any fixed $\overline{\epsilon} > 0$ and some $\underline{\epsilon} \in (0, \overline{\epsilon}]$ then Assumption 3.1 implies that functions V_i, depending on (x, μ) and the clock state that is introduced below, can be constructed to satisfy Assumption 2.1. Thus, Theorem 2.1 applies. To introduce appropriate "clock" variables, let us consider the following hybrid system:

$$\begin{aligned} \dot{\tau} &= 1 \qquad \tau \in [0, \overline{\epsilon}] \\ \tau^+ &= 0 \qquad \tau \in [\underline{\epsilon}, \overline{\epsilon}] \ . \end{aligned} \tag{33}$$

Then, it is easy to show that the hybrid time domain $E := \text{dom}\ \tau$ for any solution of the above hybrid system must satisfy (32) which implies the following:

$$\begin{aligned} j - i &\le \underline{\epsilon}^{-1}(t - s) + 1 \quad \forall (t, j), (s, i) \in E \\ &\qquad \text{with } t + j > s + i \quad (34) \\ t - s &\le \overline{\epsilon}(j - i) + \overline{\epsilon} \quad \forall (t, j), (s, i) \in E \\ &\qquad \text{with } t + j > s + i \ . \quad (35) \end{aligned}$$

Note that the hybrid domain E is different for different solutions $\tau(t, j)$ and, actually, any sequence t_i satisfying (32) corresponds to the hybrid time domain for some solution $\tau(t, j)$ of (33).

In the sequel, we imbed the impulsive system (25), (26) into an appropriate hybrid system which models a family of all impulsive systems (25), (26) with sequences t_i satisfying (32). In order to be consistent with our small-gain setting, we introduce two clock variables τ_1, τ_2 and rewrite the impulsive system (25), (26) as the following hybrid system:

$$\left.\begin{array}{l}\dot{x} = \tilde{f}(x,\mu,u)\\ \dot{\tau}_1 = 1\\ \dot{\mu} = 0\\ \dot{\tau}_2 = 1\end{array}\right\} \begin{array}{l}\tau_1 = \tau_2\\ \tau_1 \in [0,\overline{\epsilon}]\end{array} \tag{36}$$

$$\left.\begin{array}{l}x^+ = x\\ \tau_1^+ = 0\\ \mu^+ = \tilde{g}(x,\mu,u)\\ \tau_2^+ = 0\end{array}\right\} \begin{array}{l}\tau_1 = \tau_2\\ \tau_1 \in [\underline{\epsilon},\overline{\epsilon}]\end{array} . \tag{37}$$

In this case, we need to consider $C := \{(x,\tau_1,\mu,\tau_2) : \tau_1 \in [0,\overline{\epsilon}],\ \tau_1 = \tau_2\}$ and $D := \{(x,\tau_1,\mu,\tau_2) : \tau_1 \in [\underline{\epsilon},\overline{\epsilon}],\ \tau_1 = \tau_2\}$. It is not hard to see that the clock variables τ_1, τ_2 guarantee that the jumps can only occur at times t_i that satisfy (32). Moreover, we decompose the above system into two subsystems with states (x,τ_1) and (μ,τ_2) and we show next that Assumption 2.1 holds for the system (36), (37) under appropriate conditions.

Theorem 3.1: Suppose that Assumption 3.1 holds. Then, Assumption 2.1 holds for the system (36), (37) with $x_1 := (x,\tau_1)$ and $x_2 := (\mu,\tau_2)$, $h_1(x_1) := x$, $h_2(x_2) := \mu$, $C := \{(x,\tau_1) : \tau_1 \le \overline{\epsilon}\}$, $D := \{(\mu,\tau_2) : \tau_2 \in [\underline{\epsilon},\overline{\epsilon}]\}$ and

$$V_1(x_1) := e^{L_1\tau_1} W_1(x) \tag{38}$$

$$V_2(x_2) := e^{-L_2\tau_2} W_2(\mu) , \tag{39}$$

where $L_1 \in (0, \min\{c, \sigma/\overline{\epsilon}\})$ and $L_2 \in (0, \min\{d/\overline{\epsilon}, \sigma/\overline{\epsilon}\})$. ∎

The condition (32) can be relaxed and the same result proved by modifying the clock variables that are used in the hybrid model. In particular, instead of (32) one can use the notions of average and reverse average dwell times (see [10], [4]) and the following result on hybrid time domains for clock variables is useful in such cases:

Proposition 3.2: Let $\delta_1, \delta_2 \in \mathbb{R}_{\ge 0}$, $\lambda \in \mathbb{R}_{>0}$ and $N \in \mathbb{Z}_{>0}$. A hybrid time domain E satisfies

$$j - i \le \delta_1(t-s) + N \ \forall (t,j),(s,i) \in E \text{ with } t+j > s+i \tag{40}$$

$$t - s \le \delta_2(j-i) + \lambda \ \forall (t,j),(s,i) \in E \text{ with } t+j > s+i \tag{41}$$

if and only if $E = \text{dom}\,(\tau_1,\tau_2)$ for some solution (τ_1,τ_2) to the hybrid system

$$\text{for } (\tau_1,\tau_2) \in C \left\{\begin{array}{lcl}\dot{\tau}_1 & \in & [0,\delta_1]\\ \dot{\tau}_2 & = & 1\end{array}\right. \tag{42}$$

$$\text{for } (\tau_1,\tau_2) \in D \left\{\begin{array}{lcl}\tau_1^+ & \in & \tau_1 - 1\\ \tau_2^+ & = & \max\{0, \tau_2 - \delta_2\}\end{array}\right. . \tag{43}$$

where $C := [0,N] \times [0,\lambda]$ and $D := [1,N] \times [0,\lambda]$. ∎

The above result shows that a sequence of times t_i satisfying average and reverse average dwell time conditions can be reproduced by using the clock variables (τ_1,τ_2) whose hybrid model is given in Proposition 3.2. It is not hard to show that Theorem 3.1 can be modified to include the clock variables given in Proposition 3.2 but this is omitted for space reasons. Note also that (34), (35) are special cases of (40), (41) if we take $N = 1$, $\delta_2 = \lambda = \overline{\epsilon}$, $\delta_1 = \underline{\epsilon}^{-1}$.

C. Networked control systems

Motivated by results in [20], [6] we consider a class of networked control systems that are modelled as the following hybrid system:

$$\left.\begin{array}{l}\dot{x} = \tilde{f}_1(x,e,w)\\ \dot{\tau}_1 = 1\\ \dot{e} = \tilde{f}_2(x,e,w)\\ \dot{\tau}_2 = 1\\ \dot{s} = 0\end{array}\right\} \begin{array}{l}\tau_1 = \tau_2\\ \tau_1 \in [0,\overline{\epsilon}]\end{array} \tag{44}$$

$$\left.\begin{array}{l}x^+ = x\\ \tau_1^+ = 0\\ e^+ = h(s,e)\\ \tau_2^+ = 0\\ s^+ = s+1\end{array}\right\} \begin{array}{l}\tau_1 = \tau_2\\ \tau_1 \in [\underline{\epsilon},\overline{\epsilon}]\end{array} . \tag{45}$$

The above system can be obtained by following an emulation-like procedure and the variable x represents the combined states of the plant and the controller, whereas the variable e represents an error that captures the mismatch between the networked and actual values of the inputs and outputs that are sent over the network. Variables τ_1, τ_2 represent clocks and s can be thought of as the variable that counts the number of transmissions. It was shown in [20] that the jump equation for e is solely described by the network protocol. A construction similar to the one we will present below was given in [6] but we show how our general result (Theorem 2.1) applies in this case. In particular, we use the following:

Assumption 3.2: There exist continuously differentiable functions W_1, W_2 such that:

C1 There exist $\gamma_w, c > 0$, $\mathcal{K}_\infty$ functions $\overline{\psi}_{1,i}, i = 1,2$ and $\overline{\gamma}_1$ such that for all x, e, s, w we have:

$$\overline{\psi}_{11}(|x|) \le W_1(x) \le \overline{\psi}_{12}(|x|) \tag{46}$$

$$W_1(x) \ge \max\{\gamma_w W_2(s,e), \overline{\gamma}_1(|w|)\} \;\Downarrow\; \langle \nabla W_1(x), \tilde{f}_1(x,e,w)\rangle \le -cW_1(x) \tag{47}$$

C2: There exist $L, d > 0$, $\mathcal{K}_\infty$ functions $\overline{\psi}_{2,i}, i = 1,2$ and $\overline{\gamma}_2$ such that for all x, e, s, w we have:

$$\overline{\psi}_{21}(|e|) \le W_2(s,e) \le \overline{\psi}_{22}(|e|) \tag{48}$$

$$W_2(s+1, h(s,e)) \le e^{-d} W_2(s,e) \tag{49}$$

and

$$\langle \nabla W_2(e), \tilde{f}_2(x,e,w)\rangle \le L(W_2(s,e) + W_1(x)) + \overline{\gamma}_2(|w|) . \tag{50}$$

C3: $\overline{\epsilon} > 0$ is such that:

$$\overline{\epsilon} < \min\left\{\frac{d}{L_2}, \frac{1}{L_1+L_2}\ln\left(\frac{L_2-L}{\gamma_w 4L}\right)\right\} , \qquad (51)$$

where $0 < L_1 < c$ and $L_2 > L$. ■

The condition (49) characterizes the so called uniformly globally exponentially stable protocols that were introduced in [20]. We only consider ISS with linear gain in (47) in order to state an explicit condition on $\overline{\epsilon}$ in terms of γ_w and other variables in (51). The main result of this subsection is stated next:

Theorem 3.3: Suppose that Assumption 3.2 holds for the system (44), (45). Then, Assumption 2.1 holds for the system (44), (45) with $x_1 := (x, \tau_1)$ and $x_2 := (e, s, \tau_2)$, $h_1(x_1) := x$, $h_2(x_2) := e$, $C := \{(x, \tau_1) : \tau_1 \leq \overline{\epsilon}\}$, $D := \{(e, s, \tau_2) : \tau_2 \in [\underline{\epsilon}, \overline{\epsilon}]\}$ and

$$V_1(x_1) \;:=\; e^{L_1\tau_1} W_1(x) \qquad (52)$$

$$V_2(x_2) \;:=\; e^{-L_2\tau_2} W_2(s, e) \; . \qquad (53)$$

■

IV. CONCLUSIONS

We have presented a general Lyapunov small-gain theorem for a large class of hybrid systems. Continuous-time and discrete-time results are obtained as special cases of our main result. We applied our result to several examples to illustrate its generality and usefulness and we showed how introduction of certain "clock" variables aids our constructions. A result on hybrid domains for a class of hybrid systems that ensure appropriate average and reverse average dwell time conditions was presented and it may be of independent interest. Numerous other situations can be covered with our results and some these will be addressed in the journal version of this paper.

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