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Control of Acyclic Formations of Mobile Autonomous Agents

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Abstract— **This paper proposes distributed control laws for maintaining the shape of a formation of mobile autonomous agents in the plane for which the desired shape is defined in terms of prescribed distances between appropriately chosen pairs of agents. The formations considered are directed and acyclic where each given distance is maintained by only one of the associated pair of agents and there is no cycle in the sensing graph. It is shown that, except for a thin set of initial positions, the gradient-like control law can always cause a formation to converge to a finite limit in an equilibrium manifold for which all distance constraints are satisfied. The potential applicability of a control law using target positions is also discussed.**

I. Introduction

With the fast development of mobile sensor networks, intensive study has been carried out on the design of distributed control laws for the coordination of collective motion of a team of mobile autonomous agents [1], [2]. A problem of particular interest is how to maintain the overall shape of a formation given that each agent has only limited sensing capability and thus can only use information that is sensed locally [3]. Ideas from rigidity graph theory [4] have been introduced to study the formation maintenance problem for the case where each distance between chosen pairs of agents is maintained by both of the agents making up the pair [5], [6]. Since in real applications, different agents may have different sensing capabilities, the formation to be controlled might be *directed*. We say a formation is directed if each agent i can sense only the relative position of its *neighbors*. Agent i's neighbors are all other designated agents in the formation whose distances from agent i are maintained only by agent i. The notion of rigidity was generalized for directed graphs [7] to deal with directed formations, and then used to study distance constrained formation maintenance problem for directed formations [8], [9]. However, the results in [8] and [9], as well as [10], have shown that, under the proposed control laws, a formation can be guaranteed to restore its shape in the presence of only small distortions from its nominal shape. In contrast to previous work, it is the goal of this paper to analyze the characteristics of *global* convergence properties of a directed rigid formation. Although such global convergence analysis has been done for a directed, cyclic, triangular formation under a gradient-like control law [11], [12], it is known that directed formations containing cycles are, in general, difficult to control partly because of the possible instability as a result of positive feedback around cycles [10]. This paper focuses on analyzing acyclic directed formations. We refer the reader to [9] and [13] for local stability analysis for acyclic directed formations of point masses and nonholonomic robots.

The rest of the paper is organized as follows: In section II, we present a gradient-like control law for an agent with two neighbors and show convergence of the agent's position. This result is then used in section III to give global analysis of the geometry of an acyclic triangular formation and further utilized in section IV to describe the convergence behaviors of general acyclic formations in which each agent, except for the leader and the first follower, has exactly two neighbors. In section V we consider the situation when an agent has three or more neighbors and discuss a control law that makes use of target positions.

II. An agent with two neighbors

We consider a formation in the plane, shown in Figure 1, consisting of three mobile autonomous agents labelled $1, 2, 3$ in which agent 3 is required to maintain distances d_1 and d_2 from agents 1 and 2. In the sequel we write

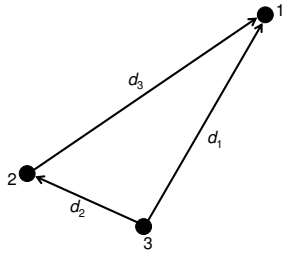

Fig. 1. Directed Triangular Point Formation

x_i for the Cartesian coordinate vector of agent i in some fixed global coordinate system in the plane, and y_{ij} for the position of agent j in some fixed coordinate system of agent i's choosing. Thus for $i \in \{1, 2, 3\}$, there is a rotation matrix R_i and a translation vector τ_i such that $y_{ij} = R_i x_j + \tau_i,\ j \in \{1, 2, 3\}$. We assume that agent i's motion is described by a simple kinematic point model of the form

$$\dot{y}_{ii} = u_i, \quad i \in \{1, 2, 3\}$$

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where u_i is agent i's control input. Thus in global coordinates,

$$\dot{x}_i = R_i^{-1} u_i, \quad i \in \{1,2,3\} \tag{1}$$

We assume that agent 3 can measure the relative positions of agents 1 and 2 in its own coordinate system. This means that agent 3 can measure the signals $R_3 z_1$ and $R_3 z_2$ where

$$z_1 = x_1 - x_3, \quad z_2 = x_2 - x_3 \tag{2}$$

As control, similar to those used in [11], [8], [9] which are gradient control laws for rigid formations, we consider

$$u_3 = R_3 z_1 e_1 + R_3 z_2 e_2, \quad i \in \{1,2,3\} \tag{3}$$

where

$$e_i = ||R_3 z_i||^2 - d_i^2, \quad i \in \{1,2\}$$

Note that the rotation matrix R_3 does not affect the definition of the e_i in that

$$e_i = ||z_i||^2 - d_i^2, \quad i \in \{1,2\} \tag{4}$$

Moreover R_3 cancels out of the update equation, so the motion of agent 3 is given by

$$\dot{x}_3 = e_1 z_1 + e_2 z_2 \tag{5}$$

Suppose that:

Assumption 1: There exist fixed points $\bar{x}_1$ and $\bar{x}_2$ such that as $t \to \infty$, there holds $\|x_1 - \bar{x}_1\| \to 0$, $\|x_2 - \bar{x}_2\| \to 0$, $\dot{x}_1 \to 0$ and $\dot{x}_2 \to 0$, with all convergence rates exponentially fast.

Assumption 2: The triangle inequalities are satisfied by the three distances d_1, d_2 and $d_3 = ||\bar{x}_1 - \bar{x}_2||$:

$$d_1 < d_2 + d_3 \quad d_2 < d_1 + d_3 \quad d_3 < d_1 + d_2 \tag{6}$$

We make two remarks on these two assumptions:

Remark 1: Assumption 1 holds in particular if agent 1 is a leader and agent 2 is a first follower with any sort of reasonable control law. We will discuss this situation in detail in section III.

Remark 2: There are two distinct triangular formations with desired distances d_i, $i = 1,2,3$. The first is as shown in Figure 1 and is referred to as a *positively-oriented* triangle. The second, called a *negatively-oriented* triangle, is the triangle which results when the triangle shown in Figure 1 is flipped over.

The main result we prove in this section is the following:

Theorem 1: Suppose Assumptions 1 and 2 are satisfied and consider the motion of agent 3 described by (5). Then x_3 converges to an equilibrium $\bar{x}_3$. Moreover, at the equilibrium, there holds

$$e_1(\bar{x}_1 - \bar{x}_3) + e_2(\bar{x}_2 - \bar{x}_3) = 0 \tag{7}$$

which implies that either $\bar{x}_1, \bar{x}_2, \bar{x}_3$ are collinear or that $e_1 = e_2 = 0$. In the event that $e_1 = e_2 = 0$, convergence is exponentially fast.

Before providing the proof of Theorem 1, we state a simple lemma.

Lemma 1: With notation as above, and in particular with $x_1(t), x_2(t)$ bounded and $||x_3||$ sufficiently large, there holds

$$e_1 z_1 + e_2 z_2 = -2||x_3||^2 x_3 + o(||x_3||^3) \tag{8}$$

Proof of Lemma 1: It is trivial to see that

$$\begin{aligned} & e_1 z_1 + e_2 z_2 \\ = \; & (||x_1 - x_3||^2 - d_1^2)(x_1 - x_3) \\ & +(||x_2 - x_3||^2 - d_2^2)(x_2 - x_3) \end{aligned}$$

and the result is immediate. ■

With this in hand, we can tackle the theorem:

Proof of Theorem 1: Because the equation for x_3 is nonlinear (and forced) existence has to be demonstrated. There is clearly a local Lipschitz property. So what must be demonstrated is that there is no finite escape time. To do this, we shall argue that solutions are bounded, and the tool for this will be a Lyapunov function. Indeed, we form a Lyapunov-like function for the motion of x_3 using ideas in [9]. Thus

$$V = \frac{1}{2}(e_1^2 + e_2^2) \tag{9}$$

It is straightforward to obtain the time derivative using the motion definition above:

$$\begin{aligned} \dot{V} &= e_1 \dot{e}_1 + e_2 \dot{e}_2 \\ &= 2e_1 z_1^T \dot{x}_1 + 2e_2 z_2^T \dot{x}_2 - 2\|e_1 z_1 + e_2 z_2\|^2 \end{aligned}$$

For large $||x_3||$, in view of Lemma 1, the last term on the right behaves as $O(||x_3||^6)$ and the first two behave as $O(||x_3||^3)$. Therefore for all sufficiently large $||x_3||$, it is clear that $\dot{V}$ is negative, which implies that V and therefore e_1, e_2 cannot be unbounded, and therefore x_3 is bounded. It follows then that we have

$$\dot{V} = \alpha(t) - 2\|e_1 z_1 + e_2 z_2\|^2 \tag{10}$$

where $\alpha(t)$ is exponentially decaying to zero. Because of the positive definite nature of V, it follows then that $e_1 z_1 + e_2 z_2$ is square integrable:

$$V(t) - V(0) = \int_0^t \alpha(s) ds - 2 \int_0^t ||e_1 z_1 + e_2 z_2||^2 ds$$

Since $V(t)$ is bounded below by zero, the square integrability property is immediate. Since also $e_1 z_1 + e_2 z_2$ is bounded and continuous and its derivative has the same property, from Barbalat's Lemma [14], this implies that $e_1 z_1 + e_2 z_2$ goes to zero. One possibility is obviously e_1 and e_2 going to zero, which would position agent 3 at correct distance from the end positions to which agents 1 and 2 converge. The second possibility is that e_1, e_2 do not go to zero; since $e_1 z_1 + e_2 z_2$ is a vector in $\mathbb{R}^2$, this means that z_1 and z_2 must be asymptotically collinear, i.e. agent 3 tends to a position that is collinear with agents 1 and 2.

Let us now argue that the number of such positions is necessarily finite. Suppose for convenience that $\bar{x}_1$ is the origin, and $\bar{x}_2$ lies at $(1,0)$. Let γ denote the x-coordinate of agent 3. Then we require

$$\lim_{t \to \infty} (\gamma(t)^2 - d_1^2)(-\gamma(t)) + ((\gamma(t) - 1)^2 - d_2^2)(1 - \gamma(t)) = 0$$

Clearly there is a limiting value, denoted by $\bar{\gamma}$, which satisfies

$$f(\bar{\gamma}) = 2\bar{\gamma}^3 - \bar{\gamma}^2 - (d_1^2 + d_2^2 - 3)\bar{\gamma} - 1 + d_2^2 = 0 \quad (11)$$

Since $f(\bar{\gamma}) \to \infty$ as $\bar{\gamma} \to \infty$ and $f(\bar{\gamma}) \to -\infty$ as $\bar{\gamma} \to -\infty$, from the continuity of $f(\bar{\gamma})$, there must exist a solution to equation (11). At the same time, since $f(\bar{\gamma})$ is cubic in $\bar{\gamma}$, there are at most three possible solutions of $\bar{\gamma}$ to equation (11). The number of possible values of $\bar{\gamma}$ depends on the values of d_1 and d_2.

In terms of the claim of the theorem, the existence of $\bar{\gamma}$ is equivalent to the existence of $\bar{x}_3$. Exponentially fast convergence is provable when the errors go to zero. Since asymptotically, z_1, z_2 are *not* collinear, it follows that for large enough t and for some constant δ, one has

$$||e_1 z_1 + e_2 z_2||^2 \geq \delta(e_1^2 + e_2^2) = \delta V$$

so that

$$\dot{V} \leq \alpha(t) - 2\delta V$$

and exponential convergence of V is immediate. This completes the proof of Theorem 1. ∎

Theorem 1 indicates that it might be possible for an acyclic triangular formation to converge to a collinear formation. Detailed study has been carried out in the past on when and how a *cyclic* triangle converge to a prescribed *non-collinear* triangular formation [11]. In the next section, we will present similar results for the acyclic triangular formation shown in Figure 1.

III. Analysis for the geometry of acyclic triangular formation

In the last section we have shown the convergence of the position of an agent with two neighbors. In this section we study the dynamics of the formation shown in Figure 1 consisting of agents 1, 2 and 3 as a whole. To simplify the analysis, we assume that agent 1 stays still, namely

$$\dot{x}_1 = 0. \quad (12)$$

This assumption can be relaxed, but we use it here for the sake of clarity as will become apparent later on. Let

$$z_3 = x_1 - x_2 \quad (13)$$

and

$$e_3 = \|z_3\|^2 - d_3^2 \quad (14)$$

Consider the control law for agent 2:

$$u_2 = z_3 e_3 \quad (15)$$

Then the closed loop system of interest is the smooth, time-invariant, dynamical system described in global coordinates by the equations

$$\begin{aligned} \dot{x}_1 &= 0 && (16)\\ \dot{x}_2 &= (x_1 - x_2)(||x_1 - x_2||^2 - d_3^2) && (17)\\ \dot{x}_3 &= (x_1 - x_3)(||x_1 - x_3||^2 - d_1^2) \\ &\quad +(x_2 - x_3)(\|x_2 - x_3\|^2 - d_2^2) && (18) \end{aligned}$$

In the sequel we shall refer this system as the *overall system*. We will show later that the overall system satisfies Assumption 1. Our aim is to study the geometry of the overall system defined by (16)-(18). Towards this end let

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \quad (19)$$

First note that as a consequence of the definitions of the z_i in (2),

$$-z_1 + z_2 + z_3 = 0 \quad (20)$$

and

$$\begin{aligned} \dot{z}_1 &= -z_1 e_1 - z_2 e_2 && (21)\\ \dot{z}_2 &= z_3 e_3 - z_1 e_1 - z_2 e_2 && (22)\\ \dot{z}_3 &= -z_3 e_3 && (23) \end{aligned}$$

Next observe that the equilibrium points of the overall system are those values of the x_i for which

$$z_3 e_3 = 0 \text{ and } z_1 e_1 + z_2 e_2 = 0 \quad (24)$$

Let $\mathcal{E}$ and $\mathcal{Z}$ denote the manifolds

$$\mathcal{E} = \{x : e = 0\} \qquad \mathcal{Z} = \{x : z = 0\} \cup \mathcal{Q} \quad (25)$$

where

$$\begin{aligned} \mathcal{Q} &= \{x : z_3 = 0, z_1 \neq 0, e_1 + e_2 = 0\} \\ &\quad \cup \{x : e_3 = 0, e_1 \text{ or } e_2 \neq 0, z_1 e_1 + z_2 e_2 = 0\} \end{aligned}$$

It is clear from (24) that every point in the manifold $\mathcal{Z} \cup \mathcal{E}$ is an equilibrium point of the overall system. The following proposition asserts that the converse is also true.

Proposition 1: The manifold $\mathcal{Z} \cup \mathcal{E}$ is the set of equilibrium points of the overall system.

Proof of Proposition 1: Since it is clear that all points in $\mathcal{Z} \cup \mathcal{E}$ are equilibrium points of the overall system, it is enough to prove that there are no others. Towards this end, consider any equilibrium point $\bar{x}$. Then from (24) it must be true that at $\bar{x}$ either $e_3 = 0$ or $z_3 = 0$.

Now suppose $e_3 = 0$. We consider the following two cases. (Case a:) $e_1 = 0$. From $z_1 e_1 + z_2 e_2 = 0$, we know $z_2 e_2 = 0$. Then either $e_2 = 0$ or $z_2 = 0$. If the former is true, then $\bar{x} \in \mathcal{E}$; if on the other hand, the latter is true, then $\bar{x} \in \{x : e_3 = 0, e_2 \neq 0, z_1 e_1 + z_2 e_2 = 0\} \subset \mathcal{Q}$. (Case b:) $e_1 \neq 0$. Then $z_1 = -\frac{e_2}{e_1} z_2$. If $z_1 = 0$, then $z_2 = 0$, and thus $\bar{x} \in \mathcal{Z}$; if on the other hand, $z_1 \neq 0$, then $e_2 \neq 0$, and thus $\bar{x} \in \{x : e_3 = 0, e_2 \neq 0, z_1 e_1 + z_2 e_2 = 0\} \subset \mathcal{Q}$. So when $e_3 = 0$, $\bar{x}$ is always in $\mathcal{E} \cup \mathcal{Z}$.

Now suppose $z_3 = 0$. Then from (20), we know $z_1 = z_2$. Then $z_1 e_1 + z_2 e_2 = 0$ implies that $z_1(e_1 + e_2) = 0$. If $z_1 = 0$, then $\bar{x} \in \{x : z = 0\} \subset \mathcal{Z}$. If on the other hand, $z_1 \neq 0$ and $e_1 + e_2 = 0$, then $\bar{x} \in \{x : z_3 = 0, z_1 \neq 0, e_1 + e_2 = 0\} \subset \mathcal{Q}$.

Combining the above discussion, we have proved that the equilibrium points of the overall system are exactly the set of points in $\mathcal{Z} \cup \mathcal{E}$. ∎

It is easy to see that $\mathcal{Z}$ and $\mathcal{E}$ are disjoint sets. In the sequel it will be shown that $\mathcal{E}$ is attractive. It is thus not unreasonable to conjecture that *all* trajectories starting outside

of $\mathcal{Z}$ might approach $\mathcal{E}$. However, we will show that this is not the case. On the other hand, the good news is that there is another manifold containing $\mathcal{Z}$, but still not large enough to intersect $\mathcal{E}$, outside of which all trajectories approach $\mathcal{E}$. The manifold to which we are referring corresponds to those formations which are collinear. To explicitly characterize this manifold, we need the following fact.

Lemma 2: The points at x_1, x_2, x_3 are collinear if and only if

$$\text{rank } \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} < 2.$$

The simple proof is omitted.

To proceed, let $\mathcal{N}$ denote the set of points in $\mathbb{R}^6$ corresponding to agent positions in the plane which are collinear. In other words

$$\mathcal{N} = \{x : \text{rank } \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} < 2,\ -z_1 + z_2 + z_3 = 0\} \tag{26}$$

Note that $\mathcal{N}$ is a closed manifold containing the $\mathcal{Z}$. Although $\mathcal{N}$ contains $\mathcal{Z}$, it is still small enough to not intersect $\mathcal{E}$:

Lemma 3: $\mathcal{N}$ and $\mathcal{E}$ are disjoint sets.

Proof of Lemma 3: Let $x \in \mathcal{N}$. Since $\mathcal{Z}$ and $\mathcal{E}$ are disjoint, it is enough to show that $\mathcal{E}$ and the complement of $\mathcal{Z}$ in $\mathcal{N}$ are disjoint. Therefore suppose that $x \notin \mathcal{Z}$ in which case $z_i \neq 0$ for some $i \in \{1, 2, 3\}$. Then there must be a number λ such that $z_j = \lambda z_i$ for some $j \in \{1, 2, 3\} - \{i\}$. Hence $z_k = -(1 + \lambda)z_i$ for $k \in \{1, 2, 3\} - \{i, j\}$. Suppose $x \in \mathcal{E}$; then $||z_i|| = d_i, \quad i \in \{1, 2, 3\}$. Thus $|\lambda| d_i = d_j$ and $|1+\lambda| d_i = d_k$. Then $d_i + d_j = d_k$ when $\lambda \geq 0$, $d_i + d_k = d_j$ when $\lambda \leq -1$, and $d_j + d_k = d_i$ when $-1 < \lambda < 0$. All of these equalities contradict (6). Therefore $\mathcal{N}$ and $\mathcal{E}$ are disjoint sets. ■

That $\mathcal{N}$ might be the place where formation control will fail is further underscored by the fact that $\mathcal{N}$ is an invariant manifold. Said differently, formations which are initially collinear, remain collinear forever. To understand why $\mathcal{N}$ is invariant, first note that for any two vectors $p, q \in \mathbb{R}^2$, $\det \begin{bmatrix} p & q \end{bmatrix} = p'Gq$ where

$$G = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

From this, (20) and the definition of $\mathcal{N}$ in (26), it follows that

$$\mathcal{N} = \{x : \det \begin{bmatrix} z_1 & z_2 \end{bmatrix} = 0\} \tag{27}$$

But along any solution to (21) – (23) for which (20) holds,

$$\frac{d}{dt} \det \begin{bmatrix} z_1 & z_2 \end{bmatrix} = -(e_1 + e_2 + e_3) \det \begin{bmatrix} z_1 & z_2 \end{bmatrix} \tag{28}$$

Thus if $\det \begin{bmatrix} z_1 & z_2 \end{bmatrix} = 0$ at $t = 0$, then $\det \begin{bmatrix} z_1 & z_2 \end{bmatrix} = 0$ for all $t > 0$. Therefore $\mathcal{N}$ is invariant as claimed.

It is interesting to note that $|\det \begin{bmatrix} z_1 & z_2 \end{bmatrix}|$ is equal to twice the area of the triangle with vertices at x_1, x_2, x_3 and for a triangle of positive area, $\text{sign}\{\det \begin{bmatrix} z_2 & z_3 \end{bmatrix}\}$ is the triangle's orientation. A proof of these elementary claims will not be given.

The dimension of $\mathcal{N}$ is less than 6 which means that "almost every" initial formation will be non-collinear. The good news is that *all* such initial formations will converge to one with the desired shape and come to rest, and moreover, the convergence will occur exponentially fast. This is in essence, the geometric interpretation of our main result on triangular formations.

Theorem 2: Every trajectory of the overall system (16) - (18) starting outside of $\mathcal{N}$, converges exponentially fast to a finite limit in $\mathcal{E}$.

The set of points $\mathbb{R}^6 - \mathcal{N}$ consists of two disjoint point sets, one for which $\det \begin{bmatrix} z_1 & z_2 \end{bmatrix} > 0$ and the other for which $\det \begin{bmatrix} z_1 & z_2 \end{bmatrix} < 0$. Once this theorem has been proved, it is easy to verify that formations starting at points such that $\det \begin{bmatrix} z_1 & z_2 \end{bmatrix} > 0$, converge to the positively-oriented triangular formations in $\mathcal{E}$ whereas formations starting at points such that $\det \begin{bmatrix} z_1 & z_2 \end{bmatrix} < 0$, converge to the corresponding negatively-oriented triangular formation in $\mathcal{E}$.

The proof of Theorem 2 involves several steps. The first is to check that the conditions in Theorem 1 are satisfied.

Lemma 4: For the overall system (16)-(18) starting outside of $\mathcal{N}$, $e_3 \to 0$ exponentially fast.

Proof of Lemma 4: Since $x(0) \notin \mathcal{N}$, we know $z_3(0) \neq 0$. Note that $\dot{e}_3 = -2||z_3||^2 e_3$. If $e_3(0) = 0$, then $e_3(t) = 0$ for all $t \geq 0$, so the conclusion holds trivially. If $e_3(0) > 0$, then for $t \geq 0$, $e_3(t) > 0$ and thus $\|z_3\| > d_3$, so $e_3(t)$ converges to 0 as fast as $e^{-2d_3 t}$, and thus the conclusion holds. If $e_3(0) < 0$, then for $t \geq 0$, $e_3(t) < 0$, and thus $||z_3(t)|| \geq ||z_3(0)||$, so $e_3(t)$ converges to 0 as fast as $e^{-2||z_3(0)||t}$. So we have proved that $e_3(t)$ always converges to 0 exponentially fast. ■

Lemma 4 implies that $\dot{x}_2 \to 0$ exponentially fast and there exists a fixed point $\bar{x}_2 \in \mathbb{R}^2$ with $\bar{x}_2 - x_1(0) = d_3$ for which $||x_2 - \bar{x}_2|| \to 0$ exponentially fast. So one can check that conditions in Theorem 1 are satisfied for the overall system (16)-(18) starting outside of $\mathcal{N}$. By applying Theorem 1 we know that any trajectory starting outside of $\mathcal{N}$ converges to a point in $\mathcal{E}$ exponentially fast if $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ are not collinear. Prompted by this, let

$$\mathcal{M} = \{x : x \in \mathcal{N},\ e_3 = 0, z_1 e_1 + z_2 e_2 = 0\}$$

Note that $\mathcal{M}$ and $\mathcal{E}$ are disjoint because $\mathcal{N}$ and $\mathcal{E}$ are. In view of Theorem 1, in order to prove Theorem 2, we need to prove that the trajectories of the overall system starting outside of $\mathcal{N}$ do not approach a limit point in $\mathcal{M}$.

We now turn to the problem of showing that all trajectories starting outside of $\mathcal{N}$ must be bounded away from $\mathcal{M}$, even in the limit as $t \to \infty$. As a first step toward this end, let us note that

$$\det[z_1(t)\ z_2(t)] = e^{-\int_\tau^t (e_1(s)+e_2(s)+e_3(s))ds} \det[z_1(\tau)\ z_2(\tau)],\ t \geq \tau \geq 0 \tag{29}$$

because of (28). In view of (27) it must therefore be true that any trajectory starting outside of $\mathcal{N}$ cannot enter $\mathcal{N}$ {and therefore $\mathcal{M}$} in finite time. It remains to be shown that any such trajectory can also not enter $\mathcal{M}$ even in the limit as $t \to \infty$. To prove that this is so we need the following fact.

Lemma 5: For any $x \in \mathcal{M}$, it must be true that

$$e_1 + e_2 + e_3 < 0$$

Proof of Lemma 5: Since for $x \in \mathcal{M}$, we have $e_3 = 0$. So we only need to prove $e_1 + e_2 < 0$ for $x \in \mathcal{M}$. It follows from $z_1 e_1 + z_2 e_2 = 0$ that

$$||z_1|||e_1| = ||z_2|||e_2|. \tag{30}$$

We consider five cases:
Case 1: $\|z_1\| = ||z_2|| + \|z_3\|$ and $||z_2|| \neq 0$. In view of (30), we know in this case $|e_1| < |e_2|$. Combining with (6), we know $e_1 > 0$ and $e_2 < 0$. Thus $e_1 + e_2 < 0$.
Case 2: $\|z_2\| = ||z_1|| + ||z_3||$ and $\|z_1\| \neq 0$. Then from (30), we know $|e_1| > |e_2|$. Again combining with (6), we have $e_1 < 0$ and $e_2 > 0$. So $e_1 + e_2 < 0$.
Case 3: $||z_1|| + \|z_2\| = ||z_3||$, $||z_1|| \neq 0$ and $\|z_2\| \neq 0$. From (6), it must be true that $e_1 < 0$ and $e_2 < 0$. So $e_1 + e_2 < 0$.
Case 4: $\|z_1\| = 0$. Then $e_1 < 0$ and $||z_2|| \neq 0$. From $z_1 e_1 + z_2 e_2 = 0$ we know $e_2 = 0$. So $e_1 + e_2 < 0$.
Case 5: $\|z_2\| = 0$. Then $e_2 < 0$ and $||z_1|| \neq 0$. From $z_1 e_1 + z_2 e_2 = 0$, we know $e_1 = 0$. So $e_1 + e_2 < 0$.
Considering the discussion of all these five cases, we conclude $e_1 + e_2 + e_3 < 0$. ■

We are now ready to show that any trajectory starting outside of $\mathcal{N}$, cannot approach $\mathcal{M}$ in the limit as $t \to \infty$. Suppose the opposite is true, namely that $x(t)$ is a trajectory starting outside of $\mathcal{N}$ which approaches $\mathcal{M}$ as $t \to \infty$. Then in view of (29), (27), and the fact that $\mathcal{M} \subset \mathcal{N}$,

$$\lim_{t\to\infty} |\det \begin{bmatrix} z_1 & z_2 \end{bmatrix}| = 0 \tag{31}$$

We will now show that this is false.

In view of Lemma 5, there must be an open set $\mathcal{V}$ containing $\mathcal{M}$ on which the inequality in the lemma continues to hold. In view of Lemma 3 and the fact that $\mathcal{M} \subset \mathcal{N}$, it is possible to choose $\mathcal{V}$ small enough so that in addition to the preceding, $\mathcal{V}$ and $\mathcal{E}$ are disjoint. For $x(t)$ to approach $\mathcal{M}$ means that for some finite time T, $x(t) \in \mathcal{V},\ t \in [T, \infty)$. This implies that $e_1 + e_2 + e_3 < 0$, $t \geq T$. In view of (29), $|\det \begin{bmatrix} z_1 & z_2 \end{bmatrix}| \geq |\det \begin{bmatrix} z_1(T) & z_2(T) \end{bmatrix}|,\ t \geq T$. But

$$|\det \begin{bmatrix} z_1(T) & z_2(T) \end{bmatrix}| = e^{-\int_0^T (e_1(s)+e_2(s)+e_3(s))ds} |\det \begin{bmatrix} z_1(0) & z_2(0) \end{bmatrix}|$$

Moreover, $|\det \begin{bmatrix} z_1(0) & z_2(0) \end{bmatrix}| > 0$ because z starts outside of $\mathcal{N}$. Therefore $|\det \begin{bmatrix} z_1 & z_2 \end{bmatrix}| > |\det \begin{bmatrix} z_1(T) & z_2(T) \end{bmatrix}| > 0,\ t \geq T$ which contradicts (31). This completes the proof of Theorem 2. ■

The preceding proves among other things that any trajectory starting outside of $\mathcal{N}$ can never enter $\mathcal{N}$. Further study reveals that equilibrium points in $\mathcal{M}$ are not stable, and thus no trajectory converges to a point in $\mathcal{M}$ in the presence of noise. Another view of the main result of this section is as follows. Disregard the fact that there are an infinity of equilibria in the set $\mathcal{E}$, which is the union of two sets of congruent and like-oriented triangles, and assume that one can regard $\mathcal{E}$ as comprising two points only, perhaps in some quotient space. These points are asymptotically stable equilibria. It then follows by arguments set out in e.g [15], that the region of attraction for each of these equilibria is an open set, the boundary of which is itself an invariant manifold. The preceding analysis then identifies the two regions of attraction as positively and negatively oriented triangles, and the boundary of the regions of attraction as collinear formations. After gaining insight into the geometry of acyclic triangular formations, we are ready to study the geometric features of general acyclic formations in which each agent has two or fewer neighbors.

IV. General acyclic formation

We can generalize our discussion on acyclic directed triangles to a class of directed, acyclic, rigid formations consisting of $n \geq 3$ agents in which each agent has two or fewer neighbors. For such formations, using *topological sorting* algorithms [16], it is possible to order the agents as $1, 2, \ldots, n$ in such a way so that agent 1 is a leader without neighbor, agent 2 is a first follower and has agent 1 as its single neighbor, and the neighbors of any agent i, $3 \leq i \leq n$, comprise exactly two agents for which the indices are all less than i. For $i \geq 3$, we write $[i]$ and $\langle i \rangle$ for the labels of agent i's two neighbors, and denote the desired distance between agents i and $[i]$ by $d_{i[i]}$ and that between agents i and $\langle i \rangle$ by $d_{i\langle i \rangle}$. Similarly, we denote the desired distance between agents 1 and 2 by d_{21}. We assume that the desired shape of the formation, defined by prescribed distances between agents and their neighbors, is realizable in the plane. Suppose there exists fixed point $\bar{x}_1$ such that as $t \to \infty$, there holds $||x_1 - \bar{x}_1|| \to 0$ and $\dot{x}_1 \to 0$ with both convergence rates exponentially fast. Now consider the gradient-like control laws that we discussed in the previous two sections, then

$$\begin{aligned} \dot{x}_2 &= (x_1 - x_2)(||x_1 - x_2||^2 - d_{21}^2) \\ \dot{x}_i &= (x_{[i]} - x_i)(||x_{[i]} - x_i||^2 - d_{i[i]}^2) \\ &\quad + (x_{\langle i \rangle} - x_i)(||x_{\langle i \rangle} - x_i||^2 - d_{i\langle i \rangle}^2),\ \ 3 \leq i \leq n \end{aligned}$$

Using an argument similar to that in the proof of Lemma 4, one can check that there exists a fixed point $\bar{x}_2$ with $||\bar{x}_2 - \bar{x}_1|| = d_{21}$ such that as $t \to \infty$, there holds $||x_2 - \bar{x}_2|| \to 0$ and $\dot{x}_2 \to 0$ with both convergence rates exponentially fast. Then using ideas similar to those in the proof of Theorem 2, one can check that if agents 1, 2 and 3 are initially in non-collinear positions, then there exists a fixed point $\bar{x}_3$ with $||\bar{x}_3 - \bar{x}_1|| = d_{31}$ and $||\bar{x}_3 - \bar{x}_2|| = d_{32}$ such that as $t \to \infty$, there holds $||x_3 - \bar{x}_3|| \to 0$ and $\dot{x}_3 \to 0$ with both covergence rates exponentially fast. Use similar arguments iteratively, one can check that for $3 < i \leq n$, if agents i, $[i]$ and $\langle i \rangle$ are initially in non-collinear positions, then there exists a fixed point $\bar{x}_i$ with $||\bar{x}_i - \bar{x}_{[i]}|| = d_{i[i]}$ and $||\bar{x}_i - \bar{x}_{\langle i \rangle}|| = d_{i\langle i \rangle}$ such that as $t \to \infty$, there holds $||x_i - \bar{x}_i|| \to 0$ and $\dot{x}_i \to 0$ with both covergence rates exponentially fast. A formal discussion of these ideas will be presented in the full length version of this paper.

In fact, the requirement that each agent has two or fewer neighbors can be relaxed. Towards this end, we introduce a different type of control laws in the next section.

V. An agent with three or more neighbors

In previous sections, we assumed that each agent in the formation has less than 3 neighbors. However, in an acyclic

graph, it may be that an agent has three or more neighbors. In this section, we will analyze this possibility. Suppose for the sake of convenience that agent 3 is required to maintain its distance from three agents 0, 1 and 2. Further, we suppose that these three agents move to fixed points $\bar{x}_0, \bar{x}_1, \bar{x}_2$ as $t \to \infty$, with convergence rates exponentially fast, and with the derivatives of x_0, x_1, x_2 also converging to zero exponentially fast. Suppose that agent 3 is required to take up a position at distances d_0, d_1, d_2 from these three agents, and these distances are consistent with the points $\bar{x}_0, \bar{x}_1, \bar{x}_2$. With the definition of $e_0 = ||z_0||^2 - d_0^2$ where $z_0 = x_3 - x_0$, and in view of the control law (3), we first take a look at the applicability of the gradient-like control law

$$\dot{x}_3 = e_0 z_0 + e_1 z_1 + e_2 z_2 \tag{32}$$

An analysis very like that of Section II suggests that an equilibrium will be attained at which

$$e_0 z_0 + e_1 z_1 + e_2 z_2 = 0 \tag{33}$$

Analyzing the equilibrium points apart from $e_0 = e_1 = e_2 = 0$ is evidently not straightforward.

On the other hand, there is a different approach than that is used in Sections II and III to design control laws for bringing agent 3 to its correct position. Given instantaneous positions of agents 1 and 2, it is possible for agent 3 to use the relative position information and the knowledge of d_1, d_2 to determine a *target position*, i.e. a point x_3^* which is at the correct distance from the current values of x_1, x_2, and given that there are in general two such points, is the closer of those points to x_3. Of course, if all three agents are collinear, there will not be a closer such point. Also, if agents 1 and 2 are initially a long way apart, there may be no such point. In this last instance, one can adopt as a target point that is on the line joining x_1 and x_2 whose join to x_3 is perpendicular to the line.

Notice that because x_1, x_2 are in generally changing with time, the target point will change with time. Notice further that after some finite time which may be time zero, and we call it t_0, there will certainly be at least one point, and in general two target points, which are at the correct distance from x_1, x_2. These will converge exponentially fast to the two points which are at distance d_1, d_2 from $\bar{x}_1, \bar{x}_2$. Assume that at this time, x_3 is not collinear with x_1, x_2. Then one can expect that x_3^* will converge exponentially to one of these points, call it $\bar{x}_3^*$.

The law governing the motion of agent 3 is

$$\dot{x}_3 = k(x_3^* - x_3) \text{ for fixed } k > 0 \tag{34}$$

and evidently we will have $x_3 \to \bar{x}_3^*$ exponentially fast.

For the case when agent 3 has agents 0, 1 and 2 as its neighbors, for any time t, one can obtain a point x_3^* with the following defining property:

$$x_3^* = argmin_{x_3}\{(e_0^2 - d_0^2)^2 + (e_1^2 - d_1^2)^2 + (e_2^2 - d_2^2)^2\} \tag{35}$$

The solution of this equation will be unique provided agents 0, 1 and 2 are not collinear, and it is reasonable to presume that their positions as $t \to \infty$ obey this property. Accordingly, for all suitably large t, collinearity will not occur. As $t \to \infty$, there holds $x_3^* \to \bar{x}_3^*$, and we use the law

$$\dot{x}_3 = k(x_3^* - x_3) \text{ for fixed } k > 0 \tag{36}$$

as before. Such type of control laws have the potential to be applied to any directed, acyclic, rigid formation.

VI. Concluding remark

In this paper we have discussed control laws that maintain the shape of a directed formation of autonomous agents. The geometric characteristics of the overall system's convergence behavior, under the proposed gradient-like control law, have been studied in detail for the case in which each agent has two, or fewer, neighbors. In fact, such a gradient-like control law can be further generalized as we have previously done for cyclic triangular formations [17]. When an agent has more than two neighbors, we have shown the implementation of a control law that makes use of target positions.

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