Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Stable Swarming by Mutual Interactions of Attraction/Alignment/Repulsion

Xiaohai Li, Jizhong Xiao and Zhijun Cai

Abstract— **In this paper we present a general decentralized controller for a swarm of agents with a dynamic topology to move in a given environment. The controller utilizes the hypothesis of Attraction/Alignment/Repulsion (A/A/R) interactions which is widely used to model fish schools in mathematical biology community. We assume that during the swarm's motion, each agent can sense and interact with its neighbors via mutual A/A/R interactions, while following the gradient force of the environment. The environment is assumed to have identical effects on all agents. With the assumption of connected graph, we show that the controller makes the velocities of all swarm members asymptotically converge to a common value. The contributions of this paper are two folds. First, the controller is general and works for any specific function of the A/A/R interactions. In other words, this paper analytically proves the common A/A/R model of fish schools in the literature. Second, all the information needed by the controller can be locally sensed, therefore, communication modules and associated problems (such as communication noise and time delay) are avoided. Simulation results are presented to verify the controller.**

I. INTRODUCTION

The natural phenomena of swarming, such as fish schooling, have invoked intensive research interests in diverse areas over decades. Some interesting phenomena were first observed and analyzed by biologists [1]-[9]. For example, tuna shoals are observed to school together with a separation of 0.16-0.25 body length in shapes of 1D "soldier", 2D "surface", and 3D "ball" [2].

In the mathematical biology community two main approaches are used to model and analyze these collective behaviors. In reference to the Lagrangian and Eulerian descriptions of fluid motion, they are referred to as Eulerian and Lagrangian approaches. The Eulerian approach applies partial differential equations to describe the evolving swarm density [1][4]; while the Lagrangian approach uses certain individual-based interaction rules or the classical Newtonian mechanics law to study each member's motion [1][3][9][10].

The typical individual-based rules used by most models of fish schools include short-distance repulsion, long-distance attraction and middle-range alignment (also called "parallel orientation") [8]-[11]. It is commonly believed that individual fish senses and adjusts its motion according to certain neighbors through the Attraction/Alighment/Repulsion (A/A/R) interactions [8]-[11]. Many different functions have been presented for the mutual A/A/R interactions [1][3][9][10]. Also many efforts have been made to propose a more reasonable or "perfect" set of functions for the A/A/R interactions by comparing the simulation results with the experimental data from real fishes [9] [11].

In [12] Jadbabaie *et al.* presented a discrete kinematic model and a decentralized controller to prove the convergence of agents' headings. In their later work [13]-[15], a continuous dynamic model and a decentralized controller are proposed for fixed and dynamic swarm topologies. The controller includes heading and velocity adjustment components, both of which are based on nearest neighbors' states. Further theoretical extensions of this work were presented in [21]. However, the controllers in [12]-[15] do not explicitly consider the environmental effects.

In [17] Gazi and Passino used a continuous kinematic model and proposed a decentralized controller for swarm aggregation. The results in [17] were extended to a class of virtual force functions in [18]. Their later works [19]-[20] demonstrated the collective behavior of swarms in different environments. In [16], Liu *et al.* used a second-order dynamic model to study the stable foraging of swarms in certain noisy environments. However, all the controllers proposed in [16]-[20] require each agent to know the global states of all other members.

In this paper, we present a general decentralized controller for a swarm of agents with dynamic topology to move in a given environment. We assume that each agent can sense and interact with its neighbors via A/A/R interactions. Moreover, according to the biological fact that many species can take advantage of the environment for their movements, for example, reef fishes can school along ocean currents [6][7] and migrating birds flock to the south by the guidance of the earth's magnetic field [5], we assume that each agent can perceive and follow the gradient force of the environmental potential during the swarm's motion. The environment is assumed to have identical effects on all agents. We assume that the swarm's topological graph is always connected. By nonsmooth stability analysis [22]-[26], we prove that the controller enables all agents' velocities to asymptotically converge to a common value.

The contributions of this paper include two aspects. First, the controller is general. No matter which specific functions the A/A/R interactions take, the collective group behavior of the swarm can be achieved. In other words, this paper analytically proves the commonly used A/A/R model of fish

This work is supported in part by U.S. Army Research Office under grant No.W911NF-05-1-0011, and U.S. National Science Foundation under grants No. CNS-0619577 and No. IIS-0644127.

Xiaohai Li is with the Department of Electrical Engineering, Graduate Center, City University of New York, New York, NY 10016, USA xli2@gc.cuny.edu

Jizhong Xiao is with the Department of Electrical Engineering, City College, City University of New York, New York, NY 10031, USA jxiao@ccny.cuny.edu

Zhijun Cai is with the Department of Electrical and Computer Engineering, University of Iowa, Iowa City, IA 52242, USA zai@engineering.uiowa.edu

978-1-4244-3124-3/08/$25.00 ©2008 IEEE

schools in the literature. Second, all the information needed by the controller can be locally sensed, therefore, communication modules are not needed for the swarm members. Subsequently, all issues related to communication links (such as time delay and communication noise) are avoided.

This paper is organized as follows. In section II we present a simplified dynamic model for individual agent and a graph representation for swarm's dynamic topology. The controller and its stability analysis are illustrated in section III. Section IV includes a few simulation results. This paper ends with some conclusions in section V.

II. Modelling of Swarms with Dynamic Topology

Consider a swarm of N agents moving in a 2D or 3D Euclidean space. For simplicity, we do not consider each agent's dimension. We assume no disturbance upon each agent. For the ith ($i = 1, 2, ..., N$) agent in the swarm, its dynamics is

$$\begin{aligned} \dot{r}_i &= v_i \\ \dot{v}_i &= u_i \end{aligned} \tag{1}$$

where $r_i \in \mathbb{R}^2$ or $\mathbb{R}^3$ is its position vector relative to ground coordinates, v_i is its velocity vector, and u_i is the control input.

Define

$$\bar{v} = \frac{1}{N}\sum_{i=1}^{N} v_i \tag{2}$$

to represent the average velocity of all swarm members. We will show that all agents' velocities converge to $\bar{v}$ by the proposed controller.

Let

$$r_{ij} = r_i - r_j, \tag{3}$$

and $\|r_{ij}\| = \|r_{ij}\|_2$ is the distance between agents i and j.

Let $r = [r_1^T, r_2^T, ..., r_N^T]^T$ and $v = [v_1^T, v_2^T, ..., v_N^T]^T$ represent the position and velocity vector of the whole swarm, respectively.

The swarm's topology can be represented by algebraic graph. According to how the information is exchanged among the agents, the graph embodies either communication or sensing relations of the swarm members. As shown in this paper, since only local and relative sensing information is needed by the proposed controller, we rather consider the swarm's topological graph as sensing graph. Also we assume the sensing relations are undirected.

Definition 2.1 (Swarm's Topological Graph) The topological graph of a swarm with dynamic topology is an undirected graph, denoted as $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, consisting of:

1) a set of vertices, $\mathcal{V} = \{1, ..., N\}$, indexed by the agents in the swarm;
2) a set of edges, $\mathcal{E} = \{(i,j) \in \mathcal{V} \times \mathcal{V} \mid \|r_{ij}\| \leq d_2\}$, in which the undirected edges represent the sensing relations between unordered pairs of vertices.

We assume that during the swarm's motion, the graph $\mathcal{G}$ is always connected.

Define

$$\mathbb{N}_i \triangleq \{j \mid (i,j) \in \mathcal{E}\} \subseteq \mathcal{V} \setminus \{i\} \tag{4}$$

to represent the set of agent i's neighbors. For a swarm with dynamic topology, $\mathbb{N}_i$ is time-variant.

III. Controller and Stability Analysis

In this section, we illustrate the general decentralized controller for swarms with dynamic topology, and prove that it enables all agents' velocities to asymptotically converge to the average.

The hypothesis of mutual A/A/R interactions for fish schools has been widely accepted in mathematical biology community for decades [1][3][8]-[11]. A fish is generally assumed to adopt different interactions (attraction, repulsion, or alignment) according to the range where the perceived neighbors are located. Fig.1 shows the diagram of two neighbored agents and the mutual interaction between them. The interaction vector $\overrightarrow{g}_{ij}$ is along the direction of r_{ij}, where $\overrightarrow{g}_{ij} \triangleq g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|}$. The interaction amplitude $g(\|r_{ij}\|)$ is a scalar function that only depends on $\|r_{ij}\|$.

Depending on the relative distance $\|r_{ij}\|$, the interaction has different dominated effects. Fig.2 shows the three non-overlapping interaction zones associated with each agent in 3D, in which d_0, d_1 and d_2 are the respective radius. Since it is not hard to implement a sensing or communication module with an omnidirectional field of view by current technology, we do not consider any blind angle with the agent as in some models of fish schools in the literature [9]-[11].

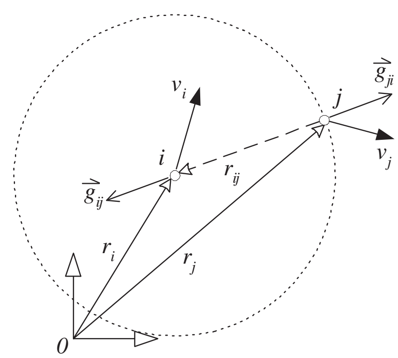

Fig. 1. Two neighbors (agent i and j) and their mutual interaction.

Many different functions have been presented for the mutual A/A/R interactions in the literature [1][3][9][10].Also many efforts have been made to propose a more reasonable or "perfect" set of functions for the A/A/R interactions by comparing the simulation results with the experimental data from real fishes [9] [11]. In this paper, the proposed controller is general and works for any set of A/A/R functions, thus, it saves the trouble in looking for "better" functions. In other words, this paper analytically proves the commonly used A/A/R model of fish schools in the literature.

On the other hand, although each agent hardly has the full knowledge about the environment, it is still reasonable to assume that it knows about the local environment around

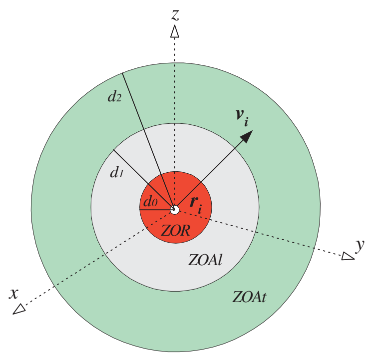

Fig. 2. Interaction zones associated with agent i: zone of repulsion (ZOR), zone of alignment (ZOAl), zone of attraction (ZOAt).

its current position. This assumption can be justified by the observations in biological systems. For example, the European robins and homing pigeons can sense the magnetic field of the earth to determine their heading directions [5], and some tropical reef fish can perceive and school along the ocean currents [6][7]. We assume that the swarm moves in an environment with a global potential function $J(r)$. The gradient of $J(r)$ at r_i is denoted by $\nabla_{r_i}J(r)$. Although each agent hardly knows $J(r)$, but local information $\nabla_{r_i}J(r)$ can be assumed to be known. We assume that the environment has identical effects on all agents, i.e., $\nabla_{r_i}J(r)$ is the same for $i = 1, ..., N$.

Based on the above discussion, we propose a general decentralized controller for each agent as

$$u_i = -k_p[v_i - \nabla_{r_i}J(r)] + \sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|}, \quad (5)$$

where $k_p > 0$ is a design constant. The implication of this controller is that each agent perceives and follows the gradient force of the environment ($\nabla_{r_i}J(r)$), and at the same time interacts with its neighbors via A/A/R forces to adjust its velocity.

For a swarm with dynamic topology, in order to avoid collision among swarm members and keep the whole group cohesive, the mutual interactions should satisfy

$$g(\|r_{ij}\|) = \begin{cases} > 0 & 0 \le \|r_{ij}\| < d_0, \\ = 0 & d_0 \le \|r_{ij}\| \le d_1, \\ < 0 & d_1 < \|r_{ij}\| \le d_2, \\ = 0 & \|r_{ij}\| > d_2. \end{cases} \quad (6)$$

Moreover, for the Lipschitz condition, let $g(\|r_{ij}\|) \ne \infty$.

For simplicity, we assume $g(\|r_{ij}\|)$ is continuous inside each interaction zone; but for generality, it is not continuous along the zone boundaries.

Note that the neighborhood defining distance in (Def.2.1) has to be the same as the radius of attraction zone (d_2). This is because agents beyond this range will not have attraction force with agent i and not be considered as its neighbors.

We will show that for any set of mutual interactions, as long as the condition in (6) is satisfied, the general controller (5) can make all agents' velocity vectors converge to a common value ($\bar{v}$).

Define error state

$$e_{v_i} = v_i - \bar{v}. \quad (7)$$

It is straight to have

$$\dot{v}_i - \dot{\bar{v}} = -k_p(v_i - \bar{v}) + \sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|} + k_p[\nabla_{r_i}J(r) - \frac{1}{N}\sum_{i=1}^{N}\nabla_{r_i}J(r)] - \frac{1}{N}\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|}.$$

Since $g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|} = -g(\|r_{ji}\|)\frac{r_{ji}}{\|r_{ji}\|}$; and because the graph $\mathcal{G}$ is assumed to be always connected and $\mathbb{N}_i$ is symmetric, so $\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|} = 0$.

Furthermore, since the environment is assumed to have identical effects on all agents, i.e., $\nabla_{r_i}J(r) = \nabla_{r_j}J(r), \forall i \ne j$. For example [20], $J(r) = \sum_{i=1}^{N} J(r_i) = \sum_{i=1}^{N} a^T \cdot r_i + b$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Then we have

$$\dot{v}_i - \dot{\bar{v}} = -k_p e_{v_i} + \sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|}. \quad (8)$$

Because $g(\|r_{ij}\|)$ is discontinuous along the interaction zone boundaries, the error dynamics is nonsmooth. So we have the following differential inclusion [23] for the error dynamics:

$$\dot{e}_{v_i} \in^{a.e.} K[e_{v_i}] = -k_p e_{v_i} + \sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|}. \quad (9)$$

Define

$$E_{ij}(\|r_{ij}\|) = \int_{\|r_{ij}\|}^{d_0} g(\tau)d\tau. \quad (10)$$

Clearly,

$$E_{ij} = \begin{cases} \|r_{ij}\| < d_0: & = \int_{\|r_{ij}\|}^{d_0} g(\tau)d\tau > 0, \\ d_0 \le \|r_{ij}\| \le d_1: & = 0, \\ d_1 < \|r_{ij}\| \le d_2: & = -\int_{d_1}^{\|r_{ij}\|} g(\tau)d\tau > 0, \\ \|r_{ij}\| > d_2: & = -\int_{d_1}^{d_2} g(\tau)d\tau = const. \end{cases} \quad (11)$$

It is not hard to see that $E_{ij}(\|r_{ij}\|)$ is discontinuous and non-differentiable at d_0 and d_1, and continuous but not differentiable at d_2. In a word, $E_{ij}(\|r_{ij}\|)$ is continuously differentiable except at d_0, d_1 and d_2.

Therefore, we have its general gradient [22] as

$$\partial E_{ij} = \begin{cases} \|r_{ij}\| < d_0: & = -g(\|r_{ij}\|), \\ \|r_{ij}\| = d_0: & = \overline{co}[-g(d_0^-),\ 0], \\ d_0 < \|r_{ij}\| < d_1: & = 0, \\ \|r_{ij}\| = d_1: & = \overline{co}[-g(d_1^+),\ 0], \\ d_1 < \|r_{ij}\| < d_2: & = -g(\|r_{ij}\|), \\ \|r_{ij}\| = d_2: & = \overline{co}[-g(d_2),\ 0], \\ \|r_{ij}\| > d_2: & = 0. \end{cases} \quad (12)$$

in which $\overline{co}[\ \cdot\ ]$ is the closed convex hull.

Theorem 3.1 Consider a swarm of N agents with dynamic topology to move in an environment that has identical effects on all agents. Assume the swarm's topological graph $\mathcal{G}$ is always connected. Then with any set of mutual interactions that satisfy the condition in (6), the decentralized controller (5) makes all agents' velocity vectors asymptotically converge to a common value ($\bar{v}$).

Proof: Use candidate Lyapunov function

$$V_t = \frac{1}{2}\sum_{i=1}^{N} e_{v_i}^T e_{v_i} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} E_{ij}(\|r_{ij}\|). \tag{13}$$

Clearly V_t is a function of e_{v_i} and r_{ij}, and smooth about e_{v_i}. But because E_{ij} is nonsmooth about $\|r_{ij}\|$, so is V_t. From (11) we know $V_t \geq 0$.

Because $g(\|r_{ij}\|) \neq \infty$, E_{ij} is locally Lipschitz, then V_t is also locally Lipschitz. From Rademacher's Theorem [22], we know that it is differentiable almost everywhere. In order to use the chain rule [24] to derive the set-valued Lie derivative of V_t [26][25], we need to show it is regular everywhere [22].

Lemma 3.1 The function V_t is regular everywhere in its domain.

Proof: Because $e_{v_i}^T e_{v_i}$ is convex, it is regular [26]; then we just need to prove E_{ij} is regular in order to show V_t is regular everywhere [26]. And since E_{ij} is smooth everywhere except at d_0, d_1 and d_2, we only need to prove it is regular at d_k $(k = 0, 1, 2)$. To show the regularity, we need to prove $E_{ij}^{\circ}(d_k, w) = E_{ij}'(d_k, w)$ [22], where $E_{ij}'(d_k, w) = \lim_{h\downarrow 0}\frac{E_{ij}(d_k+hw)-E_{ij}(d_k)}{h}$, $E_{ij}^{\circ}(d_k, w) = \lim_{y\to d_k}\sup_{h\downarrow 0}\frac{E_{ij}(y+hw)-E_{ij}(y)}{h}$, and $k = 0, 1, 2$.

For the sake of brevity, the rest of this proof is omitted. One can refer to [15] for similar details. □

Since $V_t(e_{ij}, \|r_{ij}\|)$ is locally Lipschitz, we have its generalized gradient [22] as

$$\partial V_t = \overline{co}\{\lim \nabla V_t(e_{v_i}, \|r_{ij}\|),\ \|r_{ij}\| \notin \Omega_V,\ i, j = 1, ..., N\},$$

in which Ω_V is the set with zero measure where the gradient of V_t is not defined. Specifically,

$$\partial V_t = [e_{v_1}^T, ..., e_{v_N}^T, \frac{1}{2}\partial E_{11}, ..., \frac{1}{2}\partial E_{ij}, ..., \frac{1}{2}\partial E_{NN}]^T. \tag{14}$$

For simplicity, denote $\zeta_{ij} = \frac{1}{2}\partial E_{ij}$, then

$$\partial V_t = [e_{v_1}^T, ..., e_{v_N}^T, \zeta_{11}, ..., \zeta_{ij}, ..., \zeta_{NN}]^T. \tag{15}$$

From the chain rule [24] of the set-valued Lie derivative of V_t, we know

$$\frac{dV_t}{dt} \in^{a.e.} \dot{\tilde{V}}_t, \tag{16}$$

where

$$\dot{\tilde{V}}_t = \bigcap_{\xi\in\partial V_t} \xi^T \cdot \{K[e_{v_1}], .., K[e_{v_N}], \frac{d\|r_{11}\|}{dt}, .., \frac{d\|r_{ij}\|}{dt}, .., \frac{d\|r_{NN}\|}{dt}\}^T.$$

Using (15) it becomes

$$\dot{\tilde{V}}_t = \bigcap_{\xi\in\partial V_t}\{\sum_{i=1}^{N} e_{v_i}^T \cdot K[e_{v_i}] + \sum_{i=1}^{N}\sum_{j=1}^{N}\zeta_{ij}\frac{d\|r_{ij}\|}{dt}\}. \tag{17}$$

For simplicity, let

$$\Gamma = \sum_{i=1}^{N} e_{v_i}^T \cdot K[e_{v_i}] + \sum_{i=1}^{N}\sum_{j=1}^{N}\zeta_{ij}\frac{d\|r_{ij}\|}{dt}. \tag{18}$$

To find out $\dot{\tilde{V}}_t$ on the entire domain of $\|r_{ij}\|$, we discuss it piece-wisely. Note that the nonsmoothness of both error dynamics (9) and ∂E_{ij} (12) are originated from $g(\|r_{ij}\|)$, so $K[e_{v_i}]$ and ζ_{ij} share the same nonsmooth domains.

If for $\forall i, \|r_{ij}\| > d_2$ where $j \in \{1, ..., N\} \setminus \{i\}$, then $K[e_{v_i}] = -k_p e_{v_i}$ and $\zeta_{ij} = 0$, so

$$\Gamma = \sum_{i=1}^{N} e_{v_i}^T \cdot (-k_p e_{v_i}) + \sum_{i=1}^{N}\sum_{j=1}^{N} 0 \cdot \frac{d\|r_{ij}\|}{dt} = -k_p \sum_{i=1}^{N} e_{v_i}^T e_{v_i}. \tag{19}$$

If for $\forall i$, $d_1 < \|r_{ij}\| < d_2$ or $\|r_{ij}\| < d_0$ where $j \in \mathbb{N}_i$, i.e., in the domain of attraction and repulsion zones, we have $K[e_{v_i}] = -k_p e_{v_i} + \sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|}$ and $\zeta_{ij} = -\frac{1}{2}g(\|r_{ij}\|)$. Then,

$$\Gamma = \sum_{i=1}^{N} e_{v_i}^T \cdot \{-k_p e_{v_i} + \sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|}\} - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} g(\|r_{ij}\|)\frac{d\|r_{ij}\|}{dt}. \tag{20}$$

Since for $j \notin \mathbb{N}_i$, $g(\|r_{ij}\|) = 0$, we have

$$\sum_{i=1}^{N}\sum_{j=1}^{N} g(\|r_{ij}\|)\frac{d\|r_{ij}\|}{dt} = \sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{d\|r_{ij}\|}{dt}.$$

Then equation (20) becomes

$$\begin{aligned}\Gamma &= -k_p \sum_{i=1}^{N} e_{v_i}^T e_{v_i} - \sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} \bar{v}^T g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|} + \sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} \\ &\quad g(\|r_{ij}\|) v_i^T \cdot \frac{r_{ij}}{\|r_{ij}\|} - \sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)(v_i^T \cdot \nabla_{r_{ij}}\|r_{ij}\|) \\ &= -k_p \sum_{i=1}^{N} e_{v_i}^T e_{v_i}.\end{aligned} \tag{21}$$

If for $\forall i$, $d_0 < \|r_{ij}\| < d_1$ where $j \in \mathbb{N}_i$, i.e., in the domain of alignment zone, we have $K[e_{v_i}] = -k_p e_{v_i}$ and $\zeta_{ij} = 0$, so

$$\Gamma = \sum_{i=1}^{N} e_{v_i}^T \cdot (-k_p e_{v_i}) + \sum_{i=1}^{N}\sum_{j=1}^{N} 0 \cdot \frac{d\|r_{ij}\|}{dt} = -k_p \sum_{i=1}^{N} e_{v_i}^T e_{v_i}. \tag{22}$$

If for $\forall i$, $\|r_{ij}\| = d_0$ where $j \in \mathbb{N}_i$, then $\zeta_{ij} \in \overline{co}[-\frac{1}{2}g(d_0^-),\ 0]$, and $K[e_{v_i}] = -k_p e_{v_i} + \overline{co}[g(d_0^-),\ 0]\sum_{j\in\mathbb{N}_i}\frac{r_{ij}}{\|r_{ij}\|}$. Then we have:

$$\dot{\tilde{V}}_t\,|_{d_0} = \bigcap_{\zeta_{ij}\in\overline{co}[-\frac{1}{2}g(d_0^-),\ 0]} \{\sum_{i=1}^{N} -k_p e_{v_i}^T e_{v_i} + \sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} \overline{co}[g(d_0^-),\ 0]$$

$$(v_i - \bar{v})^T \cdot \frac{r_{ij}}{\|r_{ij}\|} + \sum_{i=1}^{N} \sum_{j \in \mathbb{N}_i} \zeta_{ij} [\frac{dr_{ij}}{dt}]^T \cdot \nabla_{r_{ij}} \|r_{ij}\| \}$$
$$= \bigcap_{\zeta_{ij} \in \overline{co}[-\frac{1}{2}g(d_0^-),\, 0]} \{\sum_{i=1}^{N} -k_p e_{v_i}^T e_{v_i} + (\overline{co}[g(d_0^-),\, 0] + 2\zeta_{ij})$$
$$\sum_{i=1}^{N} \sum_{j \in \mathbb{N}_i} v_i^T \cdot \frac{r_{ij}}{\|r_{ij}\|} \} \subseteq \sum_{i=1}^{N} -k_p e_{v_i}^T e_{v_i} + \bigcap_{\zeta_{ij} \in \overline{co}[-\frac{1}{2}g(d_0^-),\, 0]}$$
$$\{(\overline{co}[g(d_0^-),\, 0] + 2\zeta_{ij}) \sum_{i=1}^{N} \sum_{j \in \mathbb{N}_i} v_i^T \cdot \frac{r_{ij}}{\|r_{ij}\|} \}.$$

Since

$$\bigcap_{\zeta_{ij} \in \overline{co}[-\frac{1}{2}g(d_0^-),\, 0]} \{\overline{co}[g(d_0^-),\, 0] + 2\zeta_{ij}\} = \{0\}, \qquad (23)$$

then

$$\dot{\tilde{V}}_t \mid_{\|r_{ij}\| = d_0} \subseteq \{\sum_{i=1}^{N} -k_p e_{v_i}^T e_{v_i}\}. \qquad (24)$$

For $\|r_{ij}\| = d_1$ where $j \in \mathbb{N}_i$, $\zeta_{ij} \in \overline{co}[-\frac{1}{2}g(d_1^+),\, 0]$, and $K[e_{v_i}] = -k_p e_{v_i} + \overline{co}[g(d_1^+),\, 0] \sum_{j \in \mathbb{N}_i} \frac{r_{ij}}{\|r_{ij}\|}$. Similarly to the above, we have:

$$\dot{\tilde{V}}_t \mid_{\|r_{ij}\| = d_1} \subseteq \{\sum_{i=1}^{N} -k_p e_{v_i}^T e_{v_i}\}. \qquad (25)$$

Similarly, for $\|r_{ij}\| = d_2$ we can have

$$\dot{\tilde{V}}_t \mid_{\|r_{ij}\| = d_2} \subseteq \{\sum_{i=1}^{N} -k_p e_{v_i}^T e_{v_i}\}. \qquad (26)$$

Therefore, on the whole domain, we have

$$\dot{\tilde{V}}_t \subseteq \{\alpha \mid \alpha = \sum_{i=1}^{N} -k_p e_{v_i}^T e_{v_i} \leq 0\}. \qquad (27)$$

And since $\frac{dV_t}{dt} \in^{a.e.} \dot{\tilde{V}}_t$ (16), then all $\frac{d}{dt}V_t \leq 0$. This means e_{v_i} is stable for any agent. Furthermore, from the nonsmooth version of Barbalat's lemma, we know that $(e_{v_i}, \|r_{ij}\|)$ approaches the largest invariant set in

$$\bar{S} = cl(\{(e_{v_i}, \|r_{ij}\|) \mid 0 \in \dot{\tilde{V}}_t,\ i,j = 1, ..., N\})$$
$$= cl(\{(0, \|r_{ij}\|), \quad i,j = 1, ..., N\}). \qquad (28)$$

where $cl(\cdot)$ is the closure of a set. This means that the velocity convergence is asymptotic. □

Remark: Note that all the information needed by the controller (5) can be locally sensed.The advantage of this configuration is that by the proposed controller, communication modules are not needed for swarm members. Subsequently all the issues related to communication setup (such as time delay and communication noise) are relieved.

IV. SIMULATIONS

In this section simulation results are presented to demonstrate the effectiveness of the proposed controller.

We select the mutual interactions to be piece-wise as: for $\|r_{ij}\| < d_0$: $g(\|r_{ij}\|) = -30\|r_{ij}\| + 320$; and for $d_2 \geq \|r_{ij}\| > d_1$: $g(\|r_{ij}\|) = -30\|r_{ij}\| + 400$, in which $d_0 = 10$, $d_1 = 14$ and $d_2 = 24$. The alignment zone lies between $d_0 \sim d_1$. Clearly $g(\|r_{ij}\|)$ is not continuous at d_0, d_1 and d_2. We assume that the environment has identical effect on all agents. Agents' initial positions and velocities are randomly given. The design constant $k_p = 5$. In the following figures, the stars and circles represent agents' initial and final positions, respectively.

Fig. 3–5 show a swarm of agents ($N = 25$) moving in a 2D linear environment. The potential profile of the environment is $\nabla_{r_i} J(r) = [-0.4, \ -0.4]^T$. Fig.3 shows the agents' trajectories on $x - y$ plane, and Fig.4 shows the convergence of their velocities. It is clear to see that all agents' velocities asymptotically converge to a common value. The swarm's steady topology is shown in Fig. 5.

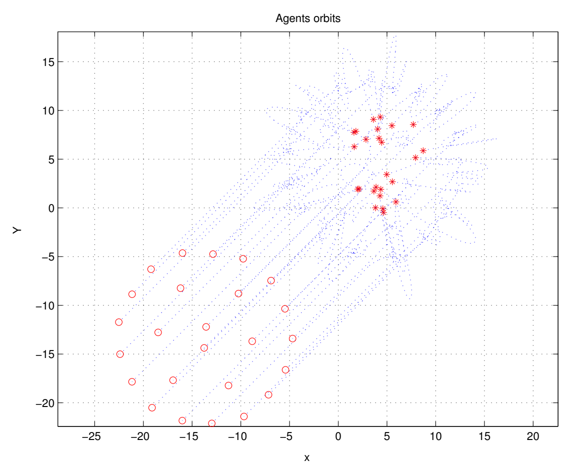

Fig. 3. Agents' trajectories on x-y plane when the swarm moves in a 2D linear environment ($N = 25$).

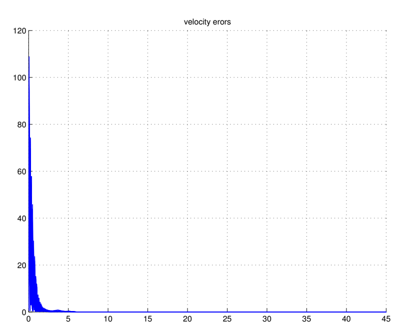

Fig. 4. Agents' velocity errors when the swarm moves in a 2D linear environment ($N = 25$).

Fig.6–7 show a swarm ($N = 50$) moving in a 2D environment with sinusoid wave profile. Fig.6 shows agent trajectories, and fig.7 shows swarm's steady topology. Note that the steady pattern of swarm's topology in Fig.7 has a

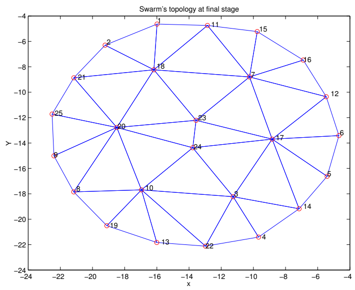

Fig. 5. Swarm's steady topology when the swarm moves in a 2D linear environment ($N = 25$).

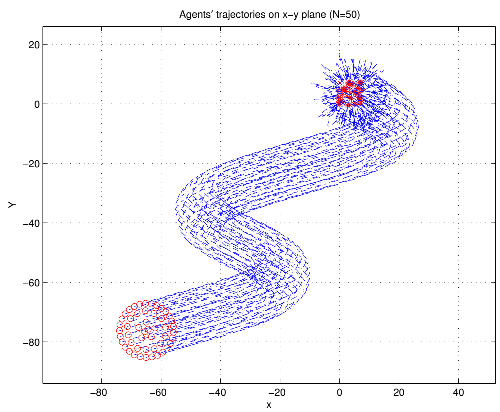

Fig. 6. Agents' trajectories on x-y plane when the swarm moves in a 2D sinusoid environment ($N = 50$).

similar structure as in Fig.5. This is interesting to be studied more.

V. CONCLUSIONS AND FUTURE WORKS

In this paper, we propose a general decentralized controller that utilizes A/A/R interactions for a swarm of agents with dynamic topology to move in given environments. With the assumption of connected graph, We show that the controller makes all agents' velocities asymptotically converge to a common value. Future work will focus on issues arising from practical applications, such as disturbance, sensing noise and fluctuation of the environment. The spacing among swarm members and swarm's steady pattern are some other important issues to be studied.

REFERENCES

[1] A. Mogilner, L. Edelstein-keshet, "A non-local model for a swarm", *J. Math. Biol.*, vol. 38, pp. 534-570, 1999.

[2] N. Newlands, PhD Dissertation, "Shoaling dynamics and abundance estimation: Atlantic Bluefin Tuna (Thunnus thynnus)", Fisheries Center, UBC, Vancouver BC, Canada, 2002.

[3] H. S. Niwa, "Newtonian dynamical approach to fish schooling", *J. Theor. Biol.*, vol. 181, pp. 47-63, 1996.

[4] E. Holmes, M. Lewis, and J. Banks, "PDE in ecology: spatial interactions and populationh dynamics", *Ecology*, 75(1), pp.17-29, 1994.

[5] W. Wiltschko and R. Wiltschko, "Magnetic orientation in birds", *J. Experim. Biol.*, 199, pp. 29-38, 1996.

[6] A. Muss, D. R. Robertson, A. Stepien, P. Wirtz and B. W. Bowen, "Phylogeography of Ophioblennius: The role of ocean currents and geography in reef fish evolution", *Evolution*, 55(3), pp. 561-571, 2001.

[7] P. R. Armsworth, "Directed motion in the sea: Efficient swimming by reef fish larvae", *J. Theor. Biol.* , (2001)210, pp. 81-91, 2001.

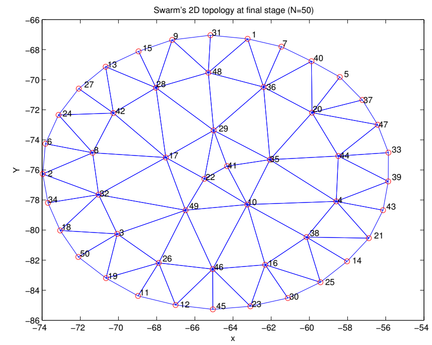

Fig. 7. Swarm's steady pattern when the swarm moves in a 2D sinusoid environment ($N = 50$).

[8] J. K. Parrish, S. V. Viscido, and D. Grunbaum, "Self-organized fish schools: An examination of emergent properties", *Biol. Bull.* 202: 296-305, June 2002.

[9] S. V. Viscido, J. K. Parrish, and D. Grunbaum, "Individual behavior and emergent properties of fish schools: a comparison of observation and theory", *Mar. Ecol. Prog. Ser.*, Vol.273: 239-249, 2004.

[10] I. D. Couzin, J. Krause, R. James, G. D. Ruxton and N. R. Franks, "Collective memory and spatial sorting in animal groups", *J. Theor. Biol.* 218: 1-11, 2002.

[11] H. Kunz and C. K. Hemelrijk, "Artificial fish schools: Collective effects of school size, body size, and body form", *Artificial Life*, 9:237-253, 2003.

[12] A. Jadbabai, J. Lin and A. S. Morse, "Coordiantion of groups of mobile autonomous agents using nearest neghborhood rules", *IEEE Trans. on Autom. Contr.*, 48(6), pp. 988-1001, 2003

[13] H. G. Tanner, A. Jababaie, and G. J. Pappas, "Flocking in fixed and switching networks", *IEEE Trans. on Autom. Contr.*, vol 52, no 5, May 2007 pp 863-868.

[14] H. G. Tanner, A. Jadbabaie and G. J. Pappas, "Stable flocking of mobile agent, Part I: Fixed toplogy", *Proc. of Conf. Decision Contr.* Maui, Hawaii, pp. 2010-2015, 2003.

[15] H. G. Tanner, A. Jadbabaie and G. J. Pappas, "Stable flocking of mobile agent, Part II: Dyanmic toplogy", *Proc. of Conf. Decision Contr.* Maui, Hawaii, pp. 2016-2021, 2003.

[16] Y. Liu and Kevin M. Passino, "Stable social foraging swarms in a noisy environment", *IEEE Trans. on Autom. Contr.*, vol. 49, no. 1, pp. 30-44, Jan 2004.

[17] V. Gazi and K. M. Passino, "Stability analysis of swarms", *IEEE Trans. on Automat. Contr.*, vol. 48, pp. 692-697, Apr. 2003

[18] V. Gazi and K. M. Passino, "A class of attraction/repulsion functions for stable swarm aggregations", *Proc. of Conf. Decision Contr.*, Los Vegas, NV, Dec 2002, pp.2842-2847.

[19] V. Gazi and K. M. Passino, "Stability analysis of social foraging swarms: combined effects of attactrant/replellent profiles", *Proc. of Conf. Decision Contr.*, Las Vegas, NV, Dec 2002, pp. 114-123.

[20] V. Gazi and K. M. Passino, "Stability analysis of social foraging swarm", *IEEE Trans. on Syst. Man, and Cyber.-Part B: Cybernetics*, vol. 34, no. 1, pp. 539-557, Feb 2004.

[21] L. Moreau, "Stability of multiagent systems with time-dependent communication links". *IEEE Trans. on Autom. Contr.*, 50(2):169-182, 2005.

[22] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Classics in Applied Mathematics: 5, SIAM, Philadelphia, 1990.

[23] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, vol. 18 of Mathematics and Its Applications (Soviet Series), Kluwer Academic Publishers, Dordrecht, The Netherlands, 1988.

[24] D. Shevitz and B. Paden, "Lyapunov Stability Theory of Nonsmooth Systems", *IEEE Trans. on Autom. Contr.*, vol. 39, No. 9, pp. 1910-1914, 1994.

[25] J. Cortes, "Achieving coordination tasks in finite time via nonsmooth gradient flows", *Proc. of Conf. Decision & Contr., and Euro. Contr. Conf.*, December 2005 Seville, Spain, pp. 6376-6381.

[26] A. Bacciotti and F. Ceragioli,"Stability and Stabilization of Discontinuous Systems and Nonsmooth Lyapunov Functions", *ESAIM J. of Contr. Optim. & Cal. of Var.*, vol.4, pp.361-376, 1999.