Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

A Locality Generalization of the NCE (Mean Field) Principle: Agent Specific Cost Interactions

Minyi Huang Peter E. Caines Roland P. Malhamé

***Abstract*— We study large population stochastic dynamic games with agent specific cost coupling where each agent assigns nonuniform weights to other agents to indicate locality related interactions. The Nash Certainty Equivalence (Mean Field) methodology is generalized to this framework to give decentralized individual strategies. The key step is the specification of a family of consistent individual controls which depend upon each agent's state and upon the aggregate effect of the other agents as locally received by that agent. This methodology has close connections with the mean field models studied by Lasry and Lions (2006, 2007) and the notion of oblivious equilibrium proposed by Weintraub, Benkard, and Van Roy (2005, 2007) via a mean field approximation.**

I. INTRODUCTION

For noncooperative games with mean field coupling, the Nash Certainty Equivalence (NCE) methodology developed in our past work [11], [14], [15], [12], [13] provides an effective analytical tool for obtaining decentralized individual strategies. The key idea of this methodology is to specify a certain consistency relationship between the individual strategies and the mass effect (i.e., the overall effect of the population on a given agent) within the population limit, and each decision-maker can ignore the fine details of the behavior of any other individual player by only focusing on the overall impact of the population. This procedure leads to decentralized strategies for the individual players in a large but finite population. For this class of game problems, a closely related approach has recently been independently developed by Lasry and Lions [19], [20], while for models of many firm industry dynamics, Weintraub, Benkard, and Van Roy proposed the notion of oblivious equilibrium by use of a mean field approximation [24], [25]. For the analysis of mean field models in the setting of mathematical physics, see [7], [23]. To see the rich economic backgrounds of noncooperative games with many players, the reader is referred to [17], [9], [8], [18] and references therein.

Although mean field models in their usual uniform aggregation form have a broad scope of application [3], [6], [18], [20], [11], they may be unable to capture structural properties in certain problems. For instance, in a vaccination mean field model, each person assesses his or her infection risk and as a rough approximation may simply refer to the vaccination coverage of the overall population [3], [6], but in reality, the different sub-populations around the respective individuals may differently impact each person. It is obvious that an individual's close friends, colleagues (or classmates) have a much higher immediate influence than those more distant in a social and physical sense. A similar situation arises in economic models. In a crowded business area, a service unit (such as a retail store, restaurant) and its nearby neighbors may strongly interact while the level of such interactions decreases with distance.

It is worthwhile briefly reviewing the extent to which game theory has dealt with the issue of locality. Blume [5] considered strategic interactions on lattice models as motivated by retailing services. Schelling [22] presented a simple line topology to examine social segregation phenomena when each agent attempts to move to a more favorable location. Despite the fact they involve very different contexts, a common feature of the above works is their investigation of the relationship between microscopic local behavior of individual agents and the resulting macroscopic phenomena (also see, e.g., [10], [21], [4]).

Motivated by these problems, we present here a generalized mean field version of the Nash Certainty Equivalence theory of our previous work (see [11], [14], [15], [16], [12]) which now takes into account the possibility of the *local* nature of agent interactions. Our approach still relies on identifying a certain consistency relationship between each individual and the mass effect but the latter may now be specific to individual agents.

The organization of the paper is as follows. The individual dynamics and costs are introduced in Section II where the uniform aggregate cost coupling [11], [12] is also briefly reviewed for comparison purposes. Section III presents the equilibrium analysis for the set of control laws calculated via the NCE equation system, and we also identify some novel features for such locality based interactions by showing an interaction radii collapse effect when the population size increases in a lattice locality model. In Section IV, we extend the NCE equation system and the equilibrium analysis to models with different sub-populations where the cost involves inter- and intra-group coupling. Finally, Section V concludes the paper.

II. THE STOCHASTIC DYNAMIC GAME MODEL

In a population of N agents, consider the dynamics for an individual agent

$$dz_i(t) = (az_i(t) + bu_i(t))dt + \sigma dW_i(t), \quad 1 \le i \le N, \quad t \ge 0,$$

M. Huang is with School of Mathematics and Statistics, Carleton University, Ottawa, ON K1S 5B6, Canada. mhuang@math.carleton.ca.
P. E. Caines is with Department of Electrical and Computer Engineering, McGill University, Montreal, QC H3A 2A7, Canada. peterc@cim.mcgill.ca.
R. P. Malhamé is with Department of Electrical Engineering, École Polytechnique de Montréal, Montreal, QC H3C 3A7, Canada. roland.malhame@polymtl.ca.

978-1-4244-3124-3/08/$25.00 ©2008 IEEE

where $b \neq 0$ and $\{W_i, 1 \leq i \leq N\}$ denotes N independent standard Wiener processes. The initial states $\{z_i(0), 1 \leq i \leq N\}$ are mutually independent and also independent of $\{W_i, 1 \leq i \leq N\}$. In addition, $E|z_i(0)|^2 < \infty$. Denote the state configuration $z = (z_1, \cdots, z_N)$, and the population average state $z^{(N)} = (1/N)\sum_{i=1}^N z_i$.

A. The NCE Principle with Mean Field Cost Coupling

We begin by giving a brief review of our previous modeling of cost coupling. The cost function is given as

$$J_i^0 = E\int_0^\infty e^{-\rho t}\left\{[z_i - \Phi_i(z^{(N)})]^2 + ru_i^2\right\}(t)dt, \qquad (1)$$

where $\rho > 0$ is a discount factor, $\Phi_i = \gamma(z^{(N)} + \eta)$, $z^{(N)} = (1/n)\sum_{i=1}^N z_i$, $\gamma > 0$, $r > 0$ and η is a constant. It should be noted that for this mean field coupling of the uniform aggregation form, Φ_i does not distinguish the ordering of the entries z_j, $1 \leq j \leq N$, within z.

Let $\Pi_a > 0$ be the solution to the algebraic Riccati equation:

$$\rho\Pi = 2a\Pi - \frac{b^2}{r}\Pi^2 + 1. \qquad (2)$$

Denote

$$\beta_1 = -a + \frac{b^2}{r}\Pi_a, \quad \beta_2 = -a + \frac{b^2}{r}\Pi_a + \rho.$$

To simplify the aggregation procedure we assume zero initial mean for all agents, i.e., $Ez_i(0) = 0$, $i \geq 1$. Also, we assume we are in the uniform case where all agents have the same dynamic parameter a in their dynamics. The NCE consistency requirement leads to the equation system:

$$\rho s_a = \frac{ds_a}{dt} + as_a - \frac{b^2}{r}\Pi_a s_a - z^*, \qquad (3)$$

$$\frac{d\bar{z}_a}{dt} = (a - \frac{b^2}{r}\Pi_a)\bar{z}_a - \frac{b^2}{r}s_a, \qquad (4)$$

$$z^* = \gamma(\bar{z}_a + \eta), \qquad (5)$$

where $\bar{z}_a(0) = 0$ corresponds to the zero initial mean assumption. See [11], [12], [14] for details on the construction of this equation system in an LQG context. In fact, the NCE equation system may take a more general form where a varies across the population and possesses an empirical distribution; see [12].

Under some mild assumptions, the equation system (3)-(5) admits a unique bounded solution $(s_a(\cdot), \bar{z}_a(\cdot))$. The function $s_a(t)$ is uniquely determined by its boundedness condition and it is unnecessary to state the initial condition $s_a(0)$ separately. In fact, $\bar{z}_a(t)$ and $s_a(t)$ may be given in an explicit form (see [14]). Let u_i^0 denote the control law

$$u_i^0 = -\frac{b}{r}(\Pi_a z_i + s_a), \qquad (6)$$

which may be interpreted as the optimal tracking control law with respect to z^* in place of $\Phi_i(z^{(N)})$ in (1). It has been shown that the set of control laws $\{u_i^0, 1 \leq i \leq N\}$ results in an ε-Nash equilibrium where the offset $\varepsilon \to 0$ when $N \to \infty$. The formal definition of an ε-Nash equilibrium will be given in Section III; also see [2].

B. The NCE Principle with Agent Specific Cost Interactions

We now generalize the basic NCE equation system to the case of agent specific cost coupling. To this end, we assign each agent with a "locality" (or "spatial") index rather than just use an integer i to label its state variable z_i. *The dynamic parameter a and the locality parameter α are completely independent of one another, and for simplicity, in the initial case discussed in this paper, explicit mention of a is suppressed*. Note that this locality index may have different interpretations and is not necessarily restricted to be a physical location. For instance, it may be used to measure to what extent the player in question is distanced from other players, and it may be used in a social interaction context [1]. We assume agent i within the N agents is assigned the locality parameter p_i.

Let the cost for the ith agent be given by

$$J_i = E\int_0^\infty e^{-\rho t}\left\{[z_i - \tilde{\Phi}_i]^2 + ru_i^2\right\}dt, \qquad (7)$$

where $\tilde{\Phi}_i = \gamma(\sum_{j=1}^N \omega_{p_i p_j}^{(N)} z_j + \eta)$ and $\rho > 0$, $\gamma > 0$, $r > 0$. The set of weight coefficients $\omega_{p_i p_j}^{(N)}$ satisfies the condition

$$\omega_{p_i p_j}^{(N)} \geq 0, \quad \forall i, j,$$
$$\sum_{j=1}^N \omega_{p_i p_j}^{(N)} = 1, \quad \forall i. \qquad (8)$$

For each fixed i, it is seen from (8) that the total weight of unit is allocated to all the N agents. In order to simplify the notation, the summation in (8) includes the index i itself. Whether or not this self-weight is included has no impact on our asymptotic analysis when $N \to \infty$.

We take a representative agent and let its locality parameter be denoted by α which takes a value from a compact interval $[\underline{\alpha}, \overline{\alpha}]$. The state process of this agent may be denoted by $z_\alpha(t)$, and we denote its mean trajectory by $\bar{z}_\alpha(t) = Ez_\alpha(t)$, where $t \geq 0$. For illustration, suppose agent i has $p_i = \alpha$; then $z_i(t)$ may be identified as $z_\alpha(t)$.

For the agent associated with the parameter α (this agent may be referred to as an α-agent), let its limiting weight allocation for $\alpha' \in [\underline{\alpha}, \overline{\alpha}]$ be described by a probability distribution $F_\alpha(\alpha')$ when the number N of agents goes to infinity. Thus, $F_\alpha(\alpha')$ is intended to reflect the following approximation within a large population:

$$\sum_{j=1, p_j \in [c,c']}^N \omega_{p_i p_j}^{(N)} \approx \int_{\alpha' \in [c,c']} dF_{p_i}(\alpha'),$$

for any $[c, c'] \subset [\underline{\alpha}, \overline{\alpha}]$ such that c, c' are continuity points of $F_{p_i}(\cdot)$. Later on we will specify related conditions.

(A1) $F_\alpha(\alpha')$: $[\underline{\alpha}, \overline{\alpha}] \times \mathbb{R} \to [0,1]$ satisfies: i) $F_\alpha(\cdot)$ is a probability distribution function for each fixed α, $\int_{\alpha' \in [\underline{\alpha}, \overline{\alpha}]} dF_\alpha(\alpha') = 1$; ii) $\int_{\alpha' \in B} dF_\alpha(\alpha')$ is a measurable function of α for each Borel subset B of $\mathbb{R}$; iii) $F_{\alpha''}(\cdot)$ converges to $F_\alpha(\cdot)$ weakly when $\alpha'' \to \alpha$, where α and α'' are in $[\underline{\alpha}, \overline{\alpha}]$. □

(A2) The constant $\beta_1 > 0$ and $(\gamma b^2)/(r\beta_1\beta_2) < 1$. □

For the given α-agent, it faces the aggregate effect of other agents described by

$$\bar{r}_\alpha(t) = \int_{\alpha'\in[\underline{\alpha},\overline{\alpha}]} \bar{z}_{\alpha'}(t)dF_\alpha(\alpha'),$$

which is intended to approximate $\sum_{j=1}^N \omega_{p_ip_j}^{(N)} z_j$ in the population limit.

Now, based on the individual and weighted mass interaction consistency relationship, we can derive the following new Nash Certainty Equivalence (Mean Field) (NCE) equation system

$$\rho s_\alpha = \frac{ds_\alpha}{dt} + as_\alpha - \frac{b^2}{r}\Pi_a s_\alpha - R_\alpha, \tag{9}$$

$$\frac{d\bar{z}_\alpha}{dt} = (a - \frac{b^2}{r}\Pi_a)\bar{z}_\alpha - \frac{b^2}{r}s_\alpha, \tag{10}$$

$$\bar{r}_\alpha(t) = \int_{\alpha'\in[\underline{\alpha},\overline{\alpha}]} \bar{z}_{\alpha'}(t)dF_\alpha(\alpha'), \tag{11}$$

$$R_\alpha = \gamma(\bar{r}_\alpha + \eta). \tag{12}$$

The interesting observation is that when the distribution function $F_\alpha(\cdot)$ does not change with α, the equation system (9)-(12) reduces to (3)-(5) with standard mean-field coupling without differentiation between neighbors. This holds since in this case $\bar{r}_\alpha$ and hence R_α are both independent of α (see Acknowledgements).

The system (9)-(12) is constructed such that an α-agent makes optimal tracking of the local mass effect R_α which, in turn, depends on locality related coupling. Equation (10) is obtained by taking expectation of the closed-loop equation of the α-agent. A consistent solution to the NCE equation system consists of a parameterized triple $(s_\alpha(\cdot), \bar{z}_\alpha(\cdot), \bar{r}_\alpha(\cdot))$ where $\alpha \in [\underline{\alpha},\overline{\alpha}]$. Each entry in the triple $(s_\alpha(\cdot), \bar{z}_\alpha(\cdot), \bar{r}_\alpha(\cdot))$ will be viewed as a function from $[\underline{\alpha},\overline{\alpha}] \times \mathbb{R}^+$ to $\mathbb{R}$.

Let $I = [\underline{\alpha},\overline{\alpha}]$. Define the function class: $C_b[I\times\mathbb{R}^+] = \{f(\alpha,t) | f \in C[I\times\mathbb{R}^+],\ |f| \triangleq \sup_{\alpha,t}|f(\alpha,t)| < \infty\}$. The two expressions $f(\alpha,t)$ and $f_\alpha(t)$ will be used interchangeably.

For each α, if $\bar{r}_\alpha$ is given, we may solve a unique bounded s_α from (9) to obtain:

$$\begin{aligned} s_\alpha(t) &= -e^{\beta_2 t}\int_t^\infty e^{-\beta_2\tau}R_\alpha(\tau)d\tau \\ &= -e^{\beta_2 t}\int_t^\infty e^{-\beta_2\tau}\gamma(\bar{r}_\alpha(\tau)+\eta)d\tau. \end{aligned}$$

We also write $\bar{r}_\alpha(t) = \bar{r}(\alpha,t)$. Next,

$$\begin{aligned} \bar{z}_{\alpha'}(t) &= \frac{b^2}{r}\int_0^t e^{-\beta_1(t-s)}e^{\beta_2 s}\int_s^\infty e^{-\beta_2\tau}\gamma(\bar{r}_{\alpha'}(\tau)+\eta)d\tau ds \\ &\triangleq (\Gamma_0\bar{r}_{\alpha'})(t) \end{aligned}$$

where Γ_0 is viewed as an operator acting on bounded continuous functions on $[0,\infty)$. Finally,

$$(\Gamma\bar{r})(\alpha,t) \triangleq \int_{\alpha'} (\Gamma_0\bar{r}_{\alpha'})(t)dF_\alpha(\alpha').$$

Note that for a general function $f(\alpha,t) \in C_b[I\times\mathbb{R}^+]$, $\Gamma_0 f_\alpha$ and Γf are defined in an obvious manner.

In order to solve the NCE equation system (9)-(12), a key step is to find a fixed point $\bar{r}$ in a suitable function space for the operator recursion corresponding to the equation

$$(\Gamma\bar{r})(\alpha,t) = \bar{r}(\alpha,t). \tag{13}$$

Lemma 1: Under (**A1**), Γ is a mapping from $C_b[I\times\mathbb{R}^+]$ to $C_b[I\times\mathbb{R}^+]$.

Proof: See appendix. □

Theorem 2: Under (**A1**)-(**A2**), there exists a unique bounded solution $(s_\alpha(\cdot), \bar{z}_\alpha(\cdot), r_\alpha(\cdot))$ to the NCE equation system (9)-(12).

Proof: By Lemma 1, we see that Γ is a linear operator from $C_b[I\times\mathbb{R}^+]$ to itself, and $C_b[I\times\mathbb{R}^+]$ is a Banach space under the norm $|f| = \sup_{\alpha,t}|f(\alpha,t)|$.

We take $f_1, f_2 \in C_b[I\times\mathbb{R}^+]$. By straightforward calculation, we obtain the estimates

$$|\Gamma f_1 - \Gamma f_2| \le \frac{\gamma b^2}{r\beta_1\beta_2}|f_1 - f_2|.$$

By (**A2**) it follows that Γ is a contraction. So there is a unique solution $\bar{r} \in C_b[I\times\mathbb{R}^+]$ satisfying equation (13). Once the above $\bar{r}(=\bar{r}_\alpha(t))$ is obtained, it is straightforward to get the other two entries in the triple $(s_\alpha(t), \bar{z}_\alpha(t), \bar{r}_\alpha(t))$. Uniqueness of the solution can be easily verified by using uniqueness of the fixed point to equation (13). □

III. The Equilibrium Analysis

For equilibrium analysis, we need the assumptions:

(**A3**) The weight allocation satisfies the condition

$$\varepsilon_N^\omega \triangleq \sup_{1\le i\le N}\sum_{j=1}^N |\omega_{p_ip_j}^{(N)}|^2 \to 0,$$

as $N\to\infty$. □

(**A4**) For each p_i, the empirical distribution

$$F_{p_i}^{(N)}(x) = \sum_{p_j<x}\omega_{p_ip_j}^{(N)}, \quad x\in\mathbb{R},$$

is associated with a distribution function $F_{p_i}(x)$ (specified in (**A1**)) such that for any $\delta > 0$, there exists a compact subset $D_{p_i}^N$ of $I = [\underline{\alpha},\overline{\alpha}]$ with Lebesgue measure meas$(D_{p_i}^N) < \delta$, and $\lim_{N\to\infty}\sup_{1\le i\le N}\sup_{x\in I\setminus D_{p_i}^N}|F_{p_i}^{(N)}(x) - F_{p_i}(x)| = 0$. □

Remark: Roughly, the last part of (**A4**) implies that $|F_{p_i}^{(N)}(x) - F_{p_i}(x)|$ tends to zero with a speed independent of p_i on I excluding a small subset $D_{p_i}^N$ (which may depend on p_i, N). Notice that (**A4**) is satisfied if $\omega_{p_ip_j}^{(N)} = 1/N$. □

Example 1: Let p_i, $1\le i\le N$, denote N locations, consecutively and uniformly spaced from left to right, on the interval $[0,1]$ where $p_1 = 0$ and $p_N = 1$. Take $\omega_{p_ip_i}^{(N)} = 0$ for each i and

$$\omega_{p_ip_j}^{(N)} = |j-i|^{-\lambda}/c_i, \qquad 1\le i\ne j\le N, \tag{14}$$

where $\lambda\in[0,1]$ and $c_i = \sum_{j=1,j\ne i}^N |j-i|^{-\lambda}$ is the normalizing factor. □

With such a choice of λ in Example 1, (**A3**) can be verified by elementary calculations. The mean field model of the uniform aggregation form corresponds to taking $\lambda = 0$

for which case the weight assignment does not distinguish locations. If $\lambda = 1$, we can also show that (**A4**) is satisfied and in this case $F_{p_i}(x) = 1$ if $x > p_i$, $F_{p_i}(x) = 0$ if $x \le p_i$.

We have the key approximation lemma.

Lemma 3: Assume (**A4**) holds. For any given bounded and continuous function $g(\alpha)$ on $\mathbb{R}$, we have

$$\lim_{N\to\infty} \sup_{1\le i\le N} \Big| \int_{-\infty}^{\infty} g(\alpha) dF_{p_i}^{(N)}(\alpha) - \int_{-\infty}^{\infty} g(\alpha) dF_{p_i}(\alpha) \Big| = 0.$$

Proof: See appendix. □

A. Discussion on the "Interaction Radii Collapse" Effect

It appears that by use of the simple weight allocation model (14) some very intriguing phenomena may be shown to be possible. We fix $p_1 = 0$. By simple calculation we can see that the associated function F_{p_1} (as a weak limit) will have very different nature. When $\lambda = 1$, F_{p_1} is just a Heaviside function with a unit jump at $x = 0$. If we go back to the NCE equation system, it means in the limit model, only the agents in an infinitesimally small neighborhood matter for the agent in question. Consequently and surprisingly, we can retrieve the usual NCE equation. When $\lambda \in [0,1)$, we can show that F_{p_1} is a continuous function connecting $(0,0)$ and $(1,1)$ via its graph. This means the effect of agents in a large range can be registered by this limit distribution function F_{p_1} and then utilized in the NCE equation system.

So, λ can be interpreted as some kind of critical parameter.

B. Properties of the NCE Based Control Laws

Within the context of a population of N agents, for any $1 \le k \le N$, the kth agent's admissible control set $\mathscr{U}_k$ consists of all feedback controls u_k adapted to the σ-algebra $\sigma(z_i(\tau), \tau \le t, 1 \le i \le N)$ (i.e., $u_k(t)$ is a function of $(t, z_1(t), \cdots, z_N(t))$) such that a unique strong solution to the closed-loop system of the N agents exists on $[0,\infty)$. Note that $\mathscr{U}_k$ itself is not restricted to be decentralized. Denote $u_{-i} = (u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_N)$.

Definition 4: A set of controls $u_k \in \mathscr{U}_k, 1 \le k \le N$, for N players is called an *ε-Nash equilibrium* with respect to the costs $J_k, 1 \le k \le N$, where $\varepsilon \ge 0$, if for any i, $1 \le i \le N$, we have

$$J_i(u_i, u_{-i}) \le J_i(u_i', u_{-i}) + \varepsilon,$$

when any alternative $u_i' \in \mathscr{U}_i$ is applied by the ith player. □

Theorem 5: Under (**A1**)-(**A4**), given any $\varepsilon > 0$, there exists N_ε such that for all $N \ge N_\varepsilon$, the set of control strategies $\{\hat{u}_i, 1 \le i \le N\}$ is an ε-Nash equilibrium where

$$\hat{u}_i = -\frac{b}{r}(\Pi_a z_i + s_{p_i})$$

and s_{p_i} is given by (9)-(12) via the substitution $\alpha = p_i$ in s_α.

Proof: Let $\bar{z}_\alpha$ be given by (9)-(12). Denote

$$R_{p_i}^{(N)}(t) = \gamma[\sum_{j=1}^N \omega_{p_i p_j}^{(N)} \bar{z}_{p_j}(t) + \eta],$$

$$\Delta_i^{(N)}(t) = \gamma \sum_{j=1}^N \omega_{p_i p_j}^{(N)} (\bar{z}_{p_j} - z_j).$$

We first write the individual cost in the form

$$J_i(u_i) = E\int_0^\infty e^{-\rho t}\{[(z_i - R_{p_i}^{(N)}) + \Delta_j^{(N)})]^2 + r u_i^2\}(t) dt.$$

Suppose all the N agents apply the controls $\hat{u}_i$, $1 \le i \le N$. Then it is straightforward to find a constant $\widehat{C}$ such that

$$\sup_N \sup_{1\le i\le N} E\int_0^\infty e^{-\rho t}(\hat{z}_i^2 + \hat{u}_i^2)(t) dt \le \widehat{C},$$

and $J_i(\hat{u}_i, \hat{u}_{-i}) \le \widehat{C}$, where we denote the state process associated with $\hat{u}_i$ by $\hat{z}_i$ and $\hat{u}_{-i} = (\hat{u}_1, \cdots, \hat{u}_{i-1}, \hat{u}_{i+1}, \cdots, \hat{u}_N)$.

In the below, when we consider alternative strategies for agent i, we may restrict that u_i satisfies

$$E\int_0^\infty e^{-\rho t}|z_i - \Phi(z^{(N)})|^2 dt \le \widehat{C},\ E\int_0^\infty e^{-\rho t} u_i^2 dt \le \widehat{C}/r. \tag{15}$$

This restriction causes no loss of generality since, otherwise, u_i will generate a cost higher than $J_i(\hat{u}_i, \hat{u}_{-i})$. Based on (15), we may further show that $E\int_0^\infty e^{-\rho t}|z_i|^2 dt \le \widehat{C}_1$ for some $\widehat{C}_1 < \infty$ independent of N.

By using (**A1**) to show that $\bar{z}_\alpha(t)$ has equicontinuity in α (w.r.t. all t), we can apply Lemma 3 to check that

$$\lim_{N\to\infty} \sup_{p_i, t} |R_{p_i}^{(N)}(t) - R_{p_i}(t)| = 0. \tag{16}$$

Also, for all u_i satisfying the prior bound (15), we use (**A3**) to show the convergence relation

$$\lim_{N\to\infty} \sup_{u_i, t, i} E|\Delta_i^{(N)}(t)|^2 = 0, \tag{17}$$

when all other agents' strategies are given by $\hat{u}_{-i}$.

Finally, for u_i satisfying (15), by use of (16)-(17) it is straightforward to show that

$$J_i(u_i, \hat{u}_{-i}) \ge J_i(\hat{u}_i, \hat{u}_{-i}) - \varepsilon_N \tag{18}$$

where $0 \le \varepsilon_N = o(1)$. By the choice of $\widehat{C}$, we see that (18) is automatically true when u_i does not satisfy (15). This completes the proof. □

IV. Cost Coupling with Heterogenous Sub-populations

In this section, we adapt the general cost structure (7) to model the interaction of agents from K groups within the population. The locality parameter p_i indicates which group the ith agent belongs to, and the cost interaction for a pair of agents is determined by either the inter-group or the intra-group coupling parameters. Suppose there is a finite set $\Theta \triangleq \{\theta^1, \cdots, \theta^K\}$ (of distinct elements) such that each p_i, $1 \le i < \infty$, takes values from Θ. The coupling weight assignment will be constructed by using the $K \times K$ matrix

$$\omega_\Theta = (\omega_{\theta^i \theta^j})_{K\times K}$$

which satisfies $\omega_{\theta^i \theta^j} \ge 0$ and $\sum_{j=1}^K \omega_{\theta^i \theta^j} = 1$ for each i. Denote

$$\sum_{i=1}^N 1_{(p_i = \theta^k)} = N_k, \qquad 1 \le k \le K.$$

If $p_i = \theta^k$ and $p_{i'} = \theta^{k'}$, then we define $\omega^{(N)}_{p_i p_{i'}} = \omega_{\theta^k \theta^{k'}}/N_{k'}$, which ensures the unit total weight condition

$$\sum_{j=1}^{N} \omega^{(N)}_{p_i p_j} = 1. \tag{19}$$

(**A5**) The sequence $\{p_i, 1 \geq 1\}$ has the limit empirical distribution

$$\lim_{N\to\infty} (1/N) \sum_{i=1}^{N} 1_{(p_i=\theta^k)} = \pi_{\theta^k}$$

where $\theta^k \in \Theta$. □

The probability vector $(\pi_{\theta^1}, \cdots, \pi_{\theta^K})$ shows the relative frequency of each of the K groups.

Now the NCE equation system takes the form:

$$\rho s_\theta = \frac{ds_\theta}{dt} + a s_\theta - \frac{b^2}{r}\Pi_a s_\theta - R_\theta, \tag{20}$$

$$\frac{d\bar{z}_\theta}{dt} = (a - \frac{b^2}{r}\Pi_a)\bar{z}_\theta - \frac{b^2}{r} s_\theta, \tag{21}$$

$$\bar{r}_\theta(t) = \sum_{\theta'\in\Theta} \pi_{\theta'} \omega_{\theta\theta'} \bar{z}_{\theta'}(t), \tag{22}$$

$$R_\theta = \gamma(\bar{r}_\theta + \eta). \tag{23}$$

where $\theta \in \Theta$ and, again, $s_\theta(t)$ is restricted to be a bounded function without the necessity of separately specifying an initial condition $s_\theta(0)$.

Theorem 6: If the two conditions (i) $\sum_{\theta'\in\Theta} \pi_{\theta'} \omega_{\theta\theta'} = 1$, (ii) $\gamma b^2/(r\beta_1\beta_2) < 1$ hold, the equation system (20)-(23) has a unique bounded solution $(s_{\theta^k}(\cdot), \bar{z}_{\theta^k}(\cdot), \bar{r}_{\theta^k}(\cdot))$, $1 \leq k \leq K$.

Proof: The theorem may be proved using a fixed point argument. □

Theorem 7: Under (**A5**) and the assumptions of Theorem 6, given any $\varepsilon > 0$, there exists N_ε such that for all $N \geq N_\varepsilon$, the set of control strategies $\{\hat{u}_i, 1 \leq i \leq N\}$ is an ε-Nash equilibrium where

$$\hat{u}_i = -\frac{b}{r}(\Pi_a z_i + s_{p_i})$$

and s_{p_i} is given by (20)-(23) via the substitution $\theta = p_i$ in s_θ.

Proof: The proof is similar to that of Theorem 5. □

V. Conclusion

In this paper we generalize our previous Nash Certainty Equivalence methodology with uniform coupling to models with locality interactions. We show that under reasonable decay rates for the interaction strength, a consistency relationship between individual strategies and local deterministic mass effects can still be specified, and this procedure leads to decentralized Nash strategies for the individual players. We also discuss how the weight allocation in the cost coupling affects the spatial spreading ability of interactions in the population limit, and we illustrate a novel interaction radii collapse phenomenon when the weight decay approaches a critical rate.

Appendix

Proof of Lemma 1. Given $f_\alpha(t) \in C_b[I \times \mathbb{R}^+]$, we have

$$(\Gamma_0 f_\alpha)(t) = \frac{b^2}{r}\int_0^t e^{-\beta_1(t-s)} e^{\beta_2 s} \int_s^\infty e^{-\beta_2\tau} \gamma(f_\alpha(\tau)+\eta) d\tau ds.$$

By the boundedness of $f_\alpha(t)$, there exists $C < \infty$ such that

$$\begin{aligned}\sup_{\alpha,t} |(\Gamma_0 f_\alpha)(t)| &\leq C \sup_{\alpha,t} \int_0^t e^{-\beta_1(t-s)} e^{\beta_2 s} \int_s^\infty e^{-\beta_2\tau} d\tau ds \\ &\leq C/(\beta_1\beta_2).\end{aligned}$$

Subsequently,

$$\begin{aligned}\sup_{\alpha,t} |(\Gamma f)(\alpha,t)| &\leq \sup_{\alpha,t} \int_{\alpha'} |(\Gamma_0 f_{\alpha'})(t)| dF_\alpha(\alpha') \\ &\leq C/(\beta_1\beta_2) \sup_{\alpha,t} \int_{\alpha'} dF_\alpha(\alpha') \\ &= C/(\beta_1\beta_2).\end{aligned}$$

Now we prove the continuity of Γf. We note the relation:

$$\begin{aligned}(\Gamma_0 f_\alpha)(t) &= \frac{b^2}{r}\int_0^t e^{-\beta_1(t-s)} e^{\beta_2 s} \int_s^\infty e^{-\beta_2\tau} \gamma(f_\alpha(\tau)+\eta) d\tau ds \\ &= \frac{b^2\gamma}{r} e^{-\beta_1 t} \int_0^t e^{\beta_1 s} \int_s^\infty e^{-\beta_2(\tau-s)} f_\alpha(\tau) d\tau ds \\ &\quad + \frac{\gamma b^2 \eta}{r\beta_1\beta_2}(1-e^{-\beta_1 t}).\end{aligned}$$

Define

$$\begin{aligned}G_0(\alpha,t) &= \int_0^t e^{\beta_1 s} \int_s^\infty e^{-\beta_2(\tau-s)} f_\alpha(\tau) d\tau ds, \\ G(\alpha,t) &= \int_{\alpha'} \int_0^t e^{\beta_1 s} \int_s^\infty e^{-\beta_2(\tau-s)} f_{\alpha'}(\tau) d\tau ds dF_\alpha(\alpha').\end{aligned}$$

Now it suffices to show the continuity of $G(\alpha,t)$ with respect to (α,t). Letting (α,t) be fixed, we pick (α_1,t_1) in a neighborhood of (α,t). Then

$$\begin{aligned}|G(\alpha_1,t_1) - G(\alpha,t)| \leq& |G(\alpha_1,t_1) - G(\alpha_1,t)| \\ &+ |G(\alpha_1,t) - G(\alpha,t)|.\end{aligned}$$

We have

$$\begin{aligned}&|G(\alpha_1,t_1) - G(\alpha_1,t)| \\ \leq& \int_{\alpha'} \left| \int_t^{t_1} e^{\beta_1 s} \int_s^\infty e^{-\beta_2(\tau-s)} f_{\alpha'}(\tau) d\tau ds \right| dF_{\alpha_1}(\alpha') \\ \leq& \int_{\alpha'} C|e^{\beta_1 t_1} - e^{\beta_1 t}| dF_{\alpha_1}(\alpha') \\ =& C|e^{\beta_1 t_1} - e^{\beta_1 t}|,\end{aligned} \tag{A.1}$$

where we may take $C = (\sup_{\alpha,t}|f_\alpha(t)|)/(\beta_1\beta_2)$.

We have

$$\begin{aligned}&|G(\alpha_1,t) - G(\alpha,t)| \\ \leq& \left| \int_{\alpha'} G_0(\alpha',t) dF_{\alpha_1}(\alpha') - \int_{\alpha'} G_0(\alpha',t) dF_\alpha(\alpha') \right|.\end{aligned}$$

For each fixed t, $\sup_{\alpha'} |G(\alpha',t)| < \infty$ and by elementary estimates we can show that $G_0(\alpha',t)$ is a continuous function of α'. Hence it follows from (**A1**) that

$$\lim_{\alpha_1\to\alpha} |G(\alpha_1,t) - G(\alpha,t)| = 0. \tag{A.2}$$

Finally, the continuity of $G(\alpha,t)$ follows from (A.1) and (A.2). The lemma follows. □

Proof of Lemma 3: It suffices to prove

$$\sup_{1\le i\le N} \Big|\int_A^B g(\alpha)dF_{p_i}^{(N)}(\alpha) - \int_A^B g(\alpha)dF_{p_i}(\alpha)\Big| \to 0,$$

as $N\to\infty$, where $-\infty < A < \underline{\alpha} < \overline{\alpha} < B < \infty$. Then clearly, after replacing $I = [\underline{\alpha},\overline{\alpha}]$ by $I_{AB} = [A,B]$, we still have

$$\lim_{N\to\infty} \sup_{1\le i\le N} \sup_{x\in I_{AB}\setminus D_{p_i}^N} |F_{p_i}^{(N)}(x) - F_{p_i}(x)| = 0. \qquad \text{(A.3)}$$

Let $\varepsilon > 0$ be any given constant. Since g is bounded and continuous, it is uniformly continuous on $[A,B]$ and hence there exists $\delta > 0$ such that $|g(x)-g(x')| \le \varepsilon$ if $|x-x'| \le \delta$, $x,x' \in [A,B]$. Note that (A.3) holds for appropriately chosen $D_{p_i}^N$ satisfying $\text{meas}(D_{p_i}^N) < \delta$. Let $A = x_1 < x_2 < \cdots < x_{l+1} = B$ be a partition of $[A,B]$ such that each x_k is a continuity point of F_{p_i} and belongs to $I_{AB}\setminus D_{p_i}^N$ and that $\max_{1\le k\le l} |x_{k+1} - x_k| \le \delta$. We may ensure $l \le 2(B-A)/\delta$.

By straightforward calculation we can show that

$$\begin{aligned}\Delta_N &\triangleq \int_A^B g(\alpha)dF_{p_i}^{(N)}(\alpha) - \int_A^B g(\alpha)dF_{p_i}(\alpha)\\ &= \sum_{k=1}^{l} \Big\{ \int_{x_k}^{x_{k+1}} [g(\alpha)-g(x_k)]dF_{p_i}^{(N)}(\alpha)\\ &+ \int_{x_k}^{x_{k+1}} [g(x_k)-g(\alpha)]dF_{p_i}(\alpha)\\ &+ g(x_k)[F_{p_i}^{(N)}(x_{k+1}) - F_{p_i}^{(N)}(x_k) - F_{p_i}(x_{k+1}) + F_{p_i}(x_k)]\Big\}.\end{aligned}$$

Denoting $C_g = \sup_x |g(x)|$, hence

$$|\Delta_N| \le 2\varepsilon + C_g \sum_{k=1}^{l} 2|F_{p_i}^{(N)}(x_k) - F_{p_i}(x_k)|.$$

On the other hand, for the above fixed pair of (ε,l), there exists $N_{\varepsilon,l} > 0$ depending on (ε,l) such that

$$\sup_{1\le i\le N} \sup_{x\in I_{AB}\setminus D_{p_i}^N} |F_{p_i}^{(N)}(x) - F_{p_i}(x)| \le \varepsilon/(2lC_g+1)$$

when $N \ge N_{\varepsilon,l}$. Therefore, for all $N \ge N_{\varepsilon,l}$, we have

$$|\Delta_N| \le 3\varepsilon.$$

By the arbitrariness of ε, the lemma follows. □

ACKNOWLEDGEMENTS

We are indebted to Arthur Lazarte for the observation following equations (9)-(12)

REFERENCES

[1] G. A. Akerlof. Social distance and social decisions. *Econometrica*, vol. 65, no. 5, pp. 1005-1027, Sept. 1997.
[2] T. Başar and G. J. Olsder. *Dynamic Noncooperative Game Theory*, 2nd ed. Academic Press, London, UK, 1995.
[3] C. T. Bauch and D. J. D. Earn. Vaccination and the theory of games. *Proc. Nat. Acad. Sci., U.S.A.*, vol. 101, pp. 13391-13394, Sept. 2004.
[4] A. Bisin, U. Horst, and O. Özgür. Rational expectations equilibria of economies with local interactions. *J. Econ. Theory*, vol. 127, pp. 74-116, 2006.
[5] L. E. Blume. The statistical mechanics of strategic interaction. *Games and Economic Behavior*, vol. 5, pp. 387-424, 1993.
[6] R. Breban, R. Vardavas, and S. Blower. Mean-field analysis of an inductive reasoning game: application to influenza vaccination. *Physical Review E*, vol. 76, 2007. DOI: 10.1103/PhysRevE.76.031127.
[7] D. A. Dawson and J. Gärtner. Large deviations from the McKean-Vlasov limit for weakly interacting diffusions. *Stochastics*, vol. 20, pp. 247-308, 1987.
[8] G. M. Erickson. Differential game models of advertsing competition. *Europ. J. Oper. Res.*, vol. 83, pp. 431-438, 1995.
[9] E. J. Green. Continuum and finite-player noncooperative models of competition. *Econometrica*, vol. 52, no. 4, pp. 975-993, 1984.
[10] U. Horsta and J. A. Scheinkman. Equilibria in systems of social interactions. *J. Economic Theory*, vol. 130, no. 1, pp. 44-77, 2006.
[11] M. Huang, P. E. Caines, and R. P. Malhamé. Individual and mass behaviour in large population stochastic wireless power control problems: centralized and Nash equilibrium solutions. *Proc. 42nd IEEE Conf. Decision Contr.*, Maui, HI, pp. 98-103, December 2003.
[12] M. Huang, P. E. Caines, and R. P. Malhamé. Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized ε-Nash equilibria. *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1560-1571, 2007.
[13] M. Huang, P. E. Caines, and R. P. Malhame. The Nash certainty equivalence principle and McKean-Vlasov systems: an invariance principle and entry adaptation. *Proc. 46th IEEE Conference on Decision and Control*, New Orleans, pp. 121-126, December 2007
[14] M. Huang, R. P. Malhamé, and P. E. Caines. Nash equilibria for large-population linear stochastic systems of weakly coupled agents, in *Analysis, Contr. Optim. Complex Dyn. Syst.*, E. K. Boukas and R. P. Malhamé eds., pp. 215-252, Springer, New York, 2005.
[15] M. Huang, R. P. Malhamé, and P. E. Caines. Nash certainty equivalence in large population stochastic dynamic games: connections with the physics of interacting particle systems. *Proc. 45th IEEE CDC Conf.*, San Diego, CA, pp. 4921-4926, Dec. 2006
[16] M. Huang, R. P. Malhamé, and P. E. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Communications in Information and Systems*, vol. 6, no. 3, pp. 221-252, 2006.
[17] M. Ali Khan and Y. Sun, Non-cooperative games with many players. In *Handbook of Game Theory with Economic Applications*, vol. 3, R. J. Aumann and S. Hart eds., North-Holland, 2002.
[18] V. E. Lambson. Self-enforcing collusion in large dynamic markets. *J. Econ. Theory*, vol. 34, pp. 282-291, 1984.
[19] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. I – Le cas stationnaire, *C. R. Acad. Sci. Paris, Ser. I*, vol. 343, pp. 619-625, 2006.
[20] J.-M. Lasry and P.-L. Lions. Mean field games. *Japan. J. Math.*, no. 2, pp. 229-260, 2007.
[21] M. A. Nowak and K. Sigmund. Games on grids. In *The Geometry of Ecological Interactions: Simplifying Spatial Complexity*, U. Dieckmann, R. Law, and J. A. J. Metz eds., pp. 135-150, Cambridge University Press, 2000.
[22] T. C. Schelling. Dynamic models of segregation. *J. Math. Sociology*, vol. 1, pp. 143-186, 1971.
[23] A.-S. Sznitman. Topics in propagation of chaos. In *Ecole d'Eté de Probabilitiés de Saint-Flour XIX -1989*, Lect. Notes Math. 1464, pp. 165-252, Springer-Verlag, Berlin, 1991.
[24] G. Y. Weintraub, C. L. Benkard, and B. Van Roy. Oblivious equilibrium: A mean field approximation for large-scale dynamic games, *Advances in Neural Information Processing Systems*, MIT Press, 2005.
[25] G. Y. Weintraub, C. L. Benkard, and B. Van Roy. Markov perfect industry dynamics with many firms, preprint, 2007.