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Stabilization of Delayed Linear Systems via Signal Reconstruction

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Abstract— **This work addresses the problem of time varying delays in systems and control. In the first part of the paper, we propose a processing scheme that allows reconstruction of a signal subject to time-varying delays. We show robustness of the scheme with respect to sensing and actuation noise. We obtain a necessary and sufficient result for stability of compensated scalar linear systems under constant time delay. Then we apply the scheme to a general n dimensional linear system subject to time-varying delay in the state feedback. We present sufficient conditions for stability of the closed loop system. These conditions provide a constructive procedure to design the compensation parameter to maximize the delay that the system can support, while remaining stable.**

I. Introduction

Many systems and processes involve delays in information sensing and/or actuation commands due to the underlining communication media. Examples are abundant in several areas such as chemical, mechanical and biological systems. In engineering applications many examples can be found such as vibration absorber in cars, spatially distributed systems communicating across a channel, and networked control systems. Time delays cause time-shift or signal distortion which results in performance degradation and possible instability. Many results on the stability analysis of such systems are summarized in [13], [8]. However, to the best of our knowledge, none of these results attempt to incorporate performance issues in the design process. This is the main focus of this paper. Existing results that try to mitigate the effect of delay differ depending on the application. Typically, when there is no demand for real time signals, data buffering can be used. More critical problems arise in control and real time applications when an instantaneous signal is required, and several methods are available that deal with specific types of delays. For example, the Smith Predictor [3] provides a viable solution in the case of constant and known delays; however, the result is sensitive to parameter variation and it requires knowledge of the model as well as the value of the delay. Model predictive control [5] requires knowledge of the model and delay, and stability is hard to guarantee on a moving horizon. An approach based on system redaction to address controllability and stability issues in delayed control linear systems was provided in [2], there the author shows how, under proper conditions, a reduction technique can provide us with a simplified delay free model, whose stability and controllability properties are equivalent to the ones in the original system.

In [17] a stochastic perturbation approach is considered in which a time varying delay is modeled as a deterministic part that is a dynamic average model for the delay, and a stochastic part that enters into the system as a perturbation. Finally in [7] the finite spectrum assignment approach is extended to perform stability design for nonlinear input delayed systems. The control law is designed based on state prediction over one delay interval. The prediction is performed using the implicit Euler method and it is based on the current state measurement and model dynamics.

In this paper we present a novel approach to deal with time varying delays. The main idea is to delay part of the transmitted signal by a fixed value which is used as design parameter, and use the delayed and the original signals to estimate future values. The stabilizing effect of delay in control systems has been addressed in [14], [1], where the authors prove that the presence of appropriate delay in the input stabilizes the system. Also in [1] it is shown that oscillatory systems can be stabilized by delayed positive feedback. In both cases a frequency domain approach is used to prove the results. Another result on the use of delayed inputs to stabilize the system can be found in [18], where the concept of Time Delay Control (TDC) is introduced. This control method is used to stabilize nonlinear systems with uncertainty in the dynamics and input disturbances. The delay is used to estimate the effect of the uncertainties and the estimate is used to cancel the effect of the uncertainties.

Then, we consider a dynamical system with delay in the sensor measurement, and prove that under certain conditions closed-loop stability is guaranteed if the control scheme is used to reconstruct the sensor measurement signal. The analysis is carried out in the time-domain using Razumikhin's theorem on time-delay systems. A main objective of the results that we present in this paper is to overcome some of the restrictions that limit existing prediction schemes. An important aspect is that the scheme allows us to handle *unknown time varying delays*. Moreover *no knowledge of the model of the dynamical system that generates the signal is needed*. Another important aspect is that our scheme can handle large time varying delays while preserving stability and good tracking performances. In the case of scalar linear systems subject to constant time delay we show that it is possible to obtain an exact upper bound on the delay that destabilizes the system. This allows comparing the system with and without compensation. A full version of the present paper is contained in [12]. The paper is organized as follows: In Section II we consider the problem of time-varying, possi-

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bly unknown, delays; we describe our processing scheme and show that the received signal is bounded within a ball around the transmitted signal. In Section III, we address robustness of the scheme with respect to noise. Section IV analyzes the case of scalar linear system under constant time delay. In Section V we recall a version of the Razumikhin-type theorem. In Section VI we apply the scheme to a linear system subject to time-varying delay in the state feedback. Simulations are provided for all the above cases within the corresponding sections.

II. Time Varying Delays: Compensation

Consider a smooth signal $f(t) : [0,\infty) \to \mathbf{R}$ transmitted from a source to a destination over a channel characterized by a time varying delay $\tau(t)$. Then, the signal received at the destination is $f(t-\tau(t))$. Throughout the paper we assume that the following condition on the delay holds:

Assumption 1 *The delay $\tau \in C^1$ is bounded $|\tau(t)| \leq B_d$ and the derivative $|\dot{\tau}(t)| < 1, \forall t \in \mathbf{R}^+$. This causality condition on the delay ensures that the time variable is a monotonically increasing function.* In what follows $F_s^*(t)$ and $F_r^*(t) = F_s^*(t-\tau(t))$ denote the processed signal being sent across the delay block and the received signal, respectively. For simplicity we shall use τ instead of $\tau(t)$ hereafter.

Theorem 1: Consider a signal $f(t) \in C^2$ and assume $|f^{(k)}| \leq M, k = 0,1,2$. Also consider a channel which introduces a time varying delay $\tau(t)$. Define a constant delay $T > 0$ that is used as design parameter to construct an estimate of the signal. Define $\sigma \in \mathbf{R}^+$ to be the argument of the function f, i.e. $f : \sigma \to f(\sigma)$, then the Taylor expansion (see [10], [15]) of the delayed signal $f(t-T)$, around $f(t)$ is defined as following:

$$f(t-T) = f(t) + \frac{df}{d\sigma}(t)(-T) + o((-T)^2),$$

Then, we define the approximate derivative of $f(t)$ as follows:

$$\frac{\hat{df}}{d\sigma}(t) = \frac{f(t)-f(t-T)}{T} = \frac{df}{d\sigma}(t) - \frac{o(T^2)}{T}, T > 0 \quad (1)$$

where $o(T^2)$ is the second order remainder in the Taylor expansion (see [10], [15]), given by $o(T^2) = \frac{|f^{(2)}(\xi)|T^2}{2}, t-T \leq \xi \leq t$. Consider the following signal being transmitted across the channel

$$F_s^*(t) = f(t) + \frac{\hat{df}}{d\sigma}(t)T \quad (2)$$

Then, the received signal satisfies $F_r^*(t) = f(t) + \varepsilon_d$, with $|\varepsilon_d| < M(\frac{T^2}{2} + \frac{\tau^2}{2} + |T-\tau|)$.

Proof *Given (1) and (2), the received signal can be written as follows:*

$$\begin{aligned} F_r^*(t) &= f(t-\tau) + \frac{\hat{df}}{d\sigma}(t-\tau)T \\ &= f(t-\tau) + \frac{df}{d\sigma}(t-\tau)\tau + \frac{df}{d\sigma}(t-\tau)(T-\tau) \\ &- o(T^2) \end{aligned}$$

The Taylor approximation of the signal $f(t)$ around $f(t-\tau)$ is given by $f(t) = f(t-\tau) + \frac{df}{d\sigma}(t-\tau)\tau + o(\tau^2)$. Then we can rewrite the received signal as

$$F_r^*(t) = f(t) - o(\tau^2) - o(T^2) + \frac{df}{d\sigma}(t-\tau)(T-\tau) \quad (3)$$

By the smoothness assumption on f it follows that:

$$|\frac{df}{d\sigma}(t-\tau)(T-\tau)| \leq M|T-\tau| \quad (4)$$

Also from the expression for the remainder in the Taylor approximation ([10], [15]) we obtain

$$\begin{aligned} |o(\tau^2)| &= \frac{|f^{(2)}(\xi)|\tau^2}{2} \leq \frac{M}{2}\tau^2, t-\tau \leq \xi \leq t \quad (5) \\ |o(T^2)| &\leq \frac{M}{2}T^2 \quad (6) \end{aligned}$$

and the result follows by combining (4) and (5).

□

III. Robustness with Respect to Noise

Next we consider the case in which the signal $f(t)$ is affected by the noise signal $n(t)$, $|n(t)| \leq \beta << \frac{1}{3}$; this results in the following compensated signal being sent across the channel $F_s^* = 2f(t) - f(t-\varepsilon) + 2n(t) - n(t-\varepsilon)$, while at the receiver side the following signal is received: $F_r^* = 2f(t-\tau) - f(t-\tau-\varepsilon) + 2n(t-\tau) - n(t-\tau-\varepsilon)$. It follows that the noise is not amplified hence

$$\begin{aligned} F_r^* &\leq 2f(t-\tau) - f(t-\tau-\varepsilon) + \beta 3 \\ &= f(t-\tau) + \varepsilon\frac{f(t-\tau)-f(t-\tau-\varepsilon)}{\varepsilon} + N \end{aligned}$$

where $N < 1$. Hence the previous analysis for stability and tracking can be repeated, including the additional noise term N.

Theorem 2: Consider a signal $f(t) \in C^2$, assume $|f^{(k)}| \leq M, k = 0,1,2$. The following signal, subject to noise $|n(t)| \leq \beta << \frac{1}{3}$ is transmitted across the channel

$$F_s^* = f(t) + f(t) - f(t-\varepsilon) + n(t) + n(t) - n(t-\tau) \quad (7)$$

Then, the received signal satisfies $F_r^*(t) = f(t) + \varepsilon_d + N$, with $|\varepsilon_d| < M(\frac{\varepsilon^2}{2} + \frac{\tau^2}{2} + \frac{2|\varepsilon-\tau|}{\varepsilon}), N < 1$.

IV. Stability of Scalar Linear Systems Under Constant Time Delay

In this section we consider a scalar linear system

$$\dot{x}(t) = -ax(t) - bx(t-\tau) \quad (8)$$

where $x, a, b, k \in \mathbf{R}$ and τ is a constant delay. For such systems it has been proven (see [13] for reference) that

- The zero delay set S_0 i.e. the set of systems (8) which are stable for $\tau = 0$ is defined by $S_0 = \{(a,b) : a+b > 0\}$.
- The delay independent set S_∞, i.e. the set of systems which are stable for all values of delays is defined by $S_\infty = \{(a,b) : a+b > 0, a \geq |b|\}$.
- The delay dependent set S_τ, i.e. the set of systems which are stable only for suitable values of delay τ is defined

by $S_\tau = \{(a,b) : a+b > 0, b \geq |a|\}$ and the maximum value of delay is

$$\tau^* = \frac{\arccos(\frac{-a}{b})}{\sqrt{b^2-a^2}} \quad (9)$$

Consider next the scalar linear system obtained by applying the processing scheme

$$\dot{x}(t) = -ax(t) - b(2x(t-\tau) - x(t-\tau-\varepsilon)) \quad (10)$$

again ε is a design parameter. The characteristic equation associated with (10) is

$$p(s) = s + a + b(2 - e^{-s\varepsilon})e^{-s\tau} \quad (11)$$

Theorem 3: The zero delay set S_0, for system (10) is

$$S_0 = \{(a,b,\varepsilon) : a+b > 0, \forall\varepsilon\} \quad (12)$$

Proof *The characteristic equation for (10) with zero delay is*

$$p(s) = s + a + b(2 - e^{-s\varepsilon}) \quad (13)$$

denote by $\alpha = a+2b, \beta = -b$, then the system is stable independently of ε if $\alpha + \beta > 0$ and $\alpha > |\beta|$. We have $\alpha + \beta = a+b+b-b$, and $\alpha - |\beta| = a+b+b-|b|$. The previous two conditions are satisfied for $a+b > 0$, hence the result follows.

□

In other words, if the original system (8) was asymptotically stable without delay, so is the compensated system (10). Next we want to compare the two systems when there is a nonzero delay. First we characterize the class of systems which are stable independent of delay, i.e. we want to define the set S_∞ for system (10).

Theorem 4: The delay-independent stability region for system (10) is given by $S_\infty = \{(a,b,\varepsilon) : a+b > 0, a > |3b|\}$.
Proof *Under the assumption $a+b > 0, a > |3b|$ we aim to prove by contradiction that the characteristic equation (11) has no solution in the right half plane for any value of the delay. Assume that there exist at least one unstable solution of (11), $\bar{\lambda} = \bar{\sigma} + j\bar{\omega}$, $\bar{\sigma} > 0$, then the characteristic polynomial becomes*

$$\begin{aligned} & p(\bar{\sigma} + j\bar{\omega}) = \bar{\sigma} + j\bar{\omega} + a + b(2 - e^{-(\bar{\sigma}+j\bar{\omega})\varepsilon})e^{-(\bar{\sigma}+j\bar{\omega})\tau} \\ = \; & \bar{\sigma} + a + 2be^{-\bar{\sigma}\tau}\cos(\bar{\omega}\tau) - be^{-\bar{\sigma}(\tau+\varepsilon)}\cos(\bar{\omega}(\tau+\varepsilon)) + \\ & j(\bar{\omega} - 2be^{-\bar{\sigma}\tau}s(\bar{\omega}\tau) - be^{-\bar{\sigma}(\tau+\varepsilon)}\sin(\bar{\omega}(\tau+\varepsilon))) \end{aligned} \quad (14)$$

$$\bar{\sigma} + a + 2be^{-\bar{\sigma}\tau}\cos(\bar{\omega}\tau) - be^{-\bar{\sigma}(\tau+\varepsilon)}\cos(\bar{\omega}(\tau+\varepsilon)) = 0$$
$$(\bar{\omega} - 2be^{-\bar{\sigma}\tau}\sin(\bar{\omega}\tau) - be^{-\bar{\sigma}(\tau+\varepsilon)}\sin(\bar{\omega}(\tau+\varepsilon))) = 0$$

from which we get the following conditions:

$$\begin{aligned} (\bar{\sigma}+a)^2 + \bar{\omega}^2 &= 4b^2e^{-2\bar{\sigma}\tau} + b^2e^{-2\bar{\sigma}(\tau+\varepsilon)} \\ & -4b^2e^{-\bar{\sigma}(2\tau+\varepsilon)}\cos(\bar{\omega}\varepsilon) = 0 \end{aligned}$$

since $\bar{\sigma} > 0$ the following bounds hold:

$$\begin{aligned} a^2 &< (\bar{\sigma}+a)^2 + \bar{\omega}^2 \leq (b^2 4 + b)^2 e^{-2\varepsilon\bar{\sigma}} + 4b^2e^{-\bar{\sigma}\varepsilon})e^{-2\bar{\sigma}\tau} \\ &\leq (2b + be^{-\bar{\sigma}\varepsilon})^2 \leq (3b)^2 \end{aligned} \quad (15)$$

this leads to $a^2 < (3b)^2$ which contradict the original assumption $a \geq |3b|$. Next we need to check if $\bar{\sigma} = 0$ is a solution of (10) in S_∞. We have $\bar{\omega} \neq 0$, since $\lambda = 0$ is not a solution of (10), then repeating the same argument as before using the fact that $|\bar{\omega}| > 0$ and $e^{2\bar{\sigma}\tau}$ we obtain $a^2 \leq (3b)^2$. Then by contradiction it must be that the characteristic polynomial does not have any unstable root in S_∞. Hence under conditions $a+b > 0, a > |3b|$ system (10) is stable independent of delay.

□

Note that the class of systems which are stable independent of delay for system (8) is larger than the class of systems which are stable independent of delay for system (10). This class of system is not of interest for us since it assumes that the open loop system is asymptotically stable and hence there is no need for feedback control. Next we consider the delay dependent case, under the assumption that the delay free system is asymptotically stable ($a+b > 0$). Note that S_τ and S_∞ are complementary with respect to the set S_0, hence $S_\tau = \{(a,b,\varepsilon) : a+b > 0, 3b \geq |a|, \forall\varepsilon\}$.

Theorem 5: The delay dependent set S_τ, for system (10) is defined by $S_\tau = \{(a,b,\varepsilon) : a+b > 0, 3b \geq |a|, \varepsilon\}$, the system is stable $\forall\tau < \tau^*$ and $\varepsilon = \varepsilon^*$ whenever the following equalities are satisfied with $\varepsilon \in \mathbf{R}^+$:

$$a + b(2cos(\bar{\omega}\tau) - \cos(\bar{\omega}(\tau+\varepsilon))) = 0$$
$$\bar{\omega} - b(2\sin(\bar{\omega}\tau) - \sin(\bar{\omega}(\tau+\varepsilon))) = 0 \quad (16)$$

explicit values of ω, τ can be calculates for $b^2(5-4cos(\varepsilon)) > a^2$

$$\bar{\omega} = \sqrt{b^2(5-4cos(\varepsilon)) - a^2}$$

$$\tau = \frac{asin\left(\frac{\sqrt{(5-4\cos(\varepsilon)) - \frac{a^2}{b^2}}}{5-4\cos(\varepsilon\sqrt{b^2(5-4\cos(\varepsilon))-a^2})}\right) - \arctan(\frac{\sin(\bar{\omega}\varepsilon)}{2-cos(\bar{\omega}\varepsilon)})}{\sqrt{b^2(5-4\cos(\varepsilon)) - a^2}}$$

Proof *Note that due to the continuity of the solution and from the fact that the free delay system is stable it is sufficient to study the case in which the system solution crosses the $j\omega$ axis, also since the system is scalar the crossing will only happens once.*

□

A. Simulation Results

We simulated the scalar linear system with no compensation $\dot{x}(t) = x(t) - 3x(t-\tau)$ from (9) we calculated the destabilizing value of delay to be $\tau^* = 0.9377$. Then we considered the system with compensation $\dot{x}(t) = x(t) - 3(2x(t-\tau) - x(t-\tau-\varepsilon))$ and from equations (16) we applied a delay of $\tau = 1.2$ and obtained a value of $\varepsilon = 0.5151$. The simulation results are plotted in Figure 3.

V. Razumikhin-Type Theorem

In this section we consider general n dimensional systems, we introduce the notion of retarded functional differential equation (RFDE) and recall the Razumikhin-Type theorem from [4], [11], which gives sufficient conditions under which a system with delay described by functional differential equation (FDE) is asymptotically stable. Also we recall the result from [16] on the ISS version of the Razumikhin-Type theorem. We consider the following retarded functional differential equation as defined in [13]

$$\dot{x}(t) = f(t, x_t), t \geq t_0 \tag{17}$$

with initial condition $x(t_0+\theta) = \phi(\theta)$, where $x_t = x(t+\theta)$, $\theta \in [-\tau, 0]$, $\tau > 0$, $x(t) \in \mathbf{R}^n$, $(t_0, \phi) \in \mathbf{R}^+ \times C^b_{n,\tau}$, $C^b_{n,\tau} = \{\phi \in C([-\tau,0],\mathbf{R}^n) : \|\phi\| < b\}$ where $C([-\tau,0],\mathbf{R}^n)$ denotes the Banach space of continuous vector functions $\phi : [-\tau, 0] \to \mathbf{R}^n$ and $b \in \mathbf{R}^+$, and f is continuous, locally Lipschitz and $f(t,0) = 0$ on $t \geq t_0$. Also we recall that a function $x : \mathbf{R} \to \mathbf{R}^n, x(t) = x(t_0, \phi)(t)$ denote the unique solution of (17). Next we recall the Razumikhin-Type theorem.

Theorem 6 (Razumikhin-Type from [4], [11]): Consider the system described by the functional differential equation (17) $x_t = x(t+\theta), \theta \in [-\tau, 0]$, $\tau > 0$, and $f(t,0) = 0$ on $t \geq t_0$. Assume that

i.) $u, v, w : \mathbf{R}^+ \to \mathbf{R}^+$ are continuous strictly increasing functions $u(s), v(s), w(s)$ positive for $s > 0$, and $u(0) = v(0) = 0$;

ii.) $q_1 > 1$ is a constant.

If there is a continuous function $V : \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}^+$ such that

(I) for $t \in \mathbf{R}^+$ and $x \in \mathbf{R}^n$

$$u(|x|) \leq V(t,x) \leq v(|x|) \tag{18}$$

(II) there is a $q_2 \geq \frac{1}{q_1}$ such that $\forall t \geq t_0$

$$\sup_{\theta \in [-\tau,0]} V(t+\theta, x(t+\theta)) < q_1 V(t, x(t)), \tag{19}$$

implies $\forall t \geq t_0$

$$\sup_{\theta \in [-\tau,0]} |x(t+\theta)| \quad \leq \quad q_1 q_2 |x(t)| < u^{-1}\left(q_1 v\left(|x(t)|\right)\right)$$

(III) for any $t_0 \in \mathbf{R}^+$
$\dot{V}[t, x(t)] \leq -w([|x(t)|])$ whenever

$$\sup_{\theta \in [-\tau,0]} |x(t+\theta)| \leq q|x(t)|, q = q_1 q_2 \geq 1 \tag{20}$$

then the zero solution of (17) is uniformly asymptotically stable.

VI. Stability of Linear Systems Under Time-Varying Delay

It is known that stable linear systems are inherently robust with respect to small delays [13], [8]. As the value of delay increase stability is no longer guaranteed. To improve the system robustness with respect to delays in the loop, we apply the scheme to a linear system subject to time-varying delay in the state feedback. We prove stability of linear systems under our scheme, the argument of the proof is based on Razumikhin's theorem [4], [11]. Please note that in the case of $n > 1$ order systems we provide an upper bound on the delay that the compensated system can support while preserving stability, we are not able to analytically evaluate performance with respect to the non-compensated system, due to the conservativeness of the used approach. We only compare the two systems in the simulation results. Future work will include performance analysis of the scheme with respect to the uncompensated system. Consider the linear time-invariant system $\dot{x}(t) = Ax(t) + Bu(t)$ where the matrices $A \in \mathbf{R}^n \times \mathbf{R}^n, B \in \mathbf{R}^n \times \mathbf{R}^m$, are such that (A, B) is stabilizable. Consider the following stabilizing state feedback control law $u(t) = Kx(t)$ such that $\tilde{A} = (A + BK)$ is Hurwitz. If the measured and transmitted state $x(t)$ is affected by time varying delay $\tau : \mathbf{R}^+ \to \mathbf{R}^+$ that satisfy assumption 1, then we have the following feedback law $u(t) = Kx(t-\tau)$. Instead of transmitting $x(t)$, we transmit the processed signal

$$x^*_s(t) = x(t) + (x(t) - x(t-\varepsilon)) \tag{21}$$

for some positive design parameter ε. Then the corresponding control input is $u(t) = Kx^*_s(t-\tau)$ and the closed loop system becomes

$$\dot{x}(t) \quad = \quad Ax(t) + BK(2x(t-\tau) - x(t-\tau-\varepsilon)) \tag{22}$$

with $\tau \in [0, \tau_m]$ where $\tau_m = \max_z \tau(z) < \infty$ and $x_t = x(t+\theta)$, $\theta \in [-\bar{\tau}, 0]$, $\bar{\tau} = \tau_m + \varepsilon$. This results in a LTI system with multiple delays, which is a special case of FDE (17), (see [9], [13] and [4], [6]). Note that A is an Hurwitz matrix and the system (22) is linear time invariant hence it satisfies all the properties of (17).

Theorem 7: Consider the system (22), with $\tau \in [0, \tau_m]$, where ε and $\tau_m < \infty$ are positive numbers and define $\tilde{A} = (A + BK)$. Assume that the delay free system is stable and hence by converse Lyapunov theorem, for an arbitrary matrix $Q > 0$ there is a unique positive definite matrix $P = P^T$ such that, $\tilde{A}^T P + P\tilde{A} = -Q$. Then the system (22) is asymptotically stable if the following upper bound on the maximum delay $\tau_m + \varepsilon$ holds: $(\tau_m + \varepsilon) \leq \bar{T} = \frac{\lambda_{min}(Q)}{2|BK|\lambda_{max}(P)A_2}$, where $A_2 = q^2(|A| + 3|BK|)$, $|.|$ denotes the induced L_2 norm, $\lambda_{min}(Q), \lambda_{max}(P)$ are the minimum and maximum eigenvalues of Q and P respectively and $q = \varepsilon^* \sqrt{\frac{\lambda_M q_1}{\lambda_m}} > 1$, where $\sqrt{\frac{\lambda_m}{\lambda_M q_1}} \leq \varepsilon^* < 1$ and $q_1 > 1$ are constants. For brevity we denote $\lambda_M = \lambda_M(P), \lambda_m = \lambda_m(P)$ as the maximum and minimum eigenvalues of P respectively and in the future $|x|$ will denote the Euclidian norm.

Proof *The proof is based on the Razumikin-Type Theorem 6. We need to prove that under the stated assumptions, the conditions of Theorem 6 are satisfied, and hence stability follows by the same theorem. The delay free system* $\dot{x} = (A + BK)x = \tilde{A}x$ *is asymptotically stable with the Lyapunov function* $V(x) = x^T Px$ *which satisfies* $u(x) = \lambda_m(P)|x|^2 \leq V(x) \leq \lambda_M(P)|x|^2 = v(x), \forall x \in \mathbf{R}^n$, *where* u, v *are the* $\mathscr{K}_\infty$ *functions defined in Theorem 6. We need to verify that condition (II) is satisfied. Assume that for some* $q_1 > 1$ *and* $\bar{\tau} = \tau_m + \varepsilon$, *the following condition holds:*

$$\sup_{s \in [-\bar{\tau},0]} V(x(t+s)) < q_1 V(x(t)), \tag{23}$$

note that we can define a constant $\sqrt{\frac{\lambda_m}{\lambda_M q_1}} \le \varepsilon^* < 1$ *such that if the inequality (23) holds, then the following one holds*

$$\sup_{s\in[-\bar{\tau},0]} V(x(t+s)) \le \varepsilon^* q_1 V(x(t)), \tag{24}$$

which implies the following:

$$\begin{aligned} \sup_{s\in[-\bar{\tau},0]} |x(t+s)| &\le \sup_{s\in[-\bar{\tau},0]} \sqrt{\frac{x(t+s)^T P x(t+s)}{\lambda_m}} \qquad (25) \\ &= \sqrt{\sup_{s\in[-\bar{\tau},0]} \frac{x(t+s)^T P x(t+s)}{\lambda_m}} \\ &\le \varepsilon^* \sqrt{\frac{q_1}{\lambda_m} x(t)^T P x(t)} \le \sqrt{\frac{q_1 \lambda_M}{\lambda_m}} |x(t)| \end{aligned}$$

Then, we define $q_2 = \frac{\varepsilon^*}{q_1}\sqrt{\frac{\lambda_M q_1}{\lambda_m}} \ge \frac{1}{q_1}$*, and from (25) obtain*

$$\begin{aligned} \sup_{s\in[-\bar{\tau},0]} |x(t+s)| &\le q_1 q_2 |x(t)| \\ &< \sqrt{\frac{q_1\lambda_M}{\lambda_m}} |x(t)| = u^{-1}(q_1 v(|x|)) \end{aligned} \tag{26}$$

Hence condition (II) is satisfied. To prove condition (III) consider the closed loop system (22) and rewrite $\tilde{A} = A + BK$ *to obtain*

$$\begin{aligned} \dot{x}(t) &= \tilde{A}x(t) + BK(x(t-\tau) - x(t) \\ &+ (x(t-\tau) - x(t-\tau-\varepsilon))) \qquad (27) \\ &= \tilde{A}x(t) + BK\left(\int_{t-\tau}^{t} -\dot{x}(s)ds + \int_{t-\tau-\varepsilon}^{t-\tau} \dot{x}(s)ds\right) \end{aligned}$$

We can upper bound the term $(\int_{t-\tau}^{t} -\dot{x}(s)ds + \int_{t-\tau-\varepsilon}^{t-\tau} \dot{x}(s)ds)$ *as follows:*

$$\begin{aligned} &\left(\int_{t-\tau}^{t} -\dot{x}(s)ds + \int_{t-\tau-\varepsilon}^{t-\tau} \dot{x}(s)ds\right) \\ \le\ & (\tau+\varepsilon) \sup_{s\in[t-\tau-\varepsilon,t]} (|\dot{x}(s)|) \end{aligned} \tag{28}$$

Then, for τ_m *and for* $q = q_1 q_2 = \varepsilon^* \sqrt{\frac{\lambda_M q_1}{\lambda_m}} > 1$ *we have that whenever* $\sup_{\alpha\in[t-\tau_m-\varepsilon,t]} |x(\alpha)| \le q|x(t)|$*, the following holds:*

$$\begin{aligned} &\sup_{\alpha\in[t-2\tau_m-2\varepsilon,t]} |x(\alpha)| = \\ &\max\{\sup_{\alpha_1\in[t-2\tau_m-2\varepsilon,t-\tau_m-\varepsilon]} |x(\alpha_1)|, \sup_{\alpha_2\in[t-\tau_m-\varepsilon,t]} |x(\alpha_2)|\} \\ &< \max\{q|x(t-\tau_m-\varepsilon)|, q|x(t)|\} \\ &< q^2|x(t)| \end{aligned}$$

from which we can bound (28) as follows:

$$\begin{aligned} &\left(\int_{t-\tau}^{t} -\dot{x}(s)ds + \int_{t-\tau-\varepsilon}^{t-\tau} \dot{x}(s)ds\right) \\ \le\ & (\tau+\varepsilon) q^2 (|A| + 3|BK|)|x(t)| \\ =\ & (\tau+\varepsilon) A_2 |x(t)| \end{aligned}$$

where $A_2 = q^2(|A| + 3|BK|)$*. Next consider the derivative of* $V(x) = x^T P x$ *along the trajectory of (27):*

$$\begin{aligned} \dot{V} &= -|x(t)|^T Q |x(t)| \\ &+ 2x(t)^T PBK \left(\int_{t-\tau}^{t} -\dot{x}(s)ds + \int_{t-\tau-\varepsilon}^{t-\tau} \dot{x}(s)ds\right) \end{aligned} \tag{29}$$

whenever $(\int_{t-\tau}^{t} -\dot{x}(s)ds + \int_{t-\tau-\varepsilon}^{t-\tau} \dot{x}(s)ds)$*, we have*

$$\begin{aligned} \dot{V} &\le -x(t)^T Q x(t) + 2P|BK||x(t)|((\tau+\varepsilon)A_2)|x(t)|) \\ &\le |x(t)|^T(-Q + 2P|BK|((\tau+\varepsilon)A_2))|x(t)| \end{aligned} \tag{30}$$

Define $\bar{T} = \frac{\lambda_{min}(Q)}{2|BK|\lambda_{max}(P)A_2}$*, then for* $(\tau_m + \varepsilon) < \bar{T}$*, we obtain* $Q - 2P|BK|((\tau+\varepsilon)A_2 = W > 0$*, and hence*

$$\dot{V} \le -|x(t)|^T W |x(t)| = -w(|x|) \tag{31}$$

where w *is the function defined in Theorem 6. Then by Razumikhin theorem we have that the closed loop system (27) is asymptotically stable.*

□

A. Simulation Results

We simulated an LTI system with $A = \begin{bmatrix} 0.1 & 1 \\ -1 & 0.1 \end{bmatrix}$, $B = [1;1]$, $K = [-0.15 - 0.15]$. The delay in the feedback loop is $\tau(t) = (1 + 0.25\sin(0.5t))$. We used the processing scheme (21) with parameters $Q = I$, $P = \begin{bmatrix} 8.6990 & -0.0765 \\ -0.0765 & 11.7602 \end{bmatrix}$ and $q_1 = 1.1$, $\varepsilon^* = 0.8199$ and $\varepsilon = .075$. The resulting state dynamics are depicted in Figure 2. If we gradually increase the delay, the compensated system preserve stability until a value of the delay $\tau(t) = (1.5 + 0.5\sin(0.5t))$. We obtain the following upper bound on the delay $\bar{T} = 0.075$. Although the sufficiency of the analysis result does not allow for an exact characterization of the robustness property of the scheme, we can observe from the simulation results that the stability of closed-loop system is preserved for large values of delays under our scheme, as compared to uncompensated delayed state feedback. We repeat the simulation introducing a random noise $n(t)$, such that the signal to noise ratio is 10. The resulting output signal preserve the same signal to noise ratio, as shown in Figure 1, hence the scheme does not amplify the noise as was shown in Section III.

VII. Conclusion and Future Work

We presented a scheme to process signals to be sent across a channel which introduces time-varying unknown delays. We proved that under appropriate conditions on the signal to be processed the received signal is guaranteed to have small error with respect to the transmitted signal. We applied our scheme to a control system with actuation delays and gave sufficient conditions for stability. For scalar linear systems under constant time-delay an exact bound on the delay which the system can support is given, this allows us to have a direct comparison with the compensation free system. A conservative upper bound on the admissible delay was also provided for the case of $n > 1$ dimensional systems. Future work will focus on considering higher order terms in the

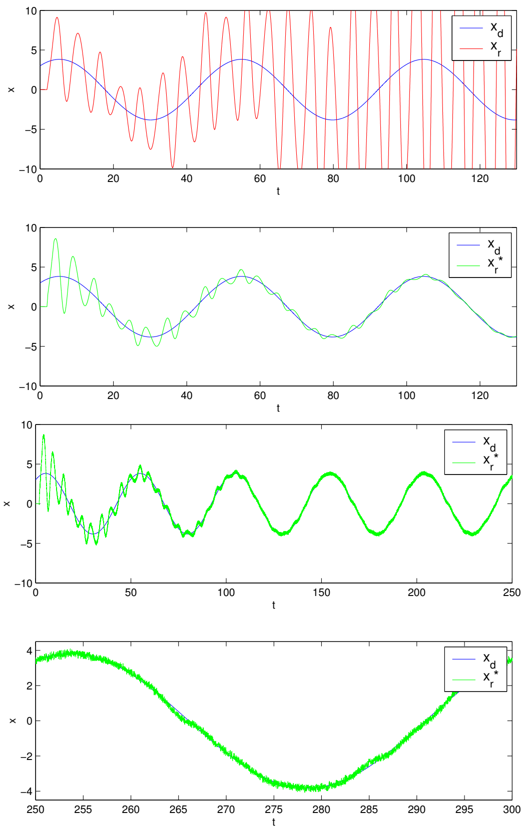

Fig. 1. From the top: 1) Noise-free closed loop system dynamics for original system x_d, and delayed system x_r. 2) Noise-free compensated system x_r^*. 3-4) Noisy closed loop system dynamics for original system x_d, and compensated system x_r^*, transient (Top) and steady state (Bottom).

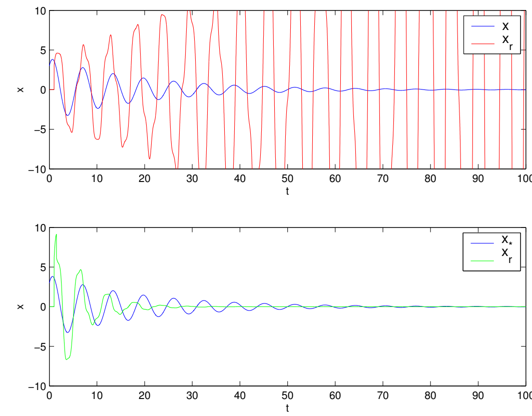

Fig. 2. Closed loop system dynamics for original system x, delayed system x_r and compensated system x_r^*.

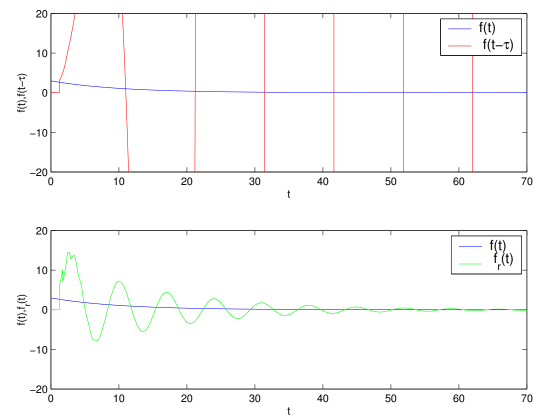

Fig. 3. Scalar system without compensation (top) and system with compensation (bottom), with a delay of $\tau = 1.2$

Taylor series expansion to achieve better estimation, this will also allow us to handle signal at high frequency. We will also consider extension of the present result to discrete time systems.

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