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Riccati Conditioning and Sensitivity for a MinMax Controlled Cable-Mass System

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Abstract— **In this paper we present a numerical study that investigates the relationship between the parameter θ, used in the design of the MinMax controller, and the conditioning of the approximate algebraic Riccati equations, the sensitivity of the eigenvalues of $I-\theta^2 P\Pi$ to θ as well as the effect of θ on the stability radia and the stability margin of the system.**

In order to guarantee accurate numerical solutions to the approximate Riccati equations, the Riccati equations must remain well-conditioned for the values of θ that are considered. This condition number reflects the combined sensitivity of the Riccati equations to the system inputs A, B, R, C and θ. In addition, we also consider the sensitivity of the eigenvalues of $I-\theta^2 P\Pi$ to θ. We study the possibility of these sensitivities serving as an indication of the largest value of θ for which $I-\theta^2 P\Pi$ remains positive definite. This sensitivity could also serve as an indication of the accuracy of the computation of $I-\theta^2 P\Pi$. Lastly, in order to design efficient low order controllers, it is important to ensure the robustness of the design. Stability radius and stability margin serve as measures of the robustness of the controller.

A one-dimensional nonlinear cable mass system is considered to illustrate these ideas and numerical results are presented.

I. INTRODUCTION

It is well known that much research attention has been devoted to the H_∞ controller since the original problem was formulated by Zames [1]. Rhee and Speyer [2] later introduced what has become known as the MinMax controller, which is really a differential game approach to solving the H_∞ control problem. As such, one of the challenges in the design of the MinMax controller is determining the "optimal" value, or range of values, of the parameter θ. Presently, a costly iterative procedure is used choose the value for θ. Chen [3] points out that, in general, the iterative procedure to determine θ is not reliable.

The value of θ should be such that $I-\theta^2 P\Pi$ is positive definite where P and Π are solutions to algebraic Riccati equations and depend on θ. To establish the positive definiteness of the finite dimensional approximation $I^N-\theta^2 P^N\Pi^N$ one could compute the eigenvalues of this matrix for different values of θ. This is for two reasons not a trivial matter. Firstly, the accuracy of the solutions of the Riccati equations, P and Π, will influence the accuracy of the eigenvalues. Secondly, these discretized problems are typically large in size and it is a challenge to obtain all the eigenvalues accurately.

As an alternative, we study the conditioning of the Riccati equations as well as the sensitivity of the eigenvalues with respect to θ. The condition number of the Riccati equation reflects the sensitivity of the solution with respect to changes in A^N, B^N, R^N, C^N and θ. The sensitivity of the eigenvalues of $I^N-\theta^2 P^N\Pi^N$ with respect to θ is investigated to determine if the sensitivity serves as an indication for which values of θ the transition takes place and $I^N-\theta^2 P^N\Pi^N$ is no longer positive definite.

In the next section we describe the cable mass system that we use as to illustrate these ideas. We present a brief description of the MinMax controller in Section III. In Sections IV, V and VI brief discussions of the conditioning of the Riccati equation, the sensitivity of the eigenvalues with respect to θ and controller robustness as a function of θ are presented. Each of these sections include numerical results that illustrate the ideas. A summary of the results and a discussion of future research follow in Section VII.

II. A STRUCTURAL VIBRATION PROBLEM

The one-dimensional nonlinear cable mass distributed parameter system considered in this paper was described in [4] and studied numerically in [5] and [6]. This model can be viewed as an elastic cable fixed at one end and attached to a mass at the other end. The mass is suspended by a spring, which contains nonlinear stiffening terms and is being forced by a sinusoidal disturbance.

The equations governing this system are as follows:

$$\rho\frac{\partial^2}{\partial t^2}w(t,s)=\frac{\partial}{\partial s}\left[\tau\frac{\partial}{\partial s}w(t,s)+\gamma\frac{\partial^2}{\partial t\partial s}w(t,s)\right],$$

for $0<s<\ell,\ t>0$, and

$$m\frac{\partial^2}{\partial t^2}w(t,\ell)\quad=-\left[\tau\frac{\partial}{\partial s}w(t,\ell)+\gamma\frac{\partial^2}{\partial t\partial s}w(t,\ell)\right]-\alpha_1 w(t,\ell)-\alpha_3[w(t,\ell)]^3+\eta(t)+u(t),$$

with boundary condition

$$w(t,0)=0,$$

and initial conditions of the form

$$w(0,s)=w_0(s),\qquad \frac{\partial}{\partial t}w(0,s)=w_1(s),$$

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where $w(t,s)$ represents the displacement of the cable at time t and position s, $w(t,\ell)$ gives the position of the mass at time t, ρ and m are the densities of the cable and mass respectively, τ is tension in the cable, and γ is the coefficient of the damping term. The spring's stiffening terms have coefficients of α_1 and α_3, with α_3 being associated with the nonlinear effects in the spring. A disturbance enters through $\eta(t)$, and $u(t)$ is a control input. This view of the cable mass system can be seen in Figure 1.

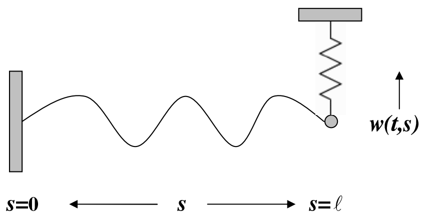

Fig. 1. Cable-mass system.

We want to use sensed information to design a feedback controller that attenuates the disturbance $\eta(t)$. We assume the control acts exclusively on the mass, and the only available measured information is the position and velocity of the mass. These two observations take the form

$$\begin{aligned} y_1(t) &= w(t,\ell), \\ y_2(t) &= \frac{\partial}{\partial t} w(t,\ell). \end{aligned}$$

III. MINMAX CONTROL DESIGN

In this section, we present a short overview of the MinMax compensator design [2]. Assume the existence of a nonlinear PDE system of the form

$$\begin{aligned} \dot{x}(t) &= \mathscr{A}_0 x(t) + \mathscr{N}(x(t)) + \mathscr{B}u(t) + \mathscr{D}\eta(t), \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $x(t) = x(t,\cdot) \in X$ is the state of the nonlinear system and X is a Hilbert space. Here, $\mathscr{A}_0$ is the system operator defined on $\mathbf{D}(\mathscr{A}_0) \subseteq X$, $\mathscr{N}$ defined on X is the nonlinearity in the system, $\mathscr{B}$ is the control operator, $\mathscr{D}$ is the disturbance operator. Both the control input, $u(t)$, and the disturbance, $\eta(t)$, are defined on Hilbert space U. It is assumed that knowledge of only part of the system can be obtained through the state measurement, y, in Hilbert space Y where

$$y(t) = \mathscr{C}x(t). \tag{2}$$

Typically, $\mathscr{C}$ is not the identity operator, so full-state feedback cannot be used to control the system. Instead, an estimate of the state is used in the control law. To provide this estimate, a compensator is used that has the form

$$\dot{x}_c(t) = \mathscr{A}_c x_c(t) + \mathscr{F}y(t), \quad x_c(0) = x_{c_0}$$

and the feedback control law is written

$$u(t) = -\mathscr{K}x_c(t)$$

where $x_c(t) = x_c(t,\cdot) \in X$ is the state estimate. Designing a controller of this type requires determining $\mathscr{A}_c, \mathscr{F}$, and $\mathscr{K}$.

The MinMax compensator is defined for linear systems, so one must first linearize the system in (1), (2). Doing so yields the linear distributed parameter control system (with state x_ℓ) defined on X

$$\dot{x}_\ell(t) = \mathscr{A}x_\ell(t) + \mathscr{B}u(t) + \mathscr{D}\eta(t), \quad x_\ell(0) = x_{\ell_0}$$

with sensed output

$$y(t) = \mathscr{C}x_\ell(t).$$

By solving the Riccati equations

$$\mathscr{A}^*\Pi + \Pi\mathscr{A} - \Pi(\mathscr{B}R^{-1}\mathscr{B}^* - \theta^2\mathscr{B}\mathscr{B}^*)\Pi + \mathscr{C}^*\mathscr{C} = 0, \tag{3}$$

where $R: U \to U$ is a weighting operator for the control of the form $R = cI$, with c a scalar and I the identity operator, and

$$\mathscr{A}P + P\mathscr{A}^* - P(\mathscr{C}^*\mathscr{C} - \theta^2\mathscr{C}^*\mathscr{C})P + \mathscr{B}\mathscr{B}^* = 0, \tag{4}$$

one can obtain the operators $\mathscr{K}$, $\mathscr{F}$, and $\mathscr{A}_c$ via

$$\begin{aligned} \mathscr{K} &= R^{-1}\mathscr{B}^*\Pi, \\ \mathscr{F} &= (I - \theta^2 P\Pi)^{-1}P\mathscr{C}^*, \\ \mathscr{A}_c &= \mathscr{A} - \mathscr{B}\mathscr{K} - \mathscr{F}\mathscr{C} + \theta^2\mathscr{B}\mathscr{B}^*\Pi. \end{aligned}$$

The resulting feedback control is applied to the original nonlinear system; the closed loop nonlinear system is then defined by

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} &= \begin{bmatrix} \mathscr{A} & -\mathscr{B}\mathscr{K} \\ \mathscr{F}\mathscr{C} & \mathscr{A}_c \end{bmatrix}\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathscr{N}(x(t)) \\ \mathscr{N}(x_c(t)) \end{bmatrix} + \begin{bmatrix} \mathscr{D} \\ 0 \end{bmatrix}\eta(t). \end{aligned} \tag{5}$$

For sufficiently small θ, there are guaranteed minimal solutions Π and P to (3) and (4), respectively, such that $(I - \theta^2 P\Pi)$ is positive definite and the linearized closed loop system, i.e. the linearized form of (5), is stable. Note that if $\theta = 0$, we are considering the classical Linear Quadratic Gaussian (LQG) compensator design.

A standard approach to computational implementation of the PDE systems defined in the previous sections is to approximate the equations in (5). When a finite element method is applied to discretize the spatial variable, one obtains the finite dimensional system

$$\begin{aligned} \begin{bmatrix} \dot{x}^N(t) \\ \dot{x}_c^N(t) \end{bmatrix} &= \begin{bmatrix} A^N & -B^N K^N \\ F^N C^N & A_c^N \end{bmatrix}\begin{bmatrix} x^N(t) \\ x_c^N(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} N^N(x^N(t)) \\ N^N(x_c^N(t)) \end{bmatrix} + \begin{bmatrix} D^N \\ F^N E^N \end{bmatrix}\eta(t) \end{aligned} \tag{6}$$

where N is related to the number of elements in the uniform mesh, and $x^N, x_c^N \in X^N \subset X$ where X^N is a finite dimensional subspace of X. Corresponding to this system, discretized forms of the algebraic Riccati equations in (3), (4) are computed. These operators are used to construct the finite dimensional versions of the MinMax controller.

TABLE I
SYSTEM PARAMETERS

ρ	τ	γ	m	ℓ	α_1	α_3
1	1	.005	1.5	2	.01	3

For our numerical experiments we use linear B-splines and convergence has been obtained for $N = 80$. The system parameter values used are given in Table I.

The eigenvalues of $I^N - \theta^2 P^N \Pi^N$ computed using the `eig` function in MATLAB® suggests that the largest possible MinMax controller parameter θ that will guarantee $(I - \theta^2 P\Pi)$ being positive definite is 0.45.

IV. RICCATI CONDITIONING

The condition number of a problem is an indication of the sensitivity of the problem to perturbations in the data. These perturbations can be due to finite precision arithmetic, corrupted data, et cetera. It is well known that in order to guarantee accurate solutions, the problem must be well-conditioned and the algorithm must be stable.

In the case of the Algebraic Riccati Equation (ARE),

$$A^*X + XA - XSX + C^*C = 0,$$

the condition number measures the sensitivity of the solution to first order perturbations in A, S (where S includes B, R and θ) and Q where $Q = C^*C$.

The relative condition number, [7], is defined by

$$\kappa(A,B,Q) = \sup_{\delta \to 0} \left\{ \frac{\|\Delta X\|}{\delta \|X\|} : \frac{\|\Delta A\|}{\|A\|} \le \delta, \ \frac{\|\Delta S\|}{\|S\|} \le \delta, \ \frac{\|\Delta Q\|}{\|Q\|} \le \delta \right\}$$

where $\Delta A = \tilde{A} - A$, $\Delta S = \tilde{S} - S$ and $\Delta Q = \tilde{Q} - Q$ and $\tilde{A}$, $\tilde{S}$ and $\tilde{Q}$ refer to the perturbed versions of A, S and Q.

If $\varepsilon\kappa(A,S,Q) \ll 1$ the problem is said to be well conditioned and very ill conditioned if $\varepsilon\kappa(A,S,Q) \approx 1$ where ε is related to the floating-point environment.

Byers, see [8], defined an approximate condition number in terms of the Frobenius norm. Kenney and Hewer, see [9], extended these ideas and obtained sharper bounds for the approximate condition number. They also extended the norms to norms other than the Frobenius norm. If κ_L and κ_U respectively denote the lower and upper bounds defined by Kenney and Hewer, then

$$\frac{\kappa_L}{3} \le \kappa \le \kappa_U$$

where the spectral norm is used. Computing κ_L and κ_U involves five Lyapunov solutions and seven matrix norms.

We compute these upper and lower bounds on the condition numbers of the discretized Riccati equations:

$$(A^N)^*\Pi^N + \Pi^N A^N - \Pi^N \left[B^N (R^N)^{-1}(B^N)^* - \theta^2 B^N (B^N)^*\right]\Pi^N + (C^N)^* C^N = 0 \quad (7)$$

and

$$A^N P^N + P^N (A^N)^* - P^N \left[(C^N)^* C^N - \theta^2 (C^N)^* C^N\right] P^N + B^N (B^N)^* = 0. \quad (8)$$

In Tables II and III we present these bounds for increasing θ.

TABLE II
CONDITIONING OF (7)

θ	κ_L	κ_U
0.00	5.0482×10^4	1.2556×10^6
0.05	5.0406×10^4	1.2543×10^6
0.10	5.0181×10^4	1.2506×10^6
0.15	4.9819×10^4	1.2447×10^6
0.20	4.9344×10^4	1.2368×10^6
0.25	4.8789×10^4	1.2274×10^6
0.30	4.8205×10^4	1.2173×10^6
0.35	4.7667×10^4	1.2075×10^6
0.40	4.7279×10^4	1.1995×10^6
0.45	4.7201×10^4	1.1952×10^6

TABLE III
CONDITIONING OF (8)

θ	κ_L	κ_U
0.00	2.2655×10^6	1.1986×10^7
0.05	2.2099×10^6	1.1899×10^7
0.10	2.0583×10^6	1.1653×10^7
0.15	1.8475×10^6	1.1292×10^7
0.20	1.6164×10^6	1.0872×10^7
0.25	1.3922×10^6	1.0455×10^7
0.30	1.1854×10^6	1.0115×10^7
0.35	0.9786×10^6	0.9972×10^7
0.40	0.6037×10^6	1.0342×10^7
0.41	0.6146×10^6	1.0552×10^7
0.42	0.8447×10^6	1.0846×10^7
0.43	1.0684×10^6	1.1264×10^7
0.44	1.3347×10^6	1.1869×10^7
0.45	1.7174×10^6	1.2793×10^7

The bounds are an order of magnitude larger for (8) than what it is for (7). This implies that P is more sensitive to changes than Π is. We also observe from Table II that the bounds for the condition number of (7) decrease as θ increases even though the bounds remain of the same order. In the case of (8), the bounds decrease up to $\theta = 0.4$ but start increasing for $\theta > 0.4$. As mentioned before, direct computations of the eigenvalues $I^N - \theta^2 P^N \Pi^N$ using the `eig` function in MATLAB® suggest that the critical value where $I^N - \theta^2 P^N \Pi^N$ is no longer positive definite, is around $\theta = 0.45$. It seems as if the condition number of (8) could serve as a verification of the value of θ where the matrix is no longer positive definite.

V. SENSITIVITY OF THE EIGENVALUES

As discussed in the introduction, we are interested in the sensitivity of the eigenvalues of $I^N - \theta^2 P^N \Pi^N$ with respect to θ. In particular, this sensitivity could serve as an indication of the values of θ for which the matrix is no longer positive definite.

For the sake of compactness, we omit the superscript N.

Consider the eigenvalue problem

$$(I - \theta^2 P\Pi)x = \lambda x.$$

Taking the derivative with respect to θ leads to

$$\begin{aligned}\frac{\partial}{\partial\theta}(I-\theta^2P\Pi)x+(I-\theta^2P\Pi)\frac{\partial x}{\partial\theta} &= \frac{\partial\lambda}{\partial\theta}x+\lambda\frac{\partial x}{\partial\theta}\\ \frac{v^T}{v^Tx}\left(-2\theta P\Pi-\theta^2\frac{\partial P}{\partial\theta}\Pi-\theta^2P\frac{\partial\Pi}{\partial\theta}\right)x &= \frac{\partial\lambda}{\partial\theta} \qquad (9)\end{aligned}$$

where v is the eigenvector of $(I-\theta^2P\Pi)^T$ associated with λ.

To obtain $\frac{\partial\Pi}{\partial\theta}$ and $\frac{\partial P}{\partial\theta}$, we consider the derivatives of (7) and (8) respectively.

In order to obtain $\frac{\partial\Pi}{\partial\theta}$ from

$$A^*\frac{\partial\Pi}{\partial\theta}+\frac{\partial\Pi}{\partial\theta}A-\left[\Pi BR^{-1}B^*\frac{\partial\Pi}{\partial\theta}+\frac{\partial\Pi}{\partial\theta}BR^{-1}B^*\Pi\right]$$
$$+\theta^2\Pi BB^*\frac{\partial\Pi}{\partial\theta}+\left[\theta^2\frac{\partial\Pi}{\partial\theta}+2\theta\Pi\right]BB^*\Pi=0$$

we have to solve a Lyapunov equation of the form

$$D_1X_1+X_1D_1^T+Q_1=0, \qquad (10)$$

where

$$\begin{aligned}X_1 &= \frac{\partial\Pi}{\partial\theta},\\ D_1 &= A^*-\Pi BR^{-1}B^*+\theta^2\Pi BB^*,\\ Q_1 &= 2\theta\Pi BB^*\Pi.\end{aligned}$$

Similarly, to obtain $\frac{\partial P}{\partial\theta}$ from

$$A\frac{\partial P}{\partial\theta}+\frac{\partial P}{\partial\theta}A^*-\left[PC^*C\frac{\partial P}{\partial\theta}+\frac{\partial P}{\partial\theta}C^*CP\right]$$
$$+\theta^2PC^*C\frac{\partial P}{\partial\theta}+\left[\theta^2\frac{\partial P}{\partial\theta}+2\theta P\right]C^*CP=0.$$

we solve a Lyapunov equation of the form

$$D_2X_2+X_2D_2^T+Q_2=0, \qquad (11)$$

where

$$\begin{aligned}X_2 &= \frac{\partial P}{\partial\theta},\\ D_2 &= A-PC^*C+\theta^2PC^*C,\\ Q_2 &= 2\theta PC^*CP.\end{aligned}$$

Both (10)and (11)are solved using the `lyap` function in MATLAB®.

In Table IV we present the sensitivity $\frac{\partial\lambda}{\partial\theta}$ for increasing θ. The sensitivity of that eigenvalue of $(I-\theta^2P\Pi)$ that changes from positive to non-negative first, is the largest of all the sensitivities and we present those values.

From Table IV we see that the absolute value of the sensitivity of that eigenvalue that has the smallest positive real part increases as θ increases. The sensitivity associated with $\theta=0.45$ implies that the value of λ will decrease 5.34 times as much as the corresponding change in θ. This sensitivity is quite significant and one may question the reliability of the eigenvalue computation which is used to verify that $(I-\theta^2P\Pi)$ is positive definite.

TABLE IV
SENSITIVITY OF EIGENVALUE

θ	$\frac{\partial\lambda}{\partial\theta}$
0.00	0
0.05	−0.3377
0.10	−0.6886
0.15	−1.0670
0.20	−1.4897
0.25	−1.9777
0.30	−2.5589
0.35	−3.2717
0.40	−4.1712
0.45	−5.3406

Even though the size of the sensitivities increases with increasing θ and the rate of change in the sensitivities also increases, there is no clear indication when $(I-\theta^2P\Pi)$ is no longer positive definite.

VI. CONTROLLER ROBUSTNESS

Though it is not the primary focus of this paper, our ultimate goal is to design low order controllers that will stabilize an original plant, not just a PDE model of the plant or a high order finite dimensional approximation of it, for a physical system of interest. To increase the likelihood that our low order controllers will stabilize the plant, a problem for which the low order controller was not originally designed, we want to ensure controller robustness. In our numerical results, we will offer two measurements of robustness by which to compare our controllers for the physical system of interest.

Since we will be comparing robustness of finite dimensional closed loop systems, the following discussion is phrased in a finite dimensional setting. One measurement is the stability margin, which measures the distance from the imaginary axis to the nearest eigenvalue for a given matrix. The second measurement is the complex stability radius as described in [10]. In computing the stability radius of a given matrix, J^N, we seek to find the distance from J^N to the nearest unstable matrix. This is achieved by searching for the minimum singular value of J^N in some rectangular region of the complex plane, thereby describing the distance of J^N from singularity.

We desire to compare robustness measures of the MinMax controlled system for varying values of θ. Thus, we compute the stability radius and stability margin for the matrix

$$\begin{bmatrix} A^N & -B^NK^N \\ F^NC^N & A_c^N \end{bmatrix}, \qquad (12)$$

for a vector of θ values: $\theta=0.00:0.05:0.45$. These computations are presented in Table V.

Note that the stability radius is on the same order of magnitude regardless of the value of θ used to compute the MinMax controller. The same is true of the stability margin.

TABLE V
ROBUSTNESS MEASURES FOR MINMAX CONTROLLED SYSTEMS

θ	Stability Radius	Stability Margin
0.00	0.03125585592333	0.03212894355587
0.05	0.03302852263542	0.03199043497269
0.10	0.03278730359443	0.03165074454705
0.15	0.03239237117549	0.03121865446290
0.20	0.03188657176539	0.03072362659148
0.25	0.03127803722047	0.03014851940854
0.30	0.03075754754857	0.02944021129271
0.35	0.03048102682009	0.02846715159222
0.40	0.03095214388041	0.02685943898823
0.45	0.03327271528664	0.02499459268316

The stability margins decrease as θ increases, but all margins are on the same order of magnitude for the interval of θ values considered.

VII. CONCLUSIONS AND FUTURE WORKS

In this paper, we have used a one-dimensional cable-mass system as a numerical testbed to examine conditioning of the MinMax Riccati equation, the sensitivity of the eigenvalues of $I-\theta^2 P\Pi$ with respect to θ and controller robustness as a function of θ. Specifically, we examined conditioning of the MinMax control and filter algebraic Riccati equations as a means to gauge sensitivity of the Riccati equations to perturbations in A^N, B^N, R^N, C^N and θ. It was found that the solution to the filter Riccati equation is more sensitive to changes in the above inputs than the solution to the control Riccati Equation. Numerical results suggest that the condition number of the filter Riccati equation could serve as a verification of the value of θ where the matrix $I^N-\theta^2 P^N\Pi^N$ is no longer positive definite. More investigation needs to be done to explore this hypothesis.

As another means to investigate values of θ where $I^N-\theta^2 P^N\Pi^N$ is no longer positive definite, the sensitivity of the eigenvalues of this matrix was also examined. The size of the sensitivities increases as θ increases and reaches a point where the eigenvalue computations can no longer be guaranteed to be accurate. Since these computations are used to verify that $I^N-\theta^2 P^N\Pi^N$ is positive definite, one could get an estimate for which values of θ this matrix is no longer guaranteed to be positive definite.

Finally, since we are interested in controller robustness, we calculated stability radii and stability margins of the MinMax controlled cable-mass system for various θ values. It was found that the stability radii and margins are all on the same order of magnitude for the domain of possible parameter values of this problem.

From the results observed here, it appears that the optimal θ is perhaps not that crucial. It seems as if there is a range of values of θ for which the Riccati conditioning, eigenvalue sensitivities for the quantity $I^N-\theta^2 P^N\Pi^N$, and controller robustness are comparable and reasonable. Preliminary studies indicate that the full order MinMax observer yields comparable performance results for the same domain of θ values considered here.

Further studies will include a set of case studies to identify examples where this approach results in better determination of the optimal minmax parameter.

VIII. ACKNOWLEDGMENTS

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