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Boundary feedback control in fluid-structure interactions

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Abstract— **We consider a boundary control system for a Fluid Structure Interaction Model. This system describes the motion of an elastic structure inside a viscous fluid with interaction taking place at the boundary of the structure, and with the possibility of controlling the dynamics from this boundary. Our aim is to construct a real time feedback control based on a solution to a Riccati Equation. The difficulty of the problem under study is due to the unboundedness of the control action, which is typical in boundary control problems. However, this class of unbounded control systems, due to its physical relevance, has attracted a lot of attention in recent literature (cf. [5], [18], [11]). It is known that Riccati feedback (unbounded) controls may develop strong singularities which destroy the well-posedness of Riccati equations. This makes computational implementations problematic, to say the least. However, as shown recently, this pathology does not happen for certain classes of unbounded control systems usually referred to as Singular Estimate Control Systems (SECS) (cf. [11], [21]). For such systems, there is a full and optimal Riccati theory in place, which leads to the well-posedness of feedback dynamics.**

Our objective is to show that the boundary control problem in question falls in the class of Singular Estimate Control Systems (SECS). Once this is accomplished, an application of the theory in [21] leads to the main result of this paper which is well-posedness of Riccati equations and of the Riccati feedback synthesis.

I. Introduction

We consider a model of fluid-structure interaction defined on a simply connected bounded domain $\Omega \in \mathbb{R}^n$, $n = 2,3$, where Ω is comprised of two open domains Ω_f and Ω_s. A stationary solid Ω_s is fully immersed in a fluid on a domain Ω_f with interaction taking place on the boundary of the solid Γ_s. The dynamics of the solid are described by a linear wave equation in the variable w, while the dynamics of the fluid are described as usual by a non-stationary Navier Stokes equation in the variable u. The interaction between the two systems takes place on the boundary Γ_s that is common to both media and is prescribed via suitable (Neumann type) transmission boundary conditions. The model presented is well established in both physical and mathematical literature (cf. [22], [9], [7], [4], [10], [8]). From the physical point of view, it is an important model arising in a variety of applications in cell biology, mechanics and fluid dynamics. From the mathematical point of view, the interest in the model stems from the rather unusual functional analytic setup of the model that is not amenable to the standard variational analysis usually employed to study Navier Stokes equations or wave equations. We have already treated well-posedness of this nonlinear model extensively in a work on weak solutions [3] and another on smooth solutions.

In this paper, we consider a boundary control system of this fluid structure interaction model. Our objective is to develop a feedback control, acting as a force on the interface between the two media, which is based on a solution to the appropriate Riccati equation. It is known that Riccati theory is a very powerful tool for designing robust feedback controls (cf. [1]) and also for numerical computations leading to effective control algorithms. This became a standard approach in finite dimensional control systems. The situation is much more complex in *infinite dimensional systems*, where various topological issues may undermine the effectiveness of the feedback control constructed from finite dimensional approximations. For this reason it is necessary to conduct first a full infinite-dimensional analysis of the relevant Riccati theory. One of the crucial questions to be answered is well-posedness of Riccati equations in an infinite-dimensional setting and boundedness of the so called "gain operator", the latter provides an effective feedback control for the system. While this kind of issues has been dealt with successfully in the case of infinite dimensional systems generated by c_0 semigroups with *bounded* control operators, the situation is much more complex in the case of *unbounded control actions*, as they arise in boundary or point control problems. Even more, there are known counter-examples demonstrating the failure of standard Riccati theory in th (cf. [24]). This, in turn, was a prime motivation for developing a generalized "extended" Riccati theory. However, the "extended" Riccati equations are shown to be well-posed only in the case of infinite horizon problems (cf. [2]). Well-posedness of standard or extended Riccati equations for the finite-horizon problems with unbounded control actions and arbitrary c_0 semigroups is still an open question.

A notable exception is *analytic semigroups*, where full and optimal Riccati theory pertaining to unbounded control actions is in place (cf. [18], [5]). In the analytic case, the difficulty of the problem due to unboundedness of the control action, is circumvented by analyticity and the resulting strong regularity of optimal solutions (cf. [18]). The model, under consideration in this paper, consists of coupled Navier Stokes and elastodynamic waves, hence it is not analytic though it possesses an analytic component. In such case there is a hope that "partial regularity" emanating from the analytic component may offset some of the singularities caused by unbounded control action.

In fact, this observation has led to the construction of a subclass of control systems referred to as Singular Estimate Control systems (SECS) where a characterization of the optimal control as a feedback control via a solution to a well posed Riccati Equation became recently available (cf. [11], [12], [21]), (the latter reference solves the problem in the most demanding case of the Bolza problem). It turns out that the control system of fluid-structure interactions in question falls into this category. Rigorous proof of this property is the main technical task of this paper. In order to proceed, we shall recall the concept of SECS (Singular Estimate Control Systems) systems. Let $\mathscr{A}$ be a generator of a c_0 semigroup $e^{\mathscr{A}t}$ on a Hilbert space $\mathscr{H}$. Let $\mathscr{B}$ - unbounded control operator- be such that $\mathscr{B} \in \mathscr{L}(U \to [D(A^*)]')$, where U is a suitable control space.

Consider the dynamics:

$$y_t = \mathscr{A}y + \mathscr{B}g \in [\mathscr{D}(\mathscr{A}^\star)]'. \quad (1)$$

With the dynamics given in (1) we associate observation operators $R \in \mathscr{L}(\mathscr{H}, Z), G \in \mathscr{L}(\mathscr{H}, W)$ where Z, W is another pair of Hilbert spaces.

Definition 1.1: We say that the system generated by the quadruple $(\mathscr{A}, \mathscr{B}, R, G)$ is SECS system iff the following singular estimate condition holds with some $0 \le \gamma < 1$.

$$|Re^{\mathscr{A}t}\mathscr{B}g|_Z + |Ge^{\mathscr{A}t}\mathscr{B}g|_W \le \frac{C}{t^\gamma}|g|_U, \quad 0 < t \le 1. \quad (2)$$

Remark 1.2: Note that when $\mathscr{B}$ is bounded from $U \to \mathscr{H}$, the singular estimate in (2) is automatically satisfied. Moreover, in the case of *analytic* semigroups, the singular estimate holds for all unbounded control operators which are relatively bounded with respect to the generator $\mathscr{A}$. Thus, SECS systems are *proper extensions* of both control systems with bounded controls and analytic systems with relatively bounded controls.

SECS systems enjoy many nice features of standard Riccati Theory. Indeed as shown in [17], [11], [12], [21], Riccati equations are well posed for this class of systems. Moreover, feedback (gain) operators are always well defined as bounded operators which are, however, in the case of Bolza problem, singular at the terminal point T. The above features allow for an effective use of Riccati theory in the construction of control algorithms. Thus, when dealing with concrete applications, the main technical issue is the verification of the validity of the singular estimate (2). This usually involves rather subtle PDE arguments and estimates.

The main technical contribution of this paper is showing that such estimate does hold for the *boundary* control system involving fluid-structure interaction in question. It turns out that in this case the index of singularity γ is equal to $1/4 + \varepsilon$. Once this is accomplished, the abstract framework presented in [21] allows one to conclude well-posedness of the Riccati equations and of the corresponding feedback (gain) operators.

II. A Boundary Control Problem for a Fluid-Structure Interaction System

The mathematical model under consideration is the following. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with an interior region Ω_s and an exterior region Ω_f. The boundary Γ_f is the outer boundary of the domain Ω while Γ_s is the boundary of the region Ω_s which also borders the exterior region Ω_f and where the interaction of the two systems take place. For the purpose of constructing feedback control, we shall consider a linearization of the original Navier-Stokes equation. Let u be a function defined on Ω_f representing the velocity of the fluid while the scalar function p represents the pressure. Additionally, let w and w_t be the displacement and the velocity functions of the solid Ω_s. We also denote by ν the unit outward normal vector with respect to the domain Ω_s. The boundary-interface control is represented by $g \in L_2([0,T]; L_2(\Gamma_s))$ and is active on the boundary Γ_s (cf. [10], [9]). Let u denote the velocity of the fluid and let p denote the pressure. We introduce the Cauchy Polya tensor which describes fluid motion. This is given by

$$\mathscr{T}(u,p) \equiv \varepsilon(u) - pI$$

where $\varepsilon(u) \equiv \nabla u + \nabla^T u$. In addition, we recall the classical stress tensor of elasticity defined by

$$\sigma(w) \equiv 2\kappa\varepsilon(w) + \lambda Tr\varepsilon(u)\delta_{ij}$$

where $\lambda > 0$ and $\kappa > 0$ are the Lamé constants. Given any $g \in L_2([0,T]; L_2(\Gamma_s))$, we are seeking a quadruple (u, w, w_t, p) that satisfy the following system:

$$\begin{cases} u_t - \operatorname{div} \mathscr{T}(u,p) = 0 & \Omega_f \times [0,T] \\ \operatorname{div} \ u = 0 & \Omega_f \times [0,T] \\ w_{tt} - \operatorname{div} \sigma(w) = 0 & \Omega_s \times [0,T] \\ u(0,\cdot) = u_0 & \Omega_f \\ w(0,\cdot) = w_0 & \Omega_s \\ w_t(0,\cdot) = w_1 & \Omega_s \\ w_t = u & \Gamma_s \times [0,T] \\ u = 0 & \Gamma_f \times [0,T] \\ \sigma(w)\cdot\nu = \mathscr{T}(u,p)\cdot\nu + g & \Gamma_s \times [0,T] \end{cases} \quad (3)$$

The control problem we address here is to minimize the terminal velocity of the fluid. This entails solving the following Bolza problem: minimize with respect to all $g \in L_2([0,T]; \Gamma_s)$ the following functional:

$$J(u,w,w_t,g) = \int_0^T |g(s)|^2_{L_2(\Gamma_s)} ds + |u(T,\cdot)|^2_{L_2(\Omega_f)}. \quad (4)$$

Remark 2.1: A distinctive feature of the control problem under consideration is the fact that control functions g actuate on the interface between the two media. This leads to very singular kernels in the integral representation of the gain operator. The latter is the main technical difficulty of the problem under study.

III. Abstract Result

We will embed the fluid structure interaction problem into a more general class of SECS systems for which feedback Riccati theory has been recently developed. We begin by recalling the main result from [21], which provides Riccati theory pertinent to SECS control systems.

Let U, $\mathscr{H}$, Z and W be given Hilbert spaces. U and $\mathscr{H}$ denote, respectively, control and state spaces while Z and W are observation spaces. Let $\mathscr{A}$ be a generator of a c_0 semigroup on $\mathscr{H}$ and let $\mathscr{B}: U \to [\mathscr{D}(A^*)]'$. We consider the dynamics governed by the state equation with a state $y(t) \in \mathscr{H}$ and control $g(t) \in U$:

$$y_t = \mathscr{A}y + \mathscr{B}g; \quad on \ [\mathscr{D}(\mathscr{A}^\star)]'; \quad y(s) = y_s \in \mathscr{H}. \quad (5)$$

The control problem considered is to minimize $J(g,y,s,y_s)$ subject to the state equation (5) over all $g \in L_2([s,T];U)$

$$J(g,y,s,y_s) = \int_s^T |Ry(t)|_Z^2 + |g(t)|_U^2 dt + |Gy(T)|_W^2. \quad (6)$$

Theorem 3.1: Consider the dynamics (5) with the functional cost given by (6) under the following assumptions:

(a) Singular Estimate Control System condition given in (2) is satisfied with some $\gamma < 1$.

(b) $R \in \mathscr{L}(\mathscr{H},Z)$ and the operator: $G \in \mathscr{L}(\mathscr{H},W)$ is such that the operator $GL_T : L_2([0,T];U) \to W$ is closeable.

Then for any initial state $y_s \in \mathscr{H}$ there exists a unique optimal control $g^0(t,s,y_s) \in L_2([s,T];U)$ and optimal trajectory $Ry^0(t,s,y_s) \in L_2([s,T];Z)$ such that $J(g^0,y^0,s,y_s) = min_{g \in L_2([s,T],U)} J(g,y(g),s,y_s)$.

Moreover, there exists a selfadjoint positive operator $P(t) \in \mathscr{L}(\mathscr{H})$ with $t \in [0,T)$ such that $(P(t)x,x)_{\mathscr{H}} = J(g^0,y^0,t,x)$.

In addition, the following properties hold:

(i) The optimal control $g^0(t)$ is continuous on $[s,T)$ but has a singularity of order gamma at the terminal time. More specifically the following estimate holds

$$|g^0(t,s,y_s)|_U \le \frac{C}{(T-t)^\gamma} |y_s|_{\mathscr{H}}, \quad s \le t < T. \quad (7)$$

(ii) The observed optimal output $Ry^0(t)$ is continuous on $[s,T]$ when $\gamma < 1/2$, but has a singularity of order $2\gamma - 1$ at the terminal time when $\gamma \ge 1/2$ and the following estimate holds :

$$|Ry^0(t,s,y_s)|_Z \le \frac{C}{(T-t)^{2\gamma-1}} |y_s|_{\mathscr{H}}, \quad s \le t < T. \quad (8)$$

(iii) $P(t)$ is continuous on $[0,T]$ and $P(t) \in \mathscr{L}(\mathscr{H}, C([0,T];\mathscr{H}))$.

(iv) $\mathscr{B}^* P(t)$ exhibits the following singularity

$$|\mathscr{B}^* P(t)x|_U \le \frac{C|x|_{\mathscr{H}}}{(T-t)^\gamma}, \quad 0 \le t < T. \quad (9)$$

(v)

$$g^0(t,s,y_s) = -\mathscr{B}^* P(t) y^0(t,s,y_s), \quad s \le t < T. \quad (10)$$

(vi) $P(t)$ satisfies the Riccati Differential equation with $t < T$, $x,y \in \mathscr{D}(\mathscr{A})$

$$\langle P_t x, y\rangle_{\mathscr{H}} + \langle \mathscr{A} P(t)x, y\rangle_{\mathscr{H}} + \langle P(t)\mathscr{A}x, y\rangle_{\mathscr{H}} + \langle Rx, Ry\rangle_Z = \langle \mathscr{B}^* P(t)x, \mathscr{B}^* P(t)y\rangle_U. \quad (11)$$

$$lim_{t \to T} P(t)x = G^* G x \quad \forall x \in \mathscr{H}. \quad (12)$$

(vii) When $\gamma < \frac{1}{2}$, the solution of the Riccati equation above is unique within the class of positive and self adjoint operators.

IV. Main Results

In order to formulate our main result on a Riccati feedback synthesis of the fluid-structure interaction system in question, we begin with putting the problem into a semigroup framework.

A. Semigroup Formulation

To this end we introduce the space

$$\mathscr{H} \equiv H \times H^1(\Omega_s) \times L_2(\Omega_s)$$

where

$$H \equiv \{u \in L_2(\Omega_f) : \text{div } u = 0, u \cdot \nu|_{\Gamma_f} = 0\}$$

We also define the space V as

$$V \equiv \{v \in H^1(\Omega_f) : \text{div}\, v = 0, u|_{\Gamma_f} = 0\}$$

In addition we will use the following notation

$$(u,v) = \int_\Omega uv\, d\Omega, \ \langle u,v\rangle = \int_{\Gamma_s} uv\, d\Gamma_s, \ D_i = \frac{\partial}{\partial x_i}$$

$$|u|_{s,D} = |u|_{H^s(D)}, \quad |u|_s = |u|_{s,\Omega}, \quad |u| = |u|_{0,\Omega}.$$

The space V is topologized with respect to an inner product and a corresponding norm given by

$$(u,v)_{1,f} \equiv \int_{\Omega_f} \varepsilon(u)\varepsilon(v) d\Omega_f, \quad |u|^2_{1,\Omega_f} = \int_{\Omega_f} |\varepsilon(u)|^2 d\Omega_f$$

Finally, the energy for the system (3) is defined as

$$E(t) = |u(t)|^2 + |\nabla w(t)|^2 + |w_t(t)|^2. \quad (13)$$

We now introduce the operator $A : V \to V'$ defined by

$$(Au, \phi) = -(\varepsilon(u), \varepsilon(\phi)), \quad \forall \phi \in V. \quad (14)$$

This allows us to consider the operator A (denoted by the same symbol) as acting on H with the domain $D(A) \equiv \{u \in V; |(\nabla u, \nabla \phi)| \le C|\phi|_H\}$. A is self adjoint and generates an analytic semigroup e^{At} on H. In particular

$$|A^\alpha e^{At}|_{\mathscr{L}(H)} \le C t^{-\alpha}, \quad 0 < t \le 1. \quad (15)$$

We also introduce N the Neumann map $N : H^{-1/2}(\Gamma_s) \to V$ defined as

$$Ng = h \Leftrightarrow \{(\varepsilon(u), \varepsilon(\phi))_{\Omega_f} = \langle g, \phi\rangle_{\Gamma_s}, \forall \phi \in V\}$$

Remark 4.1: The PDE interpretation of the map N may be given via the solver of the following Stokes problem.

$$\text{div } \mathscr{T}(h,p) = 0, \quad \text{div}\, h = 0, \text{ in } \Omega_f \quad (16)$$

$$\mathscr{T}(h,p) \cdot \nu = g, \quad \text{on } \Gamma_s \quad (17)$$

for some $p \in L_2(\Omega_f)$.

The following Proposition follows from Lax Milgram and Green's theorem:

Proposition 4.2: 1) N is continuous from $H^{-1/2}(\Gamma_s) \to V \subset H^1(\Omega_f)$.

2) $N^* A u = -u|_{\Gamma_s}$ for all $u \in V$ where the adjoint is computed with respect to L_2 topologies.

The weak formulation of (22) is

$$\begin{cases} (u_t, \phi) - (Au, \phi) + \langle \sigma(w) \cdot \nu, \phi\rangle + \langle g, \phi\rangle = 0 \\ (w_{tt}, \psi) - (\text{div}\, \sigma(w), \psi) = 0 \\ w_t|_{\Gamma_s} = u|_{\Gamma_s} \end{cases} \quad (18)$$

for all test functions $\phi \in V$ and $\psi \in H^1(\Omega_s)$ where $\text{div}\, \sigma(w)$ should be understood in a weak sense as $(\text{div}\, \sigma(w), \psi) =$

$-(\sigma(w),\varepsilon(\psi))+\langle\sigma(w)\cdot\nu,\psi\rangle$. The uncontrolled system (i.e. $g=0$) can be expressed as

$$y_t = \mathscr{A}y, \quad y_0 \in \mathscr{H}$$

where $y=(u,w,w_t)$ and

$$\mathscr{A} = \begin{pmatrix} A & AN\sigma()\cdot\nu & 0 \\ 0 & 0 & I \\ 0 & \text{div }\sigma() & 0 \end{pmatrix} \tag{19}$$

$\mathscr{D}(\mathscr{A}) = \{[u,w,z] \in \mathscr{H}, u \in V, Au + AN\sigma(w)\cdot\nu \in H; z \in H^1(\Omega_s),\ \text{div}\,\sigma(w) \in L_2(\Omega_s); z|_{\Gamma s} = u|_{\Gamma_s}\}$.

The operator $\mathscr{A}$ generates a c_0 semigroup of contractions on $\mathscr{H}$. This follows from a more general result proven in Proposition 3.1 of [3].

B. The Control Operator

We introduce the operator B defined from $L_2(\Gamma_s)\to V'$ by

$$(Bg,\phi) = -\langle g,\phi\rangle_{L_2(\Gamma_s)} = \langle g, N^*A\phi\rangle, \forall\phi\in V. \tag{20}$$

Thus, B is an unbounded operator from $L_2(\Gamma_s)\to H$ but bounded from $L_2(\Gamma_s)\to V'$. From (20) it is clear that $B=AN$ where N is the Neumann map.

We then define the control operator $\mathscr{B}: L_2(\Gamma_s)\to V'\times H^1(\Omega_s)\times L_2(\Omega_s)$ to be

$$\mathscr{B} = [AN,0,0]^T. \tag{21}$$

C. Abstract Formulation of the Control System

With the control operator introduced above, we can rewrite the original control problem as

$$y_t = \mathscr{A}y + \mathscr{B}g, \quad y_0\in\mathscr{H}. \tag{22}$$

with $\mathscr{A}$ and $\mathscr{B}$ defined in (19) and (21) respectively.

D. Main Result

Once the conditions are verified, applying Theorem 3.1 to the system in (3) subject to the control problem in (4), we obtain the main theorem of this paper

Theorem 4.3: In reference to the model in (3) and the control problem in (4), we have

1) For every initial condition $y_0 = [u_0,w_0,w_1] \in \mathscr{H}$, there exists a unique optimal control $g^0(t,\cdot) \in L_2([0,T];\Gamma_s)$ and observed optimal state $y^0(t,\cdot) = [u^0(t,\cdot),w^0(t,\cdot),w^0_t(t,\cdot)] \in L_2([0,T];H\times H^{1-\varepsilon}(\Omega_s)\times H^{-\varepsilon}(\Omega_s))$ such that $J(g^0,y^0) = \min_{g\in L_2([0,T];\Gamma_s)} J(g,y)$.
2) Moreover, there exists a positive selfadjoint $P(t) \in \mathscr{L}(\mathscr{H})$ such that $J(g^0,y^0) = (P(0)y_0,y_0)_{\mathscr{H}}$. In addition, all the properties listed in Theorem 3.1 hold with $\gamma = 1/4+\varepsilon$ with the operators $\mathscr{A}$ and $\mathscr{B}$ defined in (19) and (21). In particular, the optimal control $g^0(t,\cdot) = -\mathscr{B}^\star P(t)y^0(t)$ and the following singular estimate holds

$$|\mathscr{B}^* P(t)x|_{L_2(\Gamma_s)} \le C\frac{|x|_{\mathscr{H}}}{t^{1/4+\varepsilon}}$$

   In particular, if $P(t)x = [p_1,p_2,p_3](t)$ and $P(t)y = [\hat{p}_1,\hat{p}_2,\hat{p}_3](t)$ for given $x=[x_1,x_2,x_3]$ and $y=[y_1,y_2,y_3]\in\mathscr{D}(\mathscr{A})$ as defined in (19), then $[p_1(t),p_2(t),p_3(t)]$ and $[\hat{p}_1(t),\hat{p}_2(t),\hat{p}_3(t)]$ satisfy the Differential Riccati equation for $t\in[0,T)$:

$$\begin{aligned}&(\dot{p_1},y_1)_{\Omega_f} + (\varepsilon(\dot{p_2}),\sigma(y_2))_{\Omega_s} + (\dot{p_3},y_3)_{\Omega_s}\\ &-(\varepsilon(p_1),\varepsilon(y_1))_{\Omega_f} + \langle\sigma(p_2)\cdot\nu,y_1\rangle + \langle\sigma(p_3)\cdot\nu,y_3\rangle\\ &+(\sigma(p_3),\varepsilon(y_2))_{\Omega_s} - (\sigma(p_2),\varepsilon(y_3))_{\Omega_s}\\ &-(\sigma(x_1),\varepsilon(\hat{p}_1))_{\Omega_f} + \langle\sigma(x_2)\cdot\nu,\hat{p}_1\rangle + \langle\sigma(x_2)\cdot\nu,\hat{p}_3\rangle\\ &+(\sigma(x_3),\varepsilon(\hat{p}_2))_{\Omega_s} - (\sigma(x_2),\varepsilon(\hat{p}_3))_{\Omega_s} = \langle p_1,\hat{p}_1\rangle \end{aligned} \tag{23}$$

3) The variational formulation given in (23) leads, after projection on finite dimensional subspaces , to an effective computational algorithm for $P(t)$ (cf. [18]).

Remark 4.4: In fact, the solutions to this particular system (3), u,w,w_t are in $C([0,T];H\times H^1(\Omega_s)\times L_2(\Omega_s))$, given any $g\in L_2([0,T];L_2(\Gamma_s))$ which is more regular than guaranteed by the abstract theorem 3.1.

V. Outline of the Proof

Our aim is to apply Theorem 3.1 to the model described above. To this end, we need to verify the assumptions imposed on the control operator. The most critical and technically involved is the Singular Estimate assumption (2). We shall show that this estimate holds with the parameter $\gamma = 1/4+\varepsilon$. First of all, we must verify that $\mathscr{B}$ is bounded from $U\to[\mathscr{D}(\mathscr{A}^\star)]'$ which follows from the proposition

Proposition 5.1: $R(\lambda,\mathscr{A})\mathscr{B}\in\mathscr{L}(\mathscr{H})$, where $\lambda>0$.
and this can be verified directly.

Following the general form of the control problem for singular estimate control systems (4) as given in [21], the observation operator $R:\mathscr{H}\to Z$ is just zero for the control problem under consideration.

For the main Theorem to follow, one must verify the closability assumption on the observation operator $G=[I,0,0]$ when applied to the control to state map stated in Theorem 3.1. We first define the control to final state map $L_T: L_2([0,T];L_2(\Gamma_s))\to\mathscr{H}$ by:

$$L_T g = \int_0^T e^{\mathscr{A}(T-s)}\mathscr{B}g(s)ds. \tag{24}$$

Proposition 5.2: The operator GL_T is closeable from $L_2([0,T];U)$ into $\mathscr{H}$.
This Proposition follows from the boundedness of $A^{-1}GL_T$, where the latter can be verified computationally.

A. Singular Estimate Property

To establish Theorem 4.3 as a result of an application of Theorem 3.1 to the abstract system (22), we still need to establish the singular estimate condition (a). The form of the estimate below allows for an application of Theorem 3.1 with any bounded observation operator $R:\mathscr{H}\to Z\equiv\mathscr{H}_{-\alpha}$ and $\alpha>0$ and thus establishing Theorem 4.3 as a consequence. Here $\mathscr{H}_{-\alpha}\equiv H\times H^{1-\alpha}(\Omega_s)\times H^{-\alpha}(\Omega_s)$

Theorem 5.3: The semigroup $e^{\mathscr{A}t}$ generated by $\mathscr{A}$ when applied to the control action $\mathscr{B}$ satisfy the following singular estimate for every $g\in L_2(\Gamma_s)$ and $t\le T_0$, and $\alpha<1/4$:

$$|e^{\mathscr{A}t}\mathscr{B}g|_{\mathscr{H}_{-\alpha}} \le \frac{C}{t^{1/4+\varepsilon}}|g|_{L_2(\Gamma_s)}. \tag{25}$$

We note that the estimate in (2), in our case, automatically implies

$$|Ge^{\mathscr{A}t}\mathscr{B}g|_{\mathscr{H}} \leq \frac{C}{t^{1/4+\varepsilon}}|g|_{L_2(\Gamma_s)}$$

The estimate in Theorem 5.3 along with Proposition 5.1, when applied to Theorem 3.1 lead to the conclusions stated in Theorem 4.3. The rest of the manuscript will be devoted for the proof of Theorem 5.3.

B. Preliminary Results

The main ingredients in the proof of the singular estimate property are the following lemmas:

Lemma 5.4: Let $w_0, w_1 \in H^{\alpha+1}(\Omega_s)\times H^{\alpha}(\Omega_s)$ and let $f \in L_2([0,T];H^{1/2}(\Gamma_s))$ and w be the solution to the wave equation

$$\begin{cases} w_{tt} - \operatorname{div}\sigma(w) = 0 & \text{in } \Omega_s\times[0,T] \\ w(0,\cdot) = w_0 \text{ in } , w_t(0,\cdot) = w_1 & \text{in } \Omega_s \\ w_t = f & \text{on } \Gamma_s\times[0,T]. \end{cases} \quad (26)$$

Then w can be decomposed into $w_1 + w_2$ such that $\sigma(w_1)\cdot\nu \in C([0,T];H^{-1/2}(\Gamma_s))$ and $\sigma(w_2)\cdot\nu \in L_2(\Sigma_s) = L_2([0,T]\times\Gamma_s)$. If further $f \in H^{\alpha}(\Sigma_s)$ then $\sigma(w_2)\cdot\nu \in H^{\alpha}(\Sigma_s)$. Moreover, we have the following estimates

$$|\sigma(w_1)(t)\cdot\nu|^2_{H^{-1/2}(\Gamma_s)} \leq K[|w_0|^2_{1,s} + |w_1|^2_{0,s} + \int_0^T |f|^2_{H^{1/2}(\Gamma_s)}]. \quad (27)$$

$$|\sigma(w_2)\cdot\nu|^2_{H^{\alpha}(\Sigma_s)} \leq K[|w_0|^2_{1+\alpha,s} + |w_1|^2_{\alpha,s} + |f|^2_{H^{\alpha}(\Sigma_s)}]. \quad (28)$$

Proof: This is a hidden regularity result obtained via a microlocalization technique (cf. [3]). ■

We next prove a regularity result for u on the boundary.

Lemma 5.5: Consider the uncontrolled system (18). If in addition the initial condition $[u_0, w_0, w_1] \in L_2(\Omega_f)\times H^{1+\alpha}(\Omega_s)\times H^{\alpha}(\Omega_s)$ for $0<\alpha<1/4$, then there exists T_0 such that $u|_{\Gamma_s} \in H^{\alpha}(\Sigma_s)$ for $T \leq T_0$ and the following estimate holds

$$|u|^2_{H^{\alpha}(\Sigma_s)} \leq C(|u_0|^2_{0,f} + |w_0|^2_{1+\alpha,s} + |w_1|^2_{\alpha,s}). \quad (29)$$

Proof: From (22), we use the variation of parameters formula to express the solution u as

$$u(t,\cdot) = e^{At}u_0 + \int_0^t e^{A(t-s)}AN[\sigma(w_1)+\sigma(w_2)](s,\cdot)\cdot\nu ds$$

Here, we used Lemma 5.4 to decompose $\sigma(w)\cdot\nu$ into $\sigma(w_1)\cdot\ \nu$ and $\sigma(w_2)\cdot\ \nu$. Now, by Proposition 4.2, $N^{\star}A$ acts as the restriction on the boundary Γ_s so we express u on Γ_s and estimate each of these terms separately, letting

$$U_1 = N^{\star}Ae^{At}u_0. \quad (30)$$

$$U_2 = \int_0^t N^{\star}Ae^{A(t-s)}AN\sigma(w_1)(s,\cdot)\cdot\nu ds. \quad (31)$$

$$U_3 = \int_0^t N^{\star}Ae^{A(t-s)}AN\sigma(w_2)(s,\cdot)\cdot\nu ds. \quad (32)$$

Step 1: *Estimating* U_1- First define the space

$$H^{\alpha}(\Sigma_s) \equiv L_2([0,T];H^{\alpha}(\Gamma_s))\bigcap H^{\alpha}([0,T];L_2(\Gamma_s))$$

Note U_1 is the restriction to Γ_s of the solution generated by the analytic semigroup e^{At} which means $U_1 \in L_2([0,T];H^{1/2}(\Gamma_s))\bigcap H^{1/4}([0,T];L_2(\Gamma_s))$ given $u_0 \in H$, a well known result in parabolic theory. Thus,

$$|U_1|^2_{H^{\alpha}(\Sigma_s)} \leq K_{Tj}[|u_0|^2_{0,f} + |w_0|^2_{1+\alpha,s} + |w_1|^2_{\alpha,s}]$$

Step 2: *Estimating* U_2- Note U_2 is the restriction on the boundary of h where h solves the parabolic problem $\frac{d}{dt}h = Ah + AN\sigma(w_1)\cdot\nu$ with $h(0,\cdot) = 0$ and since $\sigma(w_1)\cdot\nu \in C([0,T];H^{-1/2}(\Gamma_s))$, a well known result in parabolic theory gives the trace $U_2 \in H^{1/2}(\Sigma_s)$. Hence

$$|U_2|^2_{H^{\alpha}(\Sigma_s)} \leq K_T[|u_0|^2_{0,f} + |w_0|^2_{1,s} + |w_1|^2_{0,s}]$$

See [20] for detailed estimates.

Step 3: *Estimating* U_3- We first observe that U_3 is the restriction on Γ_s of a function h solving the "abstract" parabolic problem

$$\frac{d}{dt}h = Ah + AN\sigma(w_2)\cdot\nu \quad (33)$$

$$h(0,\cdot) = 0 \quad (34)$$

Following the existing results in parabolic theory [14], and identifying $D(A^{\theta}) \sim H^{2\theta}(\Omega), \theta < 3/4$ we have if $\sigma(w_2)\cdot\nu \in H^{\alpha-\varepsilon}(\Sigma_s) \subset H^{\alpha-\varepsilon,\alpha/2-\varepsilon/2}(\Sigma_s)$ then $h \in H^{\alpha+3/2-\varepsilon,\alpha/2+3/4-\varepsilon/2}(\Omega_f\times[0,T])$ and consequently the trace $U_3 = h|_{\Gamma_s} \in H^{\alpha+1-\varepsilon,\alpha/2+1/2-\varepsilon/2}(\Sigma_s) \subset H^{\alpha}(\Sigma_s)$, where the last inclusion follows since $\alpha<1$ and thus $\alpha/2+1/2-\varepsilon/2 > \alpha$. Therefore:

$$|U_3|^2_{H^{\alpha}(\Sigma_s)} \leq K|\sigma(w_2)\cdot\nu|^2_{H^{\alpha-\varepsilon}(\Sigma_s)}$$

On the other hand, estimate (28) with α replaced by $\alpha-\varepsilon$ and f replaced by $u|_{\Gamma_s}$ implies

$$|\sigma(w_2)\cdot\nu|^2_{H^{\alpha-\varepsilon}(\Sigma_s)} \leq K[|u|^2_{H^{\alpha-\varepsilon}(\Sigma_s)} + |w_0|^2_{1+\alpha,s} + |w_1|^2_{\alpha,s}]$$

$$\leq K[\int_0^T |D_t^{\alpha-\varepsilon}u|^2_{L_2(\Gamma_s)}dt + |w_0|^2_{1+\alpha,s} + |w_1|^2_{\alpha,s} + |u_0|^2_{0,f}]$$

Notice in the last inequality, we used the a priori estimate for the system (3) $|u|^2_{L_2([0,T];H^{1/2}(\Gamma_s)} \leq CE(0)$ which comes from energy estimates [3]. Let $q = 1/(1-2\varepsilon)$ and its conjugate $p = 1/(1+2\varepsilon)$ then

$$|\sigma(w_2)\cdot\nu|^2_{H^{\alpha-\varepsilon}(\Sigma_s)} \leq K[\int_0^T |D_t^{\alpha-\varepsilon}u|^{2q}_{L_2(\Gamma_s)}dt]^{1/q}T^{1/p} + |y(0)|^2_{\mathscr{H}_\alpha}$$

$$\leq K[|D_t^{\alpha-\varepsilon}u|^2_{H^{\varepsilon}([0,T];L_2(\Gamma_s))}T^{1/p} + |y(0)|^2_{\mathscr{H}_\alpha}]$$

$$\leq K[|u|^2_{H^{\alpha}(\Sigma_s)}T^{1/p} + |y(0)|^2_{\mathscr{H}_\alpha}]$$

Note that in the above estimate we used the Sobolev embedding result $H^{\varepsilon}([0,T]) \subset L_{2q}([0,T])$ where again $2q = 1/(1/2-\varepsilon)$. Thus

$$|U_3|^2_{H^{\alpha}(\Sigma_s)} \leq K[|u|^2_{H^{\alpha}(\Sigma_s)}T^{1+2\varepsilon} + |y(0)|^2_{\mathscr{H}_\alpha}]. \quad (35)$$

Collecting the estimates for U_1, U_2, U_3 we obtain:

$$|u|^2_{H^{\alpha}(\Sigma_s)} \leq K[|u_0|^2_{0,f} + |y(0)|^2_{\mathscr{H}_\alpha} + T^{1+2\varepsilon}|u|^2_{H^{\alpha}(\Sigma_s)}]$$

We now choose $T = T_0$ so that $KT_0^{1+2\varepsilon} < 1$ and absorb the last term into the left hand side of the inequality to obtain the desired result. ■

C. Proof of Theorem 5.3

It is equivalent to prove the following estimate for every $y_0 = [u_0, w_0, w_1] \in H \times H^{1+\alpha}(\Omega_s) \times H^{\alpha}(\Omega_s)$:

$$|\mathscr{B}^{\star} e^{\mathscr{A}^{\star} t} y_0|_U \leq \frac{C}{t^{1/4+\varepsilon}} |y_0|_{H\times H^{1+\alpha}(\Omega_s)\times H^{\alpha}(\Omega_s)}. \tag{36}$$

This term represents the solution $[\hat{u}, \hat{w}, \hat{w}_t]$ to the adjoint system of (18), when the initial condition is $[u_0, w_0, w_1] \in H \times H^{1+\alpha}(\Omega_s) \times H^{\alpha}(\Omega_s)$. Here, the semigroup $e^{\mathscr{A}^{\star} t}$ gives the solution to the equation $y_t = \mathscr{A}^{\star} y$, expressed below

$$\begin{cases} (\hat{u}_t, \phi) = -(\varepsilon(\hat{u}), \varepsilon(\phi)) + \langle \sigma(\hat{w}) \cdot \nu, \phi \rangle \\ (\hat{w}_{tt}, \psi) = (\operatorname{div} \sigma(\hat{w}), \psi) \\ \hat{w}_t|_{\Gamma_s} = -\hat{u}|_{\Gamma_s} \end{cases} \tag{37}$$

The system above is equivalent to the system in (18). Moreover, $\mathscr{A}^{\star}$ also generates a c_0 semigroup of contractions on $\mathscr{H}$ using the same argument as that used to show that $\mathscr{A}$ generates a c_0 semigroup , (cf. [3]). Hence, we expect the same regularity for the solution $\hat{y} = [\hat{u}, \hat{w}, \hat{z}]$ to the adjoint system. We also compute $\mathscr{B}^{\star} e^{\mathscr{A}^{\star} t}$ obtaining

$$\mathscr{B}^{\star} e^{\mathscr{A}^{\star} t} y_0 = N^{\star} A \hat{u}|_{\Gamma_s} = \hat{u}|_{\Gamma_s}$$

It is sufficient then to estimate the norm of $u(t)|_{\Gamma_s}$ in $L_2(\Gamma_s)$ in our original system (18), since $\hat{u}(t)|_{\Gamma_s}$ has the same regularity. As in Lemma 5.5

$$u(t)|_{\Gamma_s} = U_1(t) + U_2(t) + U_3(t)$$

The term U_1 is precisely the source of the singular estimate

$$\begin{aligned} |U_1(t)|_{L_2(\Gamma_s)} &= |N^{\star} A^{3/4-\varepsilon} e^{At} A^{1/4+\varepsilon} u_0| \\ &\leq \frac{C}{t^{1/4+\varepsilon}} |y_0|_{H\times H^{1+\alpha}(\Omega_s)\times H^{\alpha}(\Omega_s)} \end{aligned}$$

Estimating U_2 and U_3: Using properties of A and the Neumann map N and the estimates from Lemmas 5.4 and 5.5 we have

$$\begin{aligned} |U_2(t)|_{L_2(\Gamma_s)} &\leq \int_0^t \frac{C}{(t-s)^{3/4+\varepsilon}} |A^{1/2} N \sigma(w_1)(s,\cdot) \cdot \nu|_{L_2(\Omega_f)} ds \\ &\leq C t^{1/4-\varepsilon} |\sigma(w_1) \cdot \nu|_{C([0,T];H^{-1/2}(\Gamma_s))} \\ &\leq \frac{C_T}{t^{1/4+\varepsilon}} |y_0|_{H\times H^{1+\alpha}(\Omega_s)\times H^{\alpha}(\Omega_s)} \end{aligned}$$

Since $H^{\alpha+1,\alpha/2+1/2}(\Gamma_s \times [0,T]) \subset C([0,T];L_2(\Gamma_s))$ by Sobolev embedding theorems in one dimension and U_3 is the restriction on the boundary Γ_s of h which solves problem (33) then U_3 satisfies the following estimate

$$\begin{aligned} |U_3(t)|_{L_2(\Gamma_s)} &\leq |U_3|_{H^{\alpha+1,\alpha/2+1/2}(\Gamma_s\times[0,T])} \\ \leq K|\sigma(w_2)\cdot\nu|_{H^{\alpha,\alpha/2}(\Gamma_s\times[0,T])} &\leq K|\sigma(w_2)\cdot\nu|_{H^{\alpha}(\Sigma_s)} \end{aligned}$$

We next apply the estimate (28) from Lemma 5.4 and Lemma 5.5 to obtain

$$|U_3(t)|_{L_2(\Gamma_s)} \leq K_T |y_0|_{H\times H^{1+\alpha}(\Omega_s)\times H^{\alpha}(\Omega_s)}$$

Collecting the estimates of U_1, U_2, U_3 we get

$$|\mathscr{B}^{\star} e^{\mathscr{A}^{\star} t} y_0|_{L_2(\Gamma_s)} = |u(t)|_{L_2(\Gamma_s)} \leq \frac{C}{t^{1/4+\varepsilon}} |y_0|_{\mathscr{H}_\alpha}$$

as desired.

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