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Sensitivity Relations for Optimal Control Problems with State Constraints

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***Abstract*— In optimal control theory, it is well known that the costate arc and the associated maximized Hamiltonian function can be interpreted in terms of gradients of the value function, evaluated along the optimal state trajectory. Such relations have been referred to as 'sensitivity relations' in the literature. In this paper, we announce new sensitivity relations for state constrained optimal control problems. For the class of optimal control problems considered there is no guarantee that the co-state arc is unique; a key feature of the results is that they assert 'some' choice of co-state arc can be made, for which the sensitivity relations are valid. The proof technique is to introduce a new optimal control problem that possesses a richer set of control variables than the original problem. The introduction of the additional control variables in effect enlarges the class of variations with respect to which the state trajectory under consideration is a minimizer; the extra information obtained is precisely the desired set of sensitivity relations.**

I. INTRODUCTION

We consider the following optimal control problem with state constraints:

$$(P_{S,x_0})\begin{cases} \text{Minimize } g(x(T)) \\ \text{over arcs } x(.) \in W^{1,1}([S,T];R^n) \\ \text{and measurable functions } u(.) : [S,T] \to R^m \text{ s.t.} \\ \dot{x}(t) \;=\; f(t,x(t),u(t)) \quad \text{a.e. } t\in[S,T] \\ u(t) \;\in\; U(t) \quad \text{a.e. } t \in [S,T] \\ x(t) \;\in\; A(t) \quad \text{for all } t\in[S,T] \\ x(S) \;=\; x_0 \quad . \end{cases}$$

The data for this problem comprise: an interval $[S,T]$, integers n and m, functions $g : R^n \to R$ and $f : R \times R^n \times R^m \to R^n$, a vector $x_0 \in R^n$ and multifunctions $U : [S,T] \rightsquigarrow R^m$ and $A : [S,T] \rightsquigarrow R^n$. It is assumed that the time-dependent 'state constraint' set $A(t)$ has the functional inequality representation

$$A(t) \;=\; \{x \,|\, h(t,x) \le 0\}\;, \tag{1}$$

for some integer r and some function $h : R \times R^n \to R$. Let $(\bar{x},\bar{u})$ be a minimizer for (P_{S,x_0}).

Given any $(t,x) \in [S,T] \times R^n$, we denote by $(P_{t,x})$ the modification of (P_{S,x_0}) in which the 'initial data' (t,x) replaces (S,x_0). We refer to a measurable function $u : [t,T] \to R^m$ that satisfies $u(s) \in U(s)$, a.e. as a *control function* on $[t,T]$. A pair $(x(.),u(.))$ comprising an absolutely continuous R^n valued function $x(.)$ and a control function $u(.)$ on $[t,T]$ that satisfy $\dot{x}(s) = f(s,x(s),u(s))$ a.e. is called a *process* on $[t,T]$. The first component of a process is called a *state trajectory*. A process on $[t,T]$ that satisfies the constraints of problem $(P_{t,x})$ is said to be an *admissible (or feasible) process* for $(P_{t,x})$. A *minimizer* is an admissible process that achieves the infimum value of the cost over all admissible processes. A process $(\bar{x}(.),\bar{u}(.))$ is said to be a *strong local minimizer* if it is a minimizer under the additional constraint $||x(.) - \bar{x}(.)||_{L^\infty} < \epsilon$ on admissible processes $(x(.),u(.))$.

The *value function* $V : [S,T] \times R^n \to R \cup \{+\infty\}$ is the function

$$V(t,x) \;=\; \inf\,(P_{t,x})\;,$$

where the right hand side is interpreted as the infimum cost in the case that admissible processes for $(P_{t,x})$ exist and as $+\infty$ otherwise. (In particular, the value function takes the value $+\infty$ at points (t,x) such that $x \notin A(t)$.)
Denote by $H : [S,T]\times R^n \times R^n \times R^m \to R$ the Hamiltonian function

$$H(t,x,p,u) \;=\; p \cdot f(t,x,u)$$

and by $\mathcal{H} : [S,T] \times R^n \times R^n \to R$ the maximized Hamiltonian

$$\mathcal{H}(t,x,p) \;=\; \sup_{u\in U(t)} H(t,x,p,u)\;.$$

When the state constraint is absent ($A(t) = R^n$), f and g are continuously differentiable, $f(t,x,u)$ has at most linear growth w.r.t. the x variable (uniformly over $t \in [S,T]$, $u \in U(t)$, when V is continuously differentiable on $(S,T)\times R^n$ and when $\bar{u}$ is piecewise continuous, it is well-known that V is related to the costate arc, $p(.)$ appearing in the Maximum Principle, and the maximized Hamiltonian evaluated along $\bar{x}(.)$ and $p(.)$ according to:

$$(-\mathcal{H}(t,\bar{x}(t),\; p(t)),p(t)) \;=\; \nabla V(t,\bar{x}(t)) \quad \text{a.e. } t\in[S,T]\;.$$

These relations follow, formally at least, from the Hamilton Jacobi equation (smooth form) when we identify $t \to V_x(t,\bar{x}(t))$ with the co-state arc $p(.)$. They date from the early days of optimal control theory and have been described as providing a 'sensitivity' interpretation of the Maximum Principle Lagrange 'multipliers'. (See, e.g., [2]). They are of interest because they tell us that the Pontryagin Maximum Principle can be used, not only to solve optimal control problems, but to supply first order information about how the minimum cost is affected by perturbations to the

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problem data.

For many optimal control problems of interest, the value function fails to be continuously differentiable. Under broad, unrestrictive conditions however the value function can be shown to be a (possibly discontinuous) lower semi-continuous function. So, if the sensitivity relations are to be validated in conditions of any generality, they must be couched in terms of 'nonsmooth' subdifferentials, for example

$$(\mathcal{H}(t,\bar{x}(t),p(t)),-p(t)) \;\in\; \text{co}\,\partial V(t,\bar{x}(t)) \;\text{ a.e. } t\in[S,T]\,. \tag{2}$$

(∂V denotes the subdifferential of V; see below.) If we no longer suppose that f and g are continuously differentiable in the x variable, then the co-state inclusion may have multiple solutions satisfying the maximization of the Hamiltonian condition and the transversality condition. In these circumstances it is natural to ask whether there exists *some* co-state arc that satisfies (2).

In the absence of state constraints, the validity of the partial sensitivity relation (sensitivity only w.r.t. to the x variable)

$$-p(t) \;\in\; \text{co}\;\partial_x V(t,\bar{x}(t)) \quad \text{a.e. } t\in[S,T]\,. \tag{3}$$

was proved by Clarke and Vinter [8]. The full sensitivity relation (2) was proved by Vinter in [17].

Examples are available (see [16]) showing that, in some cases, there are a number of possible choices of co-state arcs associated with (P_{S,x_0}), but some of them fail to satisfy (3).

In this paper we investigate the validity of sensitivity relations for optimal control problems with *state constraints*. The costate arc $p(.)$ involved in these relations, associated with the minimizing process $(\bar{x},\bar{u})$ of interest, is that which arises in the following version of the Maximum Principle for state constrained optimal control problems:

State Constrained Maximum Principle : There exists a function of bounded variation $p(.):[S,T]\to R^n$ and a Radon measure μ on the (Borel subsets of) $[S,T]$ such that

$$-dp(t) \;\in\; \text{co}\,\partial_x H(t,\bar{x}(t),p(t),\bar{u}(t))dt$$
$$-\nabla_x h(t,\bar{x}(t))\mu(dt) \qquad \text{on } t\in[S,T] \tag{4}$$
$$H(t,\bar{x}(t),p(t),\bar{u}(t)) \;=\; \mathcal{H}(t,\bar{x}(t),p(t)) \tag{5}$$
$$\text{supp}\,\{\mu\}\subset\{t\,|\,h(t,\bar{x}(t))=0\} \tag{6}$$
$$-p(T)\in\partial g(\bar{x}(T))\,. \tag{7}$$

Here (4) is interpreted as an integral equation: there exists an integrable function $\xi:[S,T]\to R^n$ such that

$$\xi(t) \;\in\; \text{co}\,\partial_x H(t,\bar{x}(t),p(t),\bar{u}(t)) \qquad \text{a.e. } t\in[S,T]$$

$$-p(t) \;=\; -p(S)+\int_{[S,t]}\xi(s)ds$$
$$-\int_{[S,t]}\nabla_x h(s,\bar{x}(s))\mu(ds) \;\;\forall t\in(S,T]\,.$$

The form of the above optimality conditions is that earlier employed (in the smooth case) by Ioffe and Tihomirov [11]. We refer to the function p as the true co-state arc. The optimality condition is more frequently expressed in terms of an absolutely continuous 'pseudo co-state' arc q satisfying $q(S)=p(S)$ and

$$q(t) \;:=\; p(t)-\int_{[S,t]}\nabla_x h(s,\bar{x}(s))\mu(ds) \quad \text{if } t\in(S,T]\,,$$

because q is absolutely continuous and satisfies a simple differential inclusion, namely

$$-\dot{q}(t) \;\in\; \text{co}\,\partial_x\mathcal{H}\Big(t,\bar{x}(t),q(t)$$
$$+\int_{[S,t]}\nabla_x h(s,\bar{x}(s))\mu(ds)\Big) \;\text{ a.e. } t\in[S,T]\,.$$

Maximum Principles expressed in terms of the true co-state $p(.)$ or the pseudo co-state arc $q(.)$ convey the same information about optimal controls. However, the true co-state should be the subject of sensitivity analysis because, according to formal calculations, it can, unlike q, be interpreted as the Lagrange multiplier associated with the dynamic constraint $\dot{x}=f(t,x,u)$ in (P_{t,x_0}). (See the discussion in ([16], p. 321 et seq.))

The main result of the paper is the validity of the sensitivity relations (2) and (3), under precisely stated, unrestrictive conditions, for the existence of *some* co-state arc $p(.)$ in the state constrained Maximum Principle which satisfies the full nonsmooth sensitivity relations (2).

Cernea and Frankowska [6] have earlier investigated sensitivity relations for state constrained optimal control problems, as part of a broad study which addresses also the question of when the state constrainted Maximum Principle is valid in normal form. Different hypotheses are imposed on the state constraint sets $A(t)$ and different kinds of subgradients are employed to those of this paper. However the principal differences are as follows:

(a): In [6] it is shown that, for each $t\in[S,T]$, a sensitivity relation of the form

$$-p'(t;t) \;\in\; \text{co}\;\partial_x V(t,\bar{x}(t)) \text{ for all } t\in[S,T]\,. \tag{8}$$

where $p'(s;t)$, $t\le s\le T$, is a co-state arc associated with the restriction of $(\bar{x},\bar{u})$ to $[t,T]$, regarded as a minimizing process for $(P_{t,\bar{x}(t)})$. Only in special cases (smooth data and no state constraints, for example) we can guarantee that the left endpoints of these costate arcs define a costate arc for the problem interest (P_{S,x_0}). This paper imposes no such conditions.

(b): [6] provides a version of the partial sensitivity relation (involving partial subgradients of V w.r.t. the x variable). The main theorem of the paper asserts a full sensitivity relation (involving subgradients of V w.r.t. both t and x variables).

Finally some definition and points of notation. In Euclidean space, the length of a vector x is denoted by $|x|$, and the closed unit ball $\{x \,|\, |x| \leq 1\}$ by B. The graph of a multifunction $U(.) : [S,T] \rightsquigarrow R^m$ is denoted by Graph $U(.)$. Given a set $D \subset R^k$, $\text{co}\, D$ denotes the convex hull of D.

Take an open set $\mathcal{O} \subset R^k$, a lower semicontinuous function $f : \mathcal{O} \to R \cup \{+\infty\}$ and a point $\bar{x} \in \mathcal{O}$ such that $\bar{x} \in \text{dom}\, f := \{x \,|,\ f(x) < +\infty\}$. The *subdifferential* of f at $\bar{x}$ is

$$\partial f(\bar{x}) \;=\; \Big\{ \xi \,|\, \exists\, \xi_i \to \xi \text{ and } x_i \xrightarrow{\text{dom}\, f} \bar{x} \text{ such that,} \\ \limsup_{x \to x_i} \frac{\xi_i \cdot (x - x_i) - f(x) + f(x_i)}{|x - x_i|} \leq 0 \quad \forall i \Big\} .$$

For background on nonsmooth analysis and subdifferentials, see, e.g. [1], [7], [12] or [16].

II. SENSITIVITY RELATIONS

In this section we state the main results of the paper, interpreting co-state arcs for the state constrained Maximum Principle as subgradients of the value function. The following notation will be also required:

$$h^+(t,x) \;=\; \max\{h(t,x); 0\}.$$

We shall invoke the following hypotheses:

H1: For each $(x,u) \in R^n \times R^m$ $f(.,x,u)$ is measurable, and for each $t \in [S,T]$ $f(t,.,.)$ is continuous. The multifunction $t \rightsquigarrow U(t)$ has values closed sets and is Borel measurable. g is locally Lipschitz continuous. $h(.,.)$ is of class C^{1+} (i.e. everywhere Frechet differentiable with locally Lipschitz continuous derivatives).

H2: There exist $c > 0$ and $k_f(.) \in L^1([S,T];R)$ such that for all $t \in [S,T]$, $x, x' \in R^n$ and $u \in U(t)$ we have

$$|f(t,x,u)| \;\leq\; c(1 + |x|) \\ |f(t,x,u) - f(t,x',u)| \;\leq\; k_f(t)|x - x'|.$$

H3: For any $r > 0$ there exist $\gamma,\ \rho > 0$ such that for all $(t,x) \in [S,T] \times rB$ at which $|h(t,x)| \leq \rho$ we have

$$\min_{u \in U(t)} \nabla h(t,x) \cdot (1, f(t,x,u)) \leq -\gamma.$$

Theorem 2.1: Let $(\bar{x}, \bar{u})$ be a minimizer for problem (P_{S,x_0}). Assume (H1)-(H3). Then there exists a function of bounded variation $p(.) : [S,T] \to R$, right continuous on (S,T), and a Radon measure μ on the (Borel subsets of) $[S,T]$ such that

(i): the conditions (4)- (7) of the state constrained Maximum Principle are satisfied

(ii): $(\mathcal{H}(t,\bar{x}(t),p(t)),\, -p(t)) \;\in\; \text{co}\, \partial V(t, \bar{x}(t))$ a.e. $t \in [S,T]$

(iii): $p(S) \;\in\; \partial_x\, (-V)^+(S, \bar{x}(S))$

in which $(-V)^+(.,.)$ is the extended valued function on $R \times R^n$

$$(-V)^+(t,x) \;:=\; \begin{cases} -V(t,x) & \text{if } t \in [S,T] \text{ and } x \in A(t) \\ +\infty & \text{otherwise .} \end{cases}$$

Theorem 2.2: The assertions of Thm. 2.1 remain valid (though with a possibly different co-state arc $p(.)$) when condition (ii) is replaced by

(ii)$'$: $-p(t) \;\in\; \text{co}\, \partial_x V(t, \bar{x}(t)) \quad$ a.e. $t \in [S,T]$.

III. PROOFS OF THMS. 2.1 AND 2.2

The proof technique used for both theorems is to associate with the minimizing process $(\bar{x}, \bar{u})$ for the original problem a minimizing process for a new optimal control problem involving an enriched collection of control variables. By applying the standard Maximum Principle to the new problem, we recover the customary necessary conditions of optimality. Examining the effects of the new control variables we are able also to deduce the desired sensitivity relations. Full details are supplied in a forthcoming paper [5].

Define, for each $\epsilon > 0$ the multifunction $G_\epsilon : [S,T] \to R^{1+n}$:

$$G_\epsilon(t) := \{(\alpha,\beta) \in R^{1+n} \,|\, (\alpha,\beta) \in \text{co}\, \partial V(s,y) \\ \text{for some } (s,y) \in ((t,\bar{x}) + \epsilon B) \cap ((S,T) \times R^n) \\ \text{such that } h(s,y) < 0\}$$

and, for $t \in [S,T]$, $v \in R^n$, $w \in R$,

$$\sigma_\epsilon(t,v,w) \;:=\; \sup_{(\alpha,\beta) \in G_\epsilon(t)} (\alpha,\beta) \cdot (w, -(1+w)v) \,.$$

We see that the set G_ϵ(t) captures information about convexified sub-differentials of V at points in a neighbourbood of $(t, \bar{x}(t))$ on which the state constraint is inactive. The function $\sigma_\epsilon(t,v,w)$ provides a dual description of the closed convex hull of this set. The goal will be to show that, for each $\epsilon > 0$,

$$(\mathcal{H}(t,\bar{x}(t),p(t)), -p(t)) \;\in\; \overline{\text{co}}\, G_\epsilon(t) \qquad \text{a.e.} \tag{9}$$

The desired sensitivity relation is recovered in the limit as $\epsilon \to 0$.

A key role in establishing the preceding inclusion is played by analytical techniques for constructing a *feasible* state trajectory $x(.)$ which lies 'close' to a state trajectory $\hat{x}(.)$ that violates the state constraint, and for which $x(S) = \hat{x}(S)$. Closeness is understood in the sense that the deviation of $x(.)$ is estimated by a measure of the degree constraint violation. Specifically, there exists a constant K, independent of $\hat{x}(.)$, such that

$$||x(.) - \hat{x}(.)||_{L^\infty} \;\leq\; K \max_{t \in [S,T]} h^+(t, \hat{x}(t)) \,.$$

Such techniques, which have extensive application in optimal control theory, have been widely studied. (See, e.g. [3], [4], [9], [10], [13], [14] and the references therein).

The techniques can be employed, in the present context, to establish the following facts about the optimal control: by including an extra integral term in the cost function of the original optimal control problem (P_{S,x_0}) and replacing the state constraint by an additive penalty term

$$K \max_{t\in[S,T]} h^+(t,x(t))$$

we can arrange that the optimal state trajectory is a minimizer with respect a modified set of dynamic constraints. To be precise we have:

Take any $r_0 > ||\bar{x}||_{L^\infty}$. Then, there exists $K > 0$ and $\bar{\epsilon} \in (0,1)$ with the following property: for any $\epsilon \in (0,\bar{\epsilon})$, $(\bar{x},(\bar{u}, v \equiv 0, w \equiv 0))$ is a strong local minimizer for

$$\begin{cases} \textit{Minimize } g(x(T)) + \int_S^T \sigma_\epsilon(t,v(t),w(t))dt \\ \qquad\qquad + K\, \max_{t\in[S,T]} h^+(t,x(t)) - V(S,x(S)) \\ \textit{subject to} \\ \dot{x}(t) \;=\; (1+w(t))\big(f(t,x(t),u(t)) + v(t)\big) \\ (u(t),v(t),w(t)) \in (U(t)\times \epsilon B \times [-\epsilon,\epsilon]) \\ x(S) \in A(S),\; ||x-\bar{x}||_{L^\infty} < \epsilon,\; ||x||_{L^\infty} < r_0\,. \end{cases}$$

Now apply the state constrained Maximum Principle to this problem and denote the resulting costate arc $p(.)$. Freezing the new control variables $(v(.), w(.)) \equiv (0,0)$, we recover all the assertions of the standard state constrained Maximum Principle for the original problem. (Furthermore, these optimality conditions are in 'normal' form, i.e. the cost multiplier can be set to unity.) But now allowing $v(.)$ and $w(.)$ to vary, we recover extra information from the 'maximization of the Hamiltonian' condition. It is

$$\begin{aligned} p^T(t) f(t,\bar{x}(t),\bar{u}(t)) \;=\; \\ \max_{(u,v,w)\in U(t)\times\epsilon B\times[-\epsilon,\epsilon]} (1+w)p^T(t)(f(t,\bar{x}(t),u)+v) \\ -\sigma_\epsilon(t,v,w). \end{aligned}$$

Let us examine in detail the implications of this condition. Setting $(v,w) = (0,0)$, maximization over u yields

$$H(t,\bar{x}(t),p(t),\bar{u}(t)) \;=\; \mathcal{H}(t,\bar{x}(t),p(t)) \quad \text{a.e.}$$

On the other hand fixing $u = \bar{u}(t)$, we deduce that, for all v, w such that $|w| \le \epsilon$ and $|v| \le \epsilon$ and almost every $t \in [S,T]$ we have

$$\begin{aligned} &\frac{w}{1+w}\mathcal{H} + (-p(t))^T(-v) \\ &\quad \le\; \sup_{(\alpha,\beta)\in G_\epsilon(t)} (\alpha,\beta)\cdot(\frac{w}{1+w}, -v)\,. \end{aligned}$$

Write $w' = w/(1+w)$, $v' = -v$. We note that, for some $\tilde{\epsilon} > 0$

$$|w| \le \epsilon,\; |v| \le \epsilon \text{ implies } |w'| \le \tilde{\epsilon},\; |v| \le \tilde{\epsilon}\,.$$

It follows that, for almost every t and all w', v' such that $|v'| \le \tilde{\epsilon}$, $|w'| \le \tilde{\epsilon}$

$$w'\mathcal{H} + (-p(t))^T v' \;\le\; \sup_{(\alpha,\beta)\in G_\epsilon(t)} (\alpha,\beta)\cdot(w',v')\,.$$

It follows that

$$(\mathcal{H}(t,\bar{x}(t),p(t)), -p(t)) \;\in\; \overline{\text{co}}\, G_\epsilon(t) \quad \text{a.e. } t \in [S,T]\,.$$

This is (9), the ϵ version of the desired full sensitivity relation in Thm. 2.1. the partial sensitivty relation of Thm. 2.2. is proved by the same methods, except that a modified version G_ϵ and of the integrand σ_ϵ is employed.

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