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Output-feedback control for almost global stabilization of fully-actuated rigid bodies

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Abstract— **This paper addresses the problem of stabilizing a fully-actuated rigid body. The problem is formulated considering the natural space for rigid body configurations, the Special Euclidean group SE(3). The proposed solution consists of an output-feedback controller for force and torque actuation that guarantees almost global asymptotic stability of the desired equilibrium point. As such the equilibrium point is a stable attractor for all initial conditions except for those in a nowhere dense set of measure zero. As an additional feature, the controller is required to verify prescribed bounds on the actuation.**

I. INTRODUCTION

The stabilization of a rigid body is a difficult nonlinear problem that has received a lot of attention over the years. The classical approach to the stabilization of rigid bodies in position and orientation relies on a local parametrization of the rotation matrix, such as the Euler angles. This parametrization transforms the state space into an Euclidean vector space [1], where the problem admits a trivial solution. However, results can only be proven locally and there is no guarantee that the system trajectories will not evolve to one of the singularities of the parametrization. Unit quaternions and the axis-angle representation are other widely used alternative parametrizations for rotation matrices. Examples of their application can be found in [2] and [3]. These representations are globally nonsingular and thus allow for global results, however, they cover SO(3) multiple times and, as noted by Bhat and Bernstein [4], lead to control laws that are generally not well-defined and yield closed-loop systems that exhibit unwinding. Other authors like Koditschek [5], Bullo and Murray [6], Chaturvedi and McClamroch [7] consider rotation matrices in their natural space, as elements of SO(3). In this paper, we adopt the latter approach so as to avoid problems related to singularities or multiple coverings.

Even if the control law is well-defined it is impossible to achieve global asymptotic stability (GAS) of a rigid body with a continuous feedback controller. As pointed out by several authors [4], [5], [8], and [9], for systems evolving continuously on manifolds not diffeomorphic to the Euclidean space, as is the case of SE(3), there are topological obstacles that preclude the existence of global asymptotically stable equilibrium points. The objective of GAS must then be relaxed to almost GAS (AGAS). In loose terms, this corresponds to saying the point is stable and the solutions converge asymptotically to that point for all initial conditions, except for those in a zero measure nowhere dense set.

In this paper, we consider a fully-actuated rigid body modeled as a simple mechanical control system and address the stabilization problem guaranteeing that prescribed bounds on the actuation are satisfied. Building on previous results for a kinematic model [10], [11], we specifically address the dynamics and propose an output-feedback solution defined on a setup of practical significance. It is assumed that there is a collection of landmarks fixed in the environment and that the output available for feedback are the coordinates of the landmarks' positions and the velocities expressed in the body frame. The stabilization approach followed in this paper is in line with the methods presented in [5], [6], [12], which address the attitude stabilization problem using full-state feedback control and prove stability based on total energy-like Lyapunov functions. Actuator saturation is also considered in [12] for the attitude stabilization problem.

The paper is structured as follows. Section II introduces some differential geometry background needed for the remainder of the paper. Section III describes the dynamics of the rigid body and defines the setup and the output vector considered. In Section IV we present the landmark-based error function that is used for stabilization. The stabilization control law is derived and expressed as an output-feedback law in Section V. A study of its stability properties follows in Section VI. Simulation results that illustrate the performance of the control law are presented in Section VII. Section VIII summarizes the contents of the paper and points out directions for future work.

II. MATHEMATICAL BACKGROUND

In this section we briefly introduce the mathematical formalism of differential geometry needed for the rest of the paper. The notation on differential geometry is standard and the reader is referred to the books by Lee [13], [14] for additional material regarding differential geometry. The concept of a forced simple mechanical control system and respective notation is borrowed from [15]. We start by presenting the general setup for a simple control system evolving on a Riemannian manifold and then simplify it for the case where the control system evolves on a Lie group, particularizing it for a rigid body in SE(3).

Definition 1: A forced simple mechanical control system is a 6-tuple $(\mathsf{Q},\mathbb{G},F_{\text{ext}},V,\mathcal{F},U)$, where

1) Q is a configuration manifold;
2) $\mathbb{G}$ is a Riemannian metric on Q, corresponding to the kinetic energy of the mechanical system;
3) F_{ext} is a force on Q, that can represent de drag forces acting on the rigid body;

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4) V is a potential function on Q;
5) $\mathcal{F} = \{F^1, \ldots, F^m\}$ is a collection of covector fields on Q, representing the control forces;
6) $U \subset \mathbb{R}^m$ is the control set.

The pair $(\mathsf{Q}, \mathbb{G})$ is a Riemannian manifold, therefore there exists a unique Levi-Civita connection $\overset{\mathbb{G}}{\nabla}$. Let $q(t) \in \mathsf{Q}$ denote the configuration of the mechanical system at time t and the tangent space element $\dot{q}(t) \in \mathsf{T}_{q(t)}\mathsf{Q}$ denote the velocity. Then, the governing dynamic equations for the forced simple mechanical control system are

$$\overset{\mathbb{G}}{\nabla}_{\dot{q}(t)} \dot{q}(t) = \mathbb{G}^\sharp(F_{\text{ext}}(\dot{q}) - dV(q)) + \sum_{a=1}^{m} u^a(t)(\mathbb{G}^\sharp(F^a(q))) \tag{1}$$

where $u : I \mapsto U$ are smooth control inputs and $dV(q)$ denotes the differential of the function $V(q)$. The map $\mathbb{G}^\sharp : \mathsf{T}^*_q\mathsf{Q} \mapsto \mathsf{T}_q\mathsf{Q}$ is the associated isomorphism corresponding to the Riemannian metric $\mathbb{G}$ defined as $\langle\!\langle \mathbb{G}^\sharp(\alpha_q), v_q \rangle\!\rangle = \langle \alpha_q, v_q \rangle$ where $\alpha_q \in \mathsf{T}^*_q\mathsf{Q}$, $v_q \in \mathsf{T}_q\mathsf{Q}$, $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes the inner product and $\langle \cdot, \cdot \rangle$ is the natural application between tangent vectors and covectors.

We now consider the more specific case where the configuration manifold Q is a Lie group. We assume the Lie group is endowed with a Riemannian metric $\mathbb{G} = \mathbb{G}_{\mathbb{I}}$ determined via left translation of $\mathbb{I}$, an inner product on the Lie algebra $\mathfrak{g}$. In this setup, the dynamics (1) can be simplified to Euler-Poincaré equations. Consider a simple mechanical control system evolving in a Lie group Q, subject to a potential force derived from the potential function $V(q)$ and actuated by body-fixed forces u. The Euler-Poincaré dynamic equations for such a system are

$$\mathbb{I}\dot{\xi} = \mathrm{ad}^*_\xi \, \mathbb{I}\xi + (T_e L_q)^*(F_{\text{ext}}(\dot{q}) - dV(q)) + u. \tag{2}$$

In equation (2), ξ denotes the body velocity, ad^* the dual adjoint operator, and $T_e L_q$ denotes the tangent map at the group identity of the left translation by $q \in \mathsf{Q}$. The pull-back $(T_e L_q)^*(df(q))$ can be computed without introducing coordinates by noting that

$$\mathcal{L}_{\xi_L} f(q) = \frac{d}{dt} f(q(t))|_{t=0} = \langle (T_e L_q)^*(df(q)), \xi \rangle$$

where $\mathcal{L}$ denotes the Lie derivative and ξ_L is the left-invariant vector field with $\xi_L(e) = \xi$ at the identity.

III. Problem Formulation

In this section we particularize the mathematical notation of Section II for a rigid body evolving in SE(3).

A. Equations of motion

Consider a fixed inertial frame $\{I\}$ and a frame $\{B\}$ attached to the rigid body's center of mass. The configuration of the body frame $\{B\}$ with respect to $\{I\}$ can be viewed as an element of the special euclidean group, $q = (R, \mathbf{p}) = ({}^I_B R, {}^I\mathbf{p}_B) \in$ SE(3). The kinematic equations of motion for the rigid body expressed in an inertial frame $\{I\}$ are

$$\begin{aligned} \dot{R} &= RS(\boldsymbol{\omega}) \\ \dot{\mathbf{p}} &= R\mathbf{v} \end{aligned}$$

where $\boldsymbol{\omega}$ and $\mathbf{v}$ are, respectively, the angular and linear velocities of the rigid body with respect to $\{I\}$ expressed in $\{B\}$. The vectors $\boldsymbol{\omega}$ and $\mathbf{v}$ are called the angular and linear body velocities. The map $S(\cdot) : \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ is an isomorphism between $\mathbb{R}^3$ and the Lie algebra $\mathfrak{so}(3)$ and verifies $S(\mathbf{a})\mathbf{b} = \mathbf{a} \times \mathbf{b}$.

The Euler-Poincaré equations (2) for the specific case of a rigid body in SE(3) can be decomposed in angular and linear motions as

$$\begin{aligned} \mathbb{J}\dot{\boldsymbol{\omega}} &= S(\mathbb{J}\boldsymbol{\omega})\boldsymbol{\omega} + S(\mathbb{M}\mathbf{v})\mathbf{v} \\ &\quad + \boldsymbol{\pi}_\tau\big((T_e L_q)^*(F_{\text{ext}}(\dot{q}) - dV(q))\big) + \mathbf{u}_\tau \end{aligned} \tag{3}$$

$$\mathbb{M}\dot{\mathbf{v}} = S(\mathbb{M}\mathbf{v})\boldsymbol{\omega} + \boldsymbol{\pi}_f\big((T_e L_q)^*(F_{\text{ext}}(\dot{q}) - dV(q))\big) + \mathbf{u}_f \tag{4}$$

where $\mathbb{J}$ is the moment of inertia matrix, $\mathbb{M}$ the added mass matrix, $\mathbf{u}_\tau$ the control torque and $\mathbf{u}_f$ the control force. The maps $\boldsymbol{\pi}_\tau : \mathfrak{se}(3)^* \mapsto \mathfrak{so}(3)^*$ and $\boldsymbol{\pi}_f : \mathfrak{se}(3)^* \mapsto (\mathbb{R}^3)^*$ are the canonical projections from the Lie coalgebra $\mathfrak{se}(3)^*$ to $\mathfrak{so}(3)^*$ and $(\mathbb{R}^3)^*$ respectively.

In the Euler-Poincaré equation in SE(3), the body velocity is represented by the angular and linear velocities. We have $\xi = (\boldsymbol{\omega}, \mathbf{v}) \in \mathfrak{se}(3) \simeq \mathfrak{so}(3) \times \mathbb{R}^3$. The quantities $\mathbf{u}_\tau$ and $\mathbf{u}_f$ in (3)-(4) correspond to the canonical projections of $u \in \mathfrak{se}(3)^*$ to $\mathfrak{so}(3)^*$ and $(\mathbb{R}^3)^*$, each one identified with $\mathbb{R}^3$. In the current setup, the rigid body is assumed to be fully actuated. This means there are no restrictions on the allowable directions for the torque and force vectors, i.e., $\mathbf{u}_\tau$, $\mathbf{u}_f \in \mathbb{R}^3$. The external force F_{ext} accounts for the effects of the viscous drag on the rigid body.

B. Potential Function

The setup in consideration can model several realistic situations. It can be applied to aerial vehicles as well as underwater ones. When considering underwater vehicles one generally does not take into account the existence of a potential force acting on the vehicle. However, for aerial vehicles there is a potential force one cannot ignore: gravity. Its potential function, when close to the Earth's surface, is given by

$$V(q) = -mg\mathbf{e}_3^T\mathbf{p} \tag{5}$$

where m is the total mass of the body, g the gravitational acceleration, and $\mathbf{e}_3 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$. In the definition of the potential function it is implied the use of an inertial frame whose z axis points towards the center of the Earth.

C. Problem Statement

Consider a target configuration $q^* = (R^*, \mathbf{p}^*) = ({}^I_D R, {}^I\mathbf{p}_D) \in$ SE(3) defined as the configuration of the desired frame $\{D\}$ with respect to the inertial frame $\{I\}$. The frame $\{D\}$ is assumed to be fixed in the workspace.

Fig. 1 illustrates the setup at hand, where the coordinates of n points acquired at the current and desired configurations q and q^*, respectively, are available to the system for feedback control along with the body velocities $\boldsymbol{\omega}$ and $\mathbf{v}$. In loose terms, the control objective consists of designing a control law for the actuation $\mathbf{u}_\tau$ and $\mathbf{u}_f$, which ensures the convergence of q to q^* (or, equivalently, of $\{B\}$ to $\{D\}$), with the largest possible basin of attraction and verifies some prescribed bounds.

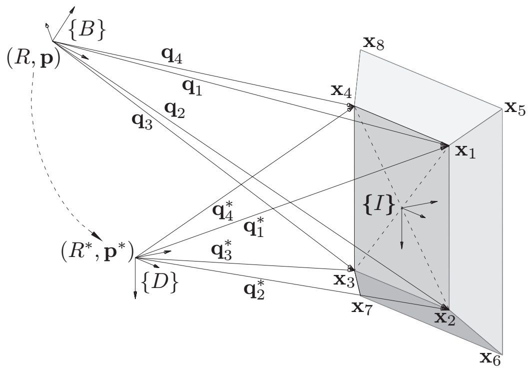

Fig. 1. Problem Setup

The control law uses measurements that come in the form of the coordinates of n fixed points expressed in the body frame. The coordinates of these points, which we call landmarks, are available both in the current body frame and in the desired body frame, as shown in Fig. 1. We also consider the body velocities $\boldsymbol{\omega}$ and $\mathbf{v}$ to be available for feedback. The landmark measurements are typically obtained from on-board sensors that produce the coordinates of the landmarks positions in the body frame.

According to Fig. 1, we define the matrix of inertial landmark coordinates $X = [\mathbf{x}_1 \dots \mathbf{x}_n] \in \mathbb{R}^{3\times n}$, where $\mathbf{x}_j \in \mathbb{R}^3$ denotes the coordinates of the jth point expressed in $\{I\}$, and the matrix of body landmark coordinates $Q = [\mathbf{q}_1 \dots \mathbf{q}_n] \in \mathbb{R}^{3\times n}$ where $\mathbf{q}_j = R^T(\mathbf{x}_j - \mathbf{p})$, $j \in \{1, 2, \dots, n\}$ denotes the coordinates of the jth point expressed in $\{B\}$. Similarly, we introduce the target matrix $Q^* = [\mathbf{q}_1^* \cdots \mathbf{q}_n^*] \in \mathbb{R}^{3\times n}$, where $\mathbf{q}_j^* = R^{*T}(\mathbf{x}_j - \mathbf{p}^*)$. Defining the vector $\mathbf{1} = [1 \cdots 1]^T \in \mathbb{R}^n$, the Q and Q^* matrices of coordinates can be rewritten as $Q = R^T(X - \mathbf{p}\mathbf{1}^T)$, $Q^* = R^{*T}(X - \mathbf{p}^*\mathbf{1}^T)$.

The landmarks are required to satisfy the following condition.

Assumption 1: The n landmarks are not coplanar.

Assumption 2: The target configuration is such that the singular values of the matrix of desired landmark coordinates Q^* are all distinct.

We can now state Proposition 3, which will prove useful in the sequel.

Proposition 3: If Assumption 1 is verified then the matrix Q^* admits

1) a vector $\mathbf{a} = [a_1 \dots a_n]^T \in \mathbb{R}^n$ such that $Q^*\mathbf{a} = 0$ and $\mathbf{1}^T\mathbf{a} = 1$,
2) another vector $\mathbf{b} = [b_1 \dots b_n]^T \in \mathbb{R}^n$ that verifies $Q^*\mathbf{b} = R^{*T}\mathbf{e}_3$, with $\mathbf{e}_3 = [0\ 0\ 1]^T$, and $\mathbf{1}^T\mathbf{b} = 1$.

To conclude the problem statement, we introduce the error configuration $q_e = (R_e, \mathbf{p_e}) \in \mathrm{SE}(3)$, with

$$\mathbf{p_e} = R^T(\mathbf{p} - \mathbf{p}^*) \in \mathbb{R}^3,$$
$$R_e = R^T R^* \in \mathrm{SO}(3),$$

and the state-space model for the error system, which can be written as

$$\dot{\mathbf{p}}_\mathbf{e} = \mathbf{v} - S(\boldsymbol{\omega})\mathbf{p_e},$$
$$\dot{R}_e = -S(\boldsymbol{\omega})R_e.$$

Using the configuration error, the output matrix of coordinates Q can be expressed as $Q = R_e Q^* - \mathbf{p_e}\mathbf{1}^T$.

IV. Landmark-Based Error Function

We wish to drive the error between the measured outputs $\mathbf{q}_j$ and the desired outputs $\mathbf{q}_j^*$ to zero. Since the system in study evolves on SE(3), we express the error as a function in SE(3) as

$$e(q_e) = \frac{n}{2}\Phi(\mathbf{p_e}^T\mathbf{p_e}) + \mathrm{tr}((I_3 - R_e)Q^*Q^{*T}) \tag{6}$$

where the function Φ is defined as

$$\Phi(x) = \frac{x}{1+\sqrt{x}}.$$

It is convenient to note that the previous error function can be expressed as a function of the output measurements. Using Proposition 3, the error function (6) can be expressed as function of the landmarks' measurements Q, as

$$e(Q) = \frac{n}{2}\Phi\big(\mathbf{a}^T Q^T Q\mathbf{a}\big) + \mathrm{tr}\left((Q^* - Q(I - \mathbf{a}\mathbf{1}^T))Q^{*T}\right).$$

Considering Assumption 2 holds, the error function (6) is a Morse function, i.e. its critical points are non-degenerate and consequently isolated. From the properties of the "modified trace" function, the error function (6) is positive definite and has a global minimum at $(R_e, \mathbf{p_e}) = (I_3, \mathbf{0})$. It has exactly four critical points: One minimum, one maximum and two saddle points. For further details the reader is referred to the discussion in [5] and references therein. As shown latter in the paper, these are important properties that will allow for the definition of an almost globally stabilizing law.

Computing the time derivative of the error function, we obtain

$$\dot{e}(q_e) = n\frac{2+\|\mathbf{p_e}\|}{2(1+\|\mathbf{p_e}\|)^2}\mathbf{p_e}^T\mathbf{v} - S^{-1}(R_eQ^*Q^{*T} - Q^*Q^{*T}R_e^T)^T\boldsymbol{\omega}$$

The differential of the error function $e(q_e)$ expressed in the body frame, identifying $\mathfrak{se}(3)$ with $\mathbb{R}^3 \times \mathbb{R}^3$, is

$$(T_eL_q)^*de(q_e) = \begin{bmatrix} -S^{-1}(R_eQ^*Q^{*T} - Q^*Q^{*T}R_e^T) \\ n\frac{2+\|\mathbf{p_e}\|}{2(1+\|\mathbf{p_e}\|)^2}\mathbf{p_e} \end{bmatrix}.$$

Notice that due to the function Φ in the error function definition, the norm of the differential is a bounded function of the configuration error q_e.

Remark 4: In the geometrical control framework, forces and torques are modeled as covectors, existing in the cotangent bundle $\mathsf{T}\mathrm{SE}(3)^*$. The cotangent bundle $\mathsf{T}\mathrm{SE}(3)^*$ can be trivialized to $\mathsf{T}\mathrm{SE}(3)^* \simeq \mathrm{SE}(3) \times \mathfrak{se}(3)^*$ and left-invariant covector fields can be identified with their value at the identity $u \in \mathfrak{se}(3)^*$. We procede to define two norms in $\mathfrak{se}(3)^*$. One that measures the total torque and one that measures the total force exerted on the rigid body, viewed on the body frame. We denote these as the torque norm $\|\cdot\|_\tau$ and the force norm $\|\cdot\|_f$, respectively. Their expressions are $\|u\|_\tau = \|\boldsymbol{\pi}_\tau(u)\|$ and $\|u\|_f = \|\boldsymbol{\pi}_f(u)\|$, where the norm $\|\cdot\|$ denotes the standard euclidean norm in $\mathbb{R}^3$ and we do the standard identifications $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$ and $(\mathbb{R}^3)^* \simeq \mathbb{R}^3$.

Proposition 5: The differential of the error function is bounded. Its torque and force norms have the following bounds

$$\|(T_eL_q)^*de(q_e)\|_\tau \le \lambda_1(Q^*Q^{*T}) + \lambda_2(Q^*Q^{*T})$$
$$\|(T_eL_q)^*de(q_e)\|_f \le n$$

where $\lambda_1(Q^*Q^{*T})$ and $\lambda_2(Q^*Q^{*T})$ denote respectively the largest and second largest eigenvalues of the matrix Q^*Q^{*T}.

V. Control Law Design

In this section we present the strategy devised for designing the control law and study the stability properties of the resulting closed-loop system. To stabilize the error system we use a proportional-derivative control law that takes the form

$$u = -(T_eL_q)^*(de(q_e)) + (T_eL_q)^*(dV(q_e)) - \Psi(\xi) \quad (7)$$

where the modified error and the potential function act as potential energy shaping terms, and the map $\Psi(\xi)$ acts as a dissipative term.

A. Potential Force

Expressing the potential (5) as a function of the configuration error we obtain $V(q_e) = -mg\mathbf{e}_3^T(R^*R_e^T\mathbf{p_e} + \mathbf{p}^*)$. The differential of the gravitational potential expressed in the body frame $\{B\}$ when the system is at configuration error q_e is given by

$$(T_eL_q)^*(dV(q_e)) = \begin{bmatrix} \mathbf{0} \\ -mgR_eR^{*T}\mathbf{e}_3 \end{bmatrix}.$$

B. Damping Force

We now introduce the dissipative force map $\Psi(\xi)$ used in the control law (7).

Let $\Psi_\tau : \mathfrak{so}(3) \mapsto \mathfrak{so}(3)^*$ be the C^1 map given by

$$\Psi_\tau(\boldsymbol{\omega}) = \frac{K_{d_\tau}\boldsymbol{\omega}}{1 + \|K_{d_\tau}\boldsymbol{\omega}\|}$$

with the usual identification $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$ and where K_{d_τ} is a positive definite matrix. It is easy to show that the natural application $\langle \Psi_\tau(\boldsymbol{\omega}), \boldsymbol{\omega}\rangle$ is a positive definite function of $\boldsymbol{\omega}$. We proceed analogously to define $\Psi_f : \mathbb{R}^3 \mapsto (\mathbb{R}^3)^*$ and obtain the force map $\Psi(\xi) : \mathfrak{se}(3) \mapsto \mathfrak{se}(3)^*$

$$\Psi(\xi) = \big(\Psi_\tau(\boldsymbol{\omega}), \Psi_f(\mathbf{v})\big). \quad (8)$$

Proposition 6: The force map as defined in (8) is bounded, and its torque and force norms verify $\|\Psi(\xi)\|_\tau \le 1$ and $\|\Psi(\xi)\|_f \le 1$.

C. Bounded Actuation

The torque and force generated by the control law (7) are bounded. However, their bounds may not be compatible with the prescribed saturation of actuators. In the sequel we will present how to derive a control law that observes some prescribed bounds in the actuation.

We define the constants M_τ and M_f as the maximum magnitude of the total torque and force available for actuation and consider the control constraint

$$\{u \in \mathfrak{se}(3)^* : \|u\|_\tau \le M_\tau \text{ and } \|u\|_f \le M_f\}$$

Necessary conditions for the stabilization of the configuration error considering an arbitrary desired configuration are

$$M_\tau > \sup_{q_e \in Q} \|(T_eL_q)^*(dV(q_e))\|_\tau = 0 \quad (9)$$
$$M_f > \sup_{q_e \in \mathsf{Q}} \|(T_eL_q)^*(dV(q_e))\|_f = mg \quad (10)$$

since to maintain the system stabilized at a given point in space it is at least necessary to counteract the potential force.

We now devise a method of obtaining a control law that verifies the prescribed force and torque limits. Given a maximum torque M_τ and force M_f satisfying (9)-(10), we define the constants δ_τ and δ_f as

$$\delta_\tau = M_\tau - \sup_{q_e \in \mathsf{Q}} \|(T_eL_q)^*(dV(q_e))\|_\tau = M_\tau$$
$$\delta_f = M_f - \sup_{q_e \in \mathsf{Q}} \|(T_eL_q)^*(dV(q_e))\|_f = M_f - mg$$

The previous constants express the total amount of force and torque available for the potential shaping and dissipative control. To specify the control law we need to define the bounds m_τ and m_f

$$m_\tau \ge \sup_{q_e \in \mathsf{Q}} \left\|-S^{-1}(R_eM - MR_e^T)\right\|$$
$$m_f \ge \sup_{q_e \in \mathsf{Q}} \left\| n\frac{2 + \|\mathbf{p_e}\|}{2(1 + \|\mathbf{p_e}\|)^2}\mathbf{p_e} \right\|.$$

We then redefine the error function as

$$e_m(q_e) = \frac{nk_f\delta_f}{2m_f}\Phi(\mathbf{p_e}^T\mathbf{p_e}) + \frac{k_\tau\delta_\tau}{m_\tau}\operatorname{tr}((I_3 - R_e)Q^*Q^{*T}) \quad (11)$$

and the dissipative force map as

$$\Psi(\xi) = \big((1 - k_\tau)\delta_\tau\Psi_\tau(\boldsymbol{\omega}),\, (1 - k_f)\delta_f\Psi_f(\boldsymbol{\omega})\big) \quad (12)$$

where $k_\tau, k_f \in (0, 1)$ are tuning parameters that control how damped the closed-loop system is. Small values for k_τ, k_f lead to highly damped closed-loop dynamics.

Proposition 7: Using the error function (11) and the force map (12) we obtain a new control law (13) that verifies the actuation bounds $\|u\|_\tau < M_\tau$ and $\|u\|_f < M_f$.

$$\begin{bmatrix} \mathbf{u}_\tau \\ \mathbf{u}_f \end{bmatrix} = \begin{bmatrix} -\frac{k_\tau\delta_\tau}{m_\tau}S^{-1}(R_eQ^*Q^{*T} - Q^*Q^{*T}R_e^T) \\ \frac{nk_f\delta_f}{m_f}\frac{2+\|\mathbf{p_e}\|}{2(1+\|\mathbf{p_e}\|)^2}\mathbf{p_e} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ -mg\mathbf{e}_3^TR^*R_e^T \end{bmatrix} - \begin{bmatrix} (1 - k_\tau)\delta_\tau\Psi_\tau(\boldsymbol{\omega}) \\ (1 - k_f)\delta_f\Psi_f(\mathbf{v}) \end{bmatrix} \quad (13)$$

D. Output-Feedback Control Law

We now express (13) in an output-feedback form. Recall that the outputs are given by the matrix of landmark coordinates Q, and the body velocities $\boldsymbol{\omega}$, $\mathbf{v}$. Regarding (13), the constants k_τ, δ_τ, m_τ, k_f, δ_f, and m_f are determined *a priori* based on the desired closed-loop characteristics and input saturation limits. Assuming that $\boldsymbol{\omega}$ and $\mathbf{v}$ are directly measured by the on-board sensors, it leaves to express $\mathbf{p_e}$, $S^{-1}(R_eQ^*Q^{*T} - Q^*Q^{*T}R_e^T)$, and $-\ mgR_eR^{*T}\mathbf{e}_3$, as functions of the outputs. These can be written as follows

$$\mathbf{p_e} = -Q\mathbf{a},$$

$$S^{-1}(R_e Q^* Q^{*T} - Q^* Q^{*T} R_e^T) = S^{-1}(QQ^{*T} - Q^* Q^T) - S(Q^*\mathbf{1})Q\mathbf{a},$$

$$-mgR_e R^{*T}\mathbf{e}_3 = -mgQ(I - \mathbf{a}\mathbf{1}^T)\mathbf{b},$$

where $\mathbf{a}$ and $\mathbf{b}$ are the vectors defined in Proposition 3. Using the previous identities, we can now express the full control law (13) from the output measurements Q, $\boldsymbol{\omega}$, and $\mathbf{v}$.

In what follows, the closed-loop autonomous system that results from the feedback interconnection of (1) and the control law obtained by left translation of (13) is denoted by Σ.

VI. Stability Analysis

In this section, we analyze the stability of the closed-loop system Σ and show that the desired equilibrium point is AGAS, in the sense that the set outside its region of attraction is nowhere dense and has measure zero [12]. The following theorem states this result.

Theorem 8: The point $(q_0, 0) \in \mathsf{TSE}(3)$, $q_0 = (I_3, 0)$, is an AGAS equilibrium point of Σ. Moreover, there exists a neighborhood of $(q_0, 0)$, such that all solutions starting inside it converge exponentially fast.

To prove Theorem 8, we follow a constructive approach that yields a succession of intermediate results. First, we show that the closed-loop trajectories converge to one of four well-defined equilibrium points of the system. We then proceed to linearize the system about each of these equilibria. Based on the stability analysis of these linearized systems, we can conclude that the initial conditions for which the system diverges from $(q_0, 0)$ form a closed set whose dimension is lower than that of the state-space $\mathsf{TSE}(3)$, meaning that it has measure zero and is nowhere dense.

Consider the total energy function $W : \mathsf{TSE}(3) \mapsto \mathbb{R}_0^+$ of the closed-loop system Σ defined as

$$W(t) = e_m(q_e(t)) + \frac{1}{2}\langle\langle \dot{q}_e(t), \dot{q}_e(t)\rangle\rangle \tag{14}$$

where $\dot{q}_e = T_e L_q(\xi)$. Using (14), we can apply LaSalle's invariance principle to obtain the following result.

Lemma 9: Under Assumptions 1 and 2, the solutions of Σ converge to one of the four equilibria in the set $M = \{(q_c, \xi_c) \in \mathsf{TSE}(3) : de_m(q_c) = 0,\ \xi_c = 0\}$.

Proof: The time derivative of W is given by

$$\begin{aligned}\dot{W}(t) &= \frac{d}{dt}\left(e_m(q_e(t)) + \frac{1}{2}\langle\langle \dot{q}_e(t), \dot{q}_e(t)\rangle\rangle\right)\\ &= \nabla_{\dot{q}_e} e_m(q_e(t)) + \langle\langle \nabla_{\dot{q}_e}\dot{q}_e, \dot{q}_e\rangle\rangle\\ &= \langle de_m(q_e), \dot{q}_e\rangle + \langle -de_m(q_e) - \widetilde{\Psi}(\dot{q}_e) + F_{\text{ext}}(\dot{q}_e), \dot{q}_e\rangle\\ &= -\langle \widetilde{\Psi}(\dot{q}_e), \dot{q}_e\rangle + \langle F_{\text{ext}}(\dot{q}_e), \dot{q}_e\rangle\end{aligned}$$

where $\widetilde{\Psi} = (T_e L_{q^{-1}})^*(\Psi)$. From the definition of $\Psi(\xi)$ given in (8) and the stricly dissipative nature of the external drag force F_{ext}, it follows immediately that $\dot{W}$ is negative semi-definite. Applying LaSalle's invariance principle, we conclude that the closed-loop trajectories converge to the largest invariant set such that $\dot{W}(q_e, \xi) = 0 \Leftrightarrow \xi = 0$. This largest invariant set is then the set of all critical points of error function e_m such that $\xi = 0$, which is exactly the set M. ∎

Until now, we have shown that, for all initial conditions, the solutions of the system converge to one of four points and that this set includes the desired equilibrium point $(q_0, 0)$. To prove that it is AGAS, we show that except for $(q_0, 0)$ all equilibrium points $(q_c, 0) \in M$ have an unstable manifold. To reach this result, we consider the linearizations of Σ about each of the four points of interest.

Let $(q_c, 0) \in M$ and consider the system (1) in closed-loop with control law obtained by left translation of (7). As shown in [15], rewriting the dynamics in first-order form and linearizing about the equilibrium points $(q_c, 0)$, using the decomposition $\mathsf{T}_{(q_c,0)}\mathsf{TSE}(3) = \mathsf{T}_{q_c}\mathrm{SE}(3) \oplus \mathsf{T}_{q_c}\mathrm{SE}(3)$, yields the linear system

$$A_\Sigma(q_c) = \begin{bmatrix} 0 & -\left((\mathbb{G}(q_c)^\sharp \circ \operatorname{Hess} e_m(q_c)\right)^T \\ \mathrm{id}_{\mathsf{T}_{q_c}\mathrm{SE}(3)} & \left(\mathbb{G}(q_c)^\sharp \circ d_{\dot{q}_e}(\widetilde{\Psi} - F_{\text{ext}})\big|_{(q_c,0)}\right)^T \end{bmatrix}^T \tag{15}$$

From the positive definiteness of the natural application $\langle \Psi(\xi), \xi\rangle$ and the dissipative nature of the external force, the tensor $d_{\dot{q}_e}(F_{\text{ext}} - \widetilde{\Psi})\big|_{(q_c,0)}$ is symmetric and negative definite. Consequently, the linear system (15) is stable (respectively unstable) if and only if the linear system

$$\dot{x} = -\operatorname{Hess}(e_m(q_c))x$$

is stable (respectively unstable) [5]. We can therefore conclude that the stability of (15) is completely determined by the Hessian matrix $\operatorname{Hess}(e_m(q_c))$.

Lemma 10: If Assumptions 1 and 2 hold, the set of initial conditions for which the solutions of Σ converge to $(q_c, 0) \in M\backslash\{(q_0, 0)\}$ is a closed set of Lebesgue measure zero, whose complement is open and dense.

Proof: As stated in the proof of Lemma 9, if Assumption 2 is satisfied, it can be shown that e_m is a Morse function with four critical points: one minimum, one maximum, and two saddle points. At the global minimum, the Hessian is positive definite, i.e. $\operatorname{Hess}(e_m(q_0)) > 0$, and at the other critical points, it is nonsingular and exhibits at least one negative eigenvalue. Consequently each equilibrium point $(q_c, 0) \in M \setminus \{(q_0, 0)\}$ has a stable manifold W^s whose dimension is smaller than that of the tangent bundle $\mathsf{TSE}(3)$. ∎

We have now gathered the ingredients needed to prove Theorem 8.

Proof: [Theorem 8] Combining Lemmas 9 and 10, the set of initial conditions for which the solutions of Σ do not converge to $(q_0, 0)$ is nowhere dense and measure zero. It follows then that $(q_0, 0)$ is almost globally asymptotically stable.

Given that $d_{\dot{q}_e}(\widetilde{\Psi} - F_{\text{ext}})\big|_{(q_c,0)}$ is positive definite, the dampening of the linear system (15) is positive definite, which implies the system is locally exponentially stable. Hence, we can also conclude that in a neighbourhood of $(q_0, 0)$, the solutions of Σ converge exponentially fast to $(q_0, 0)$. ∎

VII. Simulation Results

In this section we present simulation results for the stabilizing control law derived in the Section V. The simulation objective is to stabilize the configuration of a rigid body that starts at rest. We consider the landmark placement corresponding to

$$X = \begin{bmatrix} 5 & 0 & 0 & -5 & 0 & 0 \\ 0 & 2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 3 & 0 & 0 & -3 \end{bmatrix}$$

and a vehicle with inertia matrices $\mathbb{J} = \mathrm{diag}([1\ 2\ 3])$, $\mathbb{M} = mI_3$, and a mass $m = 2$ Kg. To model the drag we consider as external force the left-invariant covector field with $F_{\text{drag}} = (-0.01\boldsymbol{\omega}\,\|\boldsymbol{\omega}\|\,, -0.01\mathbf{v}\,\|\mathbf{v}\|)$ at the identity. The actuation is limited to 25 N for force and 5 Nm for torque, meaning that, after compensating the gravity, a force of 5 N and a torque of 5 Nm are available for stabilization. The control parameters were tuned so as to achieve a balanced closed-loop response.

The evolution of the position error and attitude error is presented in Fig. 2. For the attitude error, we consider the angle of rotation from the angle-axis representation for the error rotation matrix R_e. As expected, both errors converge asymptotically to zero.

Fig. 3 displays the torque and force actuations. Since there is no potential torque acting on the vehicle, the steady state torque is zero. As for the actuation force, its steady state is the force required to counteract gravity, given the final configuration of the vehicle. It can be observed that the torque and force actuations verify the imposed constraints.

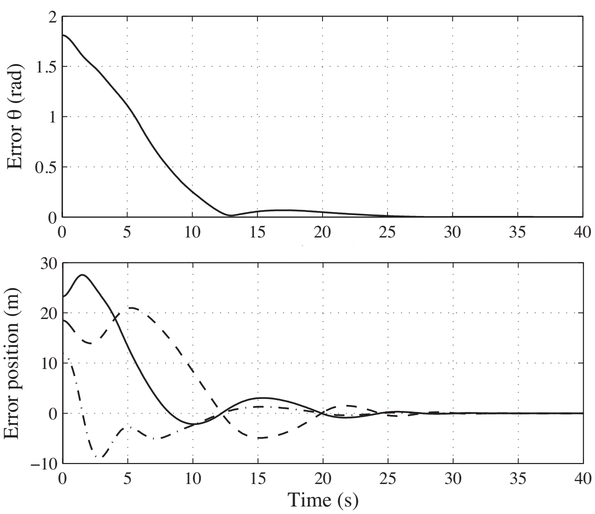

Fig. 2. Attitude and position errors

VIII. Conclusions

An output-feedback solution to the problem of stabilizing a fully-actuated rigid body while keeping the force and torque actuation within predefined bounds was presented in this paper. A landmark-based error function was introduced for potential energy shaping and combined with a dissipative force map to obtain a dissipative closed-loop system that has an AGAS equilibrium point at the minimum of the error function. The prescribed bounds on the actuation were enforced by appropriately scaling a modified version of the error function and defining a bounded dissipative force map. Future work will focus on extending these results to address

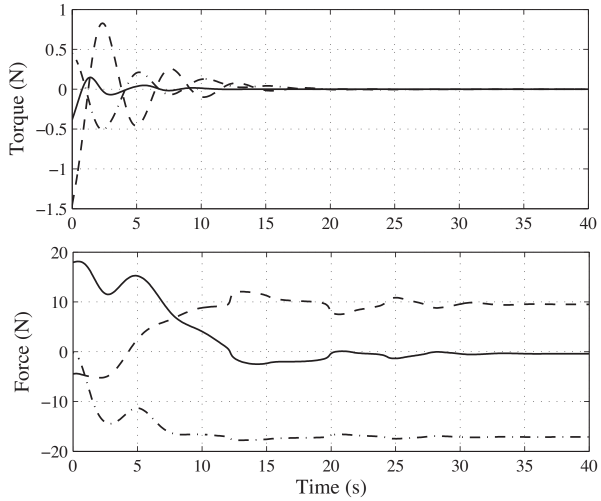

Fig. 3. Torque and force actuations

the tracking problem and advancing from a fully-actuated to an under-actuated setting.

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