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Conservation laws and open systems on higher-dimensional networks

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Abstract— **We discuss a framework for defining physical open systems on higher-dimensional complexes. We start with the formalization of the dynamics of open electrical circuits and the Kirchhoff behavior of the underlying open graph or 1-complex. It is discussed how the graph can be closed to an ordinary graph, and how this defines a Dirac structure on the extended graph. Then it is shown how this formalism can be extended to arbitrary k-complexes, which is illustrated by a discrete formulation of heat transfer on a two-dimensional spatial domain.**

I. Introduction

The work described in this paper is motivated by various developments. In physical systems modeling and simulation, recently much work has been done on developing computational tools for complex physical systems described by pde's, which retain the underlying physical characteristics and system geometry. This has been done using mixed finite element methods, or by setting up a discrete calculus that mimicks the continuous differential-geometric calculus used in the geometric description of systems of pde's (e.g., the development of discrete differential forms). Another development concerns the recent explosion in activity on networks of dynamical systems, analyzing and controlling the dynamics of systems which are interconnected by a network, usually represented as a graph. The this paper is aimed at providing a framework for the study of physical dynamics on higher-dimensional networks, represented by k-complexes. Older references in this spirit include [9], [10].

We start by considering the case of electrical circuits, where the network is described by an (oriented) graph, or equivalently by a 1-complex. We recall how Kirchhoff's laws can be rephrased in this language, and we emphasize the concept of an *open graph* and the corresponding Kirchhoff behavior, which is important for the purpose of *interconnecting* graphs. We also show how this framework fits into the setting of interconnecting systems by power-conserving interconnections as done, for example, in port-Hamiltonian systems theory [11], [12].

Next we show how this framework extends to arbitrary k-complexes, and how it allows for the definition of dynamics on open k-complexes, as illustrated by the example of (discrete) heat conduction on a 2-dimensional complex.

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II. Graphs and Kirchhoff behavior

An *oriented graph*[1] $\mathcal{G}$, see e.g. [3], consists of a finite set $\mathcal{N}$ of *nodes* (sometimes called vertices) and a finite set $\mathcal{B}$ of *branches* (also called edges). Furthermore, there is mapping from $\mathcal{B}$ to the set of ordered pairs of $\mathcal{N}$. Thus to any branch $b \in \mathcal{B}$ there corresponds an ordered pair $(n, m) \in \mathcal{N}^2$ representing the starting node n and the final node m of this branch. An oriented graph is completely specified by its *incidence matrix* D, which is an $\bar{n} \times \bar{b}$ matrix, $\bar{n}$ being the number of nodes and $\bar{b}$ being the number of branches, with (i, j)-th element d_{ij} equal to 1 if the j-th branch is a branch towards node i, equal to -1 if the j-th branch is a branch originating from node i, and equal to 0 otherwise.

Given an oriented graph we define its *node space* Λ_0 as the real vector space of all functions from $\mathcal{N}$ to $\mathbb{R}$. Clearly Λ_0 can be identified with $\mathbb{R}^{\bar{n}}$. Furthermore, we define its *branch space* Λ_1 as the vector space of all functions from $\mathcal{B}$ to $\mathbb{R}$. Again, Λ_1 can be identified with $\mathbb{R}^{\bar{b}}$.

In the context of an electrical circuit Λ_1 will be the vector space of currents through the branches in the circuit. The dual space of Λ_1 will be denoted by Λ^1, and defines the vector space of voltages over the branches. Furthermore, the duality product $< V|I >$ of a vector of currents $I \in \Lambda_1$ with a vector of voltages $V \in \Lambda^1$ is the total power over the circuit. Similarly, the dual space of Λ_0 is denoted by Λ^0 and defines the vector space of potentials at the nodes.

We note that, since both Λ_0 and Λ_1 have a canonical basis corresponding to the individual nodes, respectively, branches, there is a standard Euclidean inner product on both spaces, and thus both Λ^0 and Λ^1 can be identified with Λ_0, respectively Λ_1, such that the duality product becomes this standard inner product.

The incidence matrix D can be also regarded as the matrix representation of a map

$$\partial : \Lambda_1 \to \Lambda_0$$

called the *incidence operator*[2]. The adjoint map of the incidence operator ∂ is the linear map

$$d : \Lambda^0 \to \Lambda^1$$

which is called the *co-incidence* (or co-boundary) operator. The matrix representation of the map d is given by the transposed incidence matrix D^T.

[1] In fact, we will be considering *multi-graphs* since we allow for the existence of multiple branches between the same pair of nodes.

[2] In the literature this operator is usually called the boundary operator. Because of possible confusion with the use of the word 'boundary' for the nodes of the system that can interact with the environment we shall deviate from this use of terminology.

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The triple $(\Lambda_1, \Lambda_0, \partial)$ as defined above is, what is called in algebraic topology, a 1-*dimensional complex*, which is a special type of k-complex, with k any natural number. The elements of Λ_0 and Λ_1 are called 0-chains, respectively 1-chains. The elements of the dual spaces Λ^0 and Λ^1 are called 0-cochains, respectively 1-cochains. Formulating results in this language will facilitate the generalization to higher-dimensional complexes as will be done in Section III.

It is a well-known property [3] of any incidence operator ∂ and the corresponding incidence matrix D that

$$\mathbb{1}^T D = 0 \tag{1}$$

where $\mathbb{1}$ denotes the vector with all components equal to 1. From this property it follows that the rank of the incidence matrix D is at most $\bar{n} - 1$. In fact, the rank is given as [3]

$$\operatorname{rank} D = \bar{n} - k_{\mathcal{G}} \tag{2}$$

where $k_{\mathcal{G}}$ is the number of components of the graph $\mathcal{G}$. (A component is a maximal subgraph which is connected, that is, every two nodes are linked by a path of, -non-oriented-, branches.). Thus in case of a connected graph, $\operatorname{rank} D = \bar{n} - 1$.

A. Kirchhoff's laws for graphs

In this subsection we recall the formulation of Kirchhoff's laws for graphs as can be found e.g. in [1], [3]. Consider an oriented graph $\mathcal{G}$ specified by its incidence operator ∂ or co-incidence operator d. Kirchhoff's laws associated with the graph can be expressed as follows. Kirchhoff's current laws (KCL) are given as

$$\partial I = 0 \tag{3}$$

while Kirchhoff's voltage laws (KVL) take the form

$$V \in \operatorname{im} d. \tag{4}$$

The graph theoretic interpretation [3] is that the kernel of the incidence operator is the *cycle space* of the graph, while the image of the co-incidence operator is its *cut space*, while furthermore the cycle space is the orthogonal complement (with respect to the duality product) of the cut space. This leads to the equivalent way of formulating Kirchhoff's laws as the fact that the total current along any cut is equal to zero, and the total voltage over every cycle is zero. Equivalently we can write Kirchhoff's voltage laws as

$$V = d\phi \tag{5}$$

for some 0-cochain ϕ, which has the physical interpretation of being the *potential* at every node. Hence Kirchhoff's voltage laws express that any voltage distribution V over the graph (circuit) corresponds to a potential over every node. Using the matrix representation of the incidence operator ∂ as the incidence matrix D, Kirchhoff's laws are written in matrix notation as

$$DI = 0, \quad V = D^T \phi \tag{6}$$

Tellegen's theorem automatically follows from Kirchhoff's laws. Indeed, take any current distribution I satisfying Kirchhoff's current laws, and any voltage distribution V satisfying Kirchhoff's voltage laws. Then, by definition of the co-incidence operator d as the adjoint of the incidence operator ∂ we obtain (with $\sum_b$ and $\sum_n$ denoting the sum over all the branches, respectively nodes, of the graph)

$$\sum_b V_b I_b = \sum_b (d\phi)_b I_b = \sum_n \phi_n (\partial I)_n = 0 \tag{7}$$

since I satisfies Kirchhoff's current laws $\partial I = 0$. In particular, Tellegen's theorem implies that for any actual current and voltage distribution over the circuit the total power is equal to zero.

We summarize the Kirchhoff behavior $B_K(\mathcal{G})$ of a graph $\mathcal{G}$ with incidence operator ∂ as

$$B_K(\mathcal{G}) := \{(I, V) \in \Lambda_1 \times \Lambda^1 \mid I \in \ker \partial, V \in \operatorname{im} d\} \tag{8}$$

It immediately follows that the Kirchhoff behavior defines a *Dirac structure*. Recall [5], [11], [12] that a subspace $D \subset V \times V^*$ for some vector space V defines a Dirac structure if $D = D^\perp$ where $^\perp$ denotes the orthogonal complement with respect to the indefinite inner product $<<,>>$ on $V \times V^*$ defined as

$$<< (v_1, v_1^*), (v_2, v_2^*) >> := < v_1^* | v_2 > + < v_2^* | v_1 >,$$

with $v_1, v_2 \in V, v_1^*, v_2^* \in V^*$, where $<|>$ denotes the duality product between V and V^*. For finite-dimensional vector spaces V one can equivalently characterize a Dirac structure by the property that $< v^* | v >= 0$ for every $(v, v^*) \in D$ and that $\dim D = \dim V$.

B. Kirchhoff's laws for open graphs

Next we wish to define *open graphs* $\mathcal{G}$ and their resulting Kirchhoff behavior. An open graph $\mathcal{G}$ is obtained from an ordinary graph with set of nodes $\mathcal{N}$ by identifying a subset $\mathcal{N}_e \subset \mathcal{N}$ of *external nodes*. The interpretation of $\mathcal{N}_e$ is that these are the nodes that are open to interconnection with other graphs. The complementary subset $\mathcal{N}_i := \mathcal{N} - \mathcal{N}_e$ are the *internal nodes* of the open graph.

Kirchhoff's current laws apply to an open graph $\mathcal{G}$ in a different manner than to an ordinary (closed) graph, since the ordinary Kirchhoff's current laws would imply that the currents over every branch incident on a terminal node are *zero*, which is *not* what we want for interconnection. Furthermore, by Tellegen's theorem, the ordinary Kirchhoff's laws would imply that the total power in the circuit is equal to zero, thus implying that there cannot be any ingoing or outgoing power flow. Hence we will modify Kirchhoff's current laws by requiring that the incidence operator ∂ maps the vector of currents I to a vector that has zero components corresponding to the internal nodes, while for the external nodes the image is equal to (minus) the *external current* I_e

$$\partial I + \begin{bmatrix} I_e \\ 0 \end{bmatrix} = 0, \tag{9}$$

where for simplicity of notation we have reordered the nodes in such a manner that the external nodes come first. The dimension of the vector I_e of external currents is equal to the number $\bar{n}_e$ of external nodes in the set $\mathcal{N}_e$. The vector I_e can

be regarded to be belonging to the space of chains associated with the *leaves*[3] emanating from the external nodes $\mathcal{N}_e$. (The external nodes together with the leaves are called *terminals* in [13].) We will denote the vector space of external currents by Λ_e. Kirchhoff's voltage laws remain unchanged, and will be written as

$$V = d\phi = d\begin{bmatrix}\phi_e\\ \phi_i\end{bmatrix} \tag{10}$$

where ϕ_e denotes the vector of the potentials at the external nodes and ϕ_i the vector of potentials at the internal nodes. This results in the following Kirchhoff behavior for open graphs:

$$\begin{aligned} B_K(\mathcal{G}) \quad := \quad & \{(I,V,I_e) \in \Lambda_1 \times \Lambda^1 \times \Lambda_e \mid \\ & \partial I + \begin{bmatrix}I_e\\ 0\end{bmatrix} = 0,\ \exists \phi \text{ s.t. } V = d\phi\} \end{aligned} \tag{11}$$

By computing as before, cf. (7), the total power we now obtain (with $n_i \in \mathcal{N}_i$ and $n_e \in \mathcal{N}_e$ denoting the internal, respectively, external nodes)

$$\begin{aligned} &\textstyle\sum_b V_b I_b = \sum_b (d\phi)_b I_b = \sum_n \phi_n (\partial I)_n = \\ &\textstyle\sum_{n_i} \phi_{n_i}(\partial I)_{n_i} + \sum_{n_e} \phi_{n_e}(\partial I)_{n_e} = -\sum_{n_e} \phi_{n_e} I_{n_e} \end{aligned} \tag{12}$$

since I satisfies the Kirchhoff's current laws $(\partial I)_{n_i} = 0$ and $(\partial I)_{n_e} = -I_{n_e}$, where I_{n_e} denotes the current along the leaf emanating from the external node n_e. Thus, for open graphs the total power is not zero but equal to $-\sum_{n_e} \phi_{n_e} I_{n_e}$.

Even though the vector of potentials ϕ_e at the external nodes is *not* uniquely determined by the vector V of voltages, the expression $\sum_{n_e} \phi_{n_e} I_{n_e}$ *is* uniquely determined, and in a circuit context has the interpretation of being the outgoing *power* (through the external nodes). In fact, the freedom in the choice of the total vector ϕ corresponding to the same vector of voltages V is given by all vectors ψ such that $d\psi = 0$, or in matrix notation $D^T\psi = 0$. Writing as before $\phi = \begin{bmatrix}\phi_e\\ \phi_i\end{bmatrix}$ we see that the freedom in the choice of ϕ_e is given by all vectors ψ_e such that for some ψ_i it holds that $D_e^T\psi_e + D_i^T\psi_i = 0$, where D_e is the submatrix of D consisting of the first $\bar{n}_e$ rows (corresponding to the external nodes) and D_i is the submatrix consisting of the last $\bar{n} - \bar{n}_e$ rows. Then, because of the modified Kirchhoff's laws (9) it follows that for any such ψ_e we have

$$\psi_e^T I_e = -\psi_e^T D_e I = \psi_i^T D_i I = 0 \tag{13}$$

and thus the right-hand side of (12) is indeed independent of the choice of ϕ_e.

Note that the kernel of D^T can be easily characterized. Indeed, by the property (1) $\mathbb{1} \in \ker D^T$, and thus by (9)

$$0 = \mathbb{1}^T D I = \sum_{n_e} I_{n_e} \tag{14}$$

Hence the external part of the Kirchhoff behavior of an open graph is constrained by the obvious fact that all external currents sum up to zero.

[3]The definition of an oriented graph can be extended by allowing for 'branches' (properly called *leaves*) that are incident on only one node.

In general, see e.g. [3], the rank of D is equal to $\bar{n} - k_{\mathcal{G}}$, where $k_{\mathcal{G}}$ denotes the number of connected components of the graph $\mathcal{G}$. For example, if $\mathcal{G}$ consists of two connected components, then, if we reorder the nodes in two subsets in such a way that the first nodes correspond to the first component, and the last ones to the second, then the vectors $\begin{bmatrix}\mathbb{1}\\ 0\end{bmatrix}$ and $\begin{bmatrix}0\\ \mathbb{1}\end{bmatrix}$ span the kernel of D^T. In the case of external nodes this implies that both the sum of the currents corresponding to the external nodes belonging to the first component as well as the sum of the currents of the external nodes belonging to the second component are equal to zero, which is obvious from a circuit point of view (see also [13]).

An important consequence of the above is that we may always *close* an *open* graph $\mathcal{G}$ to an ordinary graph $\bar{\mathcal{G}}$. Consider first the case that $\mathcal{G}$ is connected. Then we may add one virtual node n_0, and virtual edges from this virtual ('ground') node to every external node $n_e \in \mathcal{N}_e$, in such a manner that the Kirchhoff behavior of this graph $\bar{\mathcal{G}}$ *extends* the Kirchhoff behavior of the open graph $\mathcal{G}$. In fact, to the virtual node n_0 we may associate an arbitrary potential ϕ_{n_0} (a ground-potential), and we may rewrite the righthand-side of (12) as (since $\sum_e I_{n_e} = 0$)

$$-\sum_{n_e} (\phi_{n_e} - \phi_{n_0}) I_{n_e} = -\sum_{n_e} V_{n_e} I_{n_e} \tag{15}$$

where $V_{n_e} := \phi_{n_e} - \phi_{n_0}$ and I_{n_e} denote the voltage, respectively current, over the virtual edge towards the external node n_e (corresponding to the leaf of the open graph). It follows that

Proposition 2.1: Consider an open connected graph $\mathcal{G}$. The projection of the Kirchhoff behavior $B_K(\bar{\mathcal{G}})$ of the extended graph $\bar{\mathcal{G}}$ (in the variables I, V, I_e, V_e) onto the variables I, V, I_e is equal to the Kirchhoff behavior $B_K(\mathcal{G})$. If the open graph $\mathcal{G}$ consists of more than one component, one extends the graph by adding a virtual node to *every* component containing external nodes, while the above proposition remains to hold.

The consequence of this 'closing' operation of an open graph with external nodes is that one can associate to the open graph a Dirac structure, involving the pairs of dual variables I, V and I_e, V_e. This Dirac structure encodes the power structure of the graph, and since $< V_e | I_e >$ equals external power the pair I_e, V_e constitutes a power port.

C. Constitutive relations

Next to the Kirchhoff behavior an electrical circuit consists of constitutive relations for every branch. They specify for a branch b a relation between the current variable I_b and the voltage variable V_b. The simplest case is a *resistive* relation between I_b and V_b such that $V_b I_b \leq 0$. In the case of a *capacitive* relation one defines an additional energy variable Q_b (denoting the charge) together with a real function $H_b(Q_b)$ denoting the electric energy stored in the capacitor. The complete constitutive relations are then given by

$$\dot{Q}_b = -I_b, \quad V_b = \frac{dH_b}{dQ_b}(Q_b) \tag{16}$$

Alternatively, in the case of an inductor one specifies the magnetic energy $H_b(\varphi_b)$, where φ_b denotes the flux, together with the dynamic relations

$$\dot{\varphi}_b = -V_b, \quad I_b = \frac{dH_b}{d\varphi_b}(\varphi_b) \tag{17}$$

Substituting these constitutive relations into the Kirchhoff behavior (and corresponding Dirac structure) defined by the graph results in a port-Hamiltonian system description, see e.g. [11], [12].

D. Interconnection of graphs and Kirchhoff behavior

Let us consider two open graphs $\mathcal{G}^A$ and $\mathcal{G}^B$ with respective sets of external nodes $\mathcal{N}_e^A$ and $\mathcal{N}_e^B$. Suppose we want to interconnect them over a set of *shared external nodes*

$$\bar{\mathcal{N}} \subset \mathcal{N}_e^A \cap \mathcal{N}_e^B$$

This may be done by identifying the shared nodes corresponding to the two graphs, leading to an interconnected open graph $\mathcal{G}^A \parallel \mathcal{G}^B$ with resulting set of internal nodes

$$\mathcal{N}_i^A \cup \mathcal{N}_i^B \cup \bar{\mathcal{N}}$$

(after the interconnection we will regard the shared external nodes as being internal), thus leading to a remaining set of external nodes

$$(\mathcal{N}_e^A - \bar{\mathcal{N}}) \cup (\mathcal{N}_e^B - \bar{\mathcal{N}})$$

The total set of nodes of the interconnected graph will be denoted as $\mathcal{N}^A \parallel \mathcal{N}^B$, and the resulting space of 0-chains by $\Lambda_0^A \parallel \Lambda_0^B$.

The resulting incidence operator $\partial^A \parallel \partial^B : \Lambda_1^A \times \Lambda_1^B \to \Lambda_0^A \parallel \Lambda_0^B$ is given as follows. Reorder the set of nodes $\mathcal{N}^A \parallel \mathcal{N}^B$ of $\mathcal{G}^A \parallel \mathcal{G}^B$ in such a manner that the nodes corresponding to only $\mathcal{G}^A$ come first, those corresponding to only $\mathcal{G}^B$ come second, while the shared nodes $\bar{\mathcal{N}}$ come last. Then define

$$\partial^A \parallel \partial^B(I^A, I^B) := \begin{bmatrix} \partial^A(I^A) \\ \partial^B(I^B) \\ \partial^A(I^A) + \partial^B(I^B) \end{bmatrix} \tag{18}$$

where we consider the appropriate components of ∂^A and ∂^B. The interconnected Kirchhoff behavior $B_K(\mathcal{G}^A) \parallel B_K(\mathcal{G}^B)$ is simply defined as the Kirchhoff behavior of the interconnected graph $\mathcal{G}^A \parallel \mathcal{G}^B$. The following proposition is easy to prove.

Proposition 2.2: Let $\mathcal{G}^A$ and $\mathcal{G}^B$ be open graphs. Then

$$B_K(\overline{\mathcal{G}^A \parallel \mathcal{G}^B}) = B_K(\bar{\mathcal{G}}^A) \circ B_K(\bar{\mathcal{G}}^B) \tag{19}$$

where $\circ$ denotes the composition of Dirac structures, cf. [4].

III. Conservation laws on higher-dimensional complexes

A. Kirchhoff behavior on k-complexes

Kirchhoff's laws for electrical circuits can be regarded as a prime example of *conservation laws*, defined on 1-complexes. In this section we will extend the theory of the previous section to higher-dimensional complexes.

An k-complex Λ is specified by a sequence of real linear spaces[4] $\Lambda_0, \Lambda_1, \cdots, \Lambda_k$, together with a sequence of incidence operators

$$\Lambda_k \xrightarrow{\partial_k} \Lambda_{k-1} \xrightarrow{\partial_{k-1}} \cdots \Lambda_1 \xrightarrow{\partial_1} \Lambda_0 \xrightarrow{\partial_0} 0$$

with the property that

$$\partial_{j-1} \circ \partial_j = 0, \quad j = 1, \cdots, k$$

The vector spaces Λ_j, $j = 0, 1 \cdots, k$, are called the spaces of j-chains. Each Λ_j is generated by a finite set of j-cells (like branches and nodes for graphs) in the sense that Λ_j is the set of functions from the j-cells to $\mathbb{R}$. A typical example of a k-complex is a triangularization of a k-dimensional manifold, with the j-cells, $j = 0, 1, \cdots, k$, being the sets of vertices, edges, faces, etc..

Denoting the dual linear spaces by Λ^j, $j = 0, 1 \cdots, k$, we have the following dual sequence

$$\Lambda^0 \xrightarrow{d_1} \Lambda^1 \xrightarrow{d_2} \Lambda^2 \cdots \Lambda^{k-1} \xrightarrow{d_k} \Lambda^k$$

having the analogous property

$$d_j \circ d_{j-1} = 0, \quad j = 2, \cdots, k$$

The elements of Λ^j are called j-cochains.

Consider any k-complex Λ, with k-chains $\alpha \in \Lambda_k$ and k-cochains $\beta \in \Lambda^k$. We define, similarly as in the case of a graph (1-complex) its Kirchhoff behavior as

$$\begin{aligned} B_K(\Lambda) \quad := \quad & \{(\alpha, \beta) \in \Lambda_k \times \Lambda^k \mid \\ & \partial_k \alpha = 0, \exists \phi \in \Lambda^{k-1} \text{ s.t. } \beta = d_k \phi\} \end{aligned} \tag{20}$$

We will still refer to $\partial_k \alpha = 0$ as Kirchhoff's current laws (KCL), and to $\beta = d_k \phi$ as Kirchhoff's voltage laws (KVL). As before, it is immediately seen that $B_K(\Lambda) \subset \Lambda_k \times \Lambda^k$ is a *Dirac structure*. In particular, it follows that $< \beta \mid \alpha >_k = 0$ for every $(\alpha, \beta) \in B_K(\Lambda)$, where $< \cdot | \cdot >_k$ denotes the duality product between the dual linear spaces Λ_k and Λ^k.

B. Open k-complexes

Next we consider an *open* k-complex, by identifying a subset $\mathcal{N}_{(k-1)}^e$ of the set of all $(k-1)$-cells, called the *external* $(k-1)$-cells. Define the linear space of functions from this subset of $(k-1)$-cells to $\mathbb{R}$ as $\bar{\Lambda} \subset \Lambda_{k-1}$. As before, Kirchhoff's voltage laws remain unchanged

$$\beta = d_k \phi, \tag{21}$$

while Kirchhoff's current laws are modified into

$$\partial_k \alpha + \begin{bmatrix} \alpha_e \\ 0 \end{bmatrix} = 0 \tag{22}$$

where α_e denotes the vector of external currents associated to the external $(k-1)$-cells. By computing as before the total power we obtain for any α and β satisfying (21, 22)

$$\begin{aligned} & < \beta \mid \alpha >_k = < d_k \phi \mid \alpha >_k = < \phi \mid \partial_k \alpha >_{k-1} = \\ & < \phi \mid \begin{bmatrix} -\alpha_e \\ 0 \end{bmatrix} >_{k-1} = - < \phi_e \mid \alpha_e >_{k-1} \end{aligned} \tag{23}$$

[4] In algebraic topology [6] one usually starts with *abelian groups* Λ_j.

where ϕ_e is the vector of potentials at the external $(k-1)$-cells.

Note that there exist many ϕ with $d_k\phi = \beta$ for the same β, and thus in general there will exist many ϕ_e for the same β in the left-hand side. Recall from the case of graphs that the value of the righthand-side actually does not depend on this choice of ϕ_e. The same holds in the k-complex case too. In fact, the freedom in choosing ϕ such that $d_k\phi = \beta$ for a given β is given by all $\psi \in \Lambda^{k-1}$ such that $d_k\psi = 0$, or in matrix notation $D_k^T\psi = 0$, where D_k is the matrix representation of the incidence operator ∂_k. Hence, with D_{ke} denoting the submatrix of D_k corresponding to the external $(k-1)$-cells and D_{ki} denoting the submatrix corresponding to the internal $(k-1)$-cells, the freedom in ϕ_e is given by all ψ_e such that for some ψ_i it holds tha $D_{ke}^T\psi_e + D_{ki}^T\psi_i = 0$. Thus, since $D_{ke}\alpha = -\alpha_e$ and $D_{ki}\alpha = 0$,

$$< \psi_e|\alpha_e >_{k-1} = \psi_e^T\alpha_e = -\psi_e^T D_{ke}\alpha = \psi_e^T D_{ki}\alpha = 0 \tag{24}$$

and we conclude that $< \phi_e \mid \alpha_e >_{k-1}$ is independent of the choice of ϕ_e.

Similar to graphs it follows that the Kirchhoff current laws for open k-complexes $D_{ke}\alpha = -\alpha_e$ imply certain constraints on the incoming 'currents' α_e. Indeed, by the fact that

$$\partial_{k-1} \circ \partial_k = 0 \tag{25}$$

it follows that $D_{(k-1)e}\alpha_e = 0$. Furthermore, the existence of any other map L such that $LD_{ke} = 0$ will yield additional constraints $L\alpha_e = 0$.

As in the case of graphs, this allows us to *close* an open k-complex. This is done by completing the open k-complex (Λ, Λ_e) by an additional set of $(k-1)$-cells and k-cells, without changing the Kirchhoff behavior. Because of space limitations we leave the details to another paper. For *interconnection* of open k-complexes we may proceed in the same way as for graphs: we start from two k-complexes Λ^A and Λ^B, with *external* $(k-1)$-cells $\mathcal{N}_{k-1}^A$, respectively $\mathcal{N}_{k-1}^B$, and identify a subset of shared $(k-1)$-cells. Details are also left to a future paper.

IV. Dissipative port-Hamiltonian dynamics on k-complexes

Consider an (open) k-complex Λ, together with its Kirchhoff behavior as defined before. Dynamics on the k-complex can be defined in various ways. Similar to the case of electrical circuits we can define constitutive relations for every k-cell, by specifying a relation between every component of Λ_k and Λ^k. As in the case of an electrical circuit this can be a relation of static resistive type, or a dynamic relation.

In this section we will define dynamics in a different way by specifying one type of dynamical relations between Λ_k and Λ^k, together with resistive relations between Λ_{k-1} and Λ^{k-1}. This will define a port-Hamiltonian dynamics, which is dissipative (because of the presence of resistive relations) and of *relaxation* type since there is only one type of physical energy (and thus no oscillations between different types of physical energy occur).

The definition of dissipative port-Hamiltonian dynamics used in this section is as follows. (Other options are left for a future paper.) On the k-complex Λ, with $\partial_k : \Lambda_k \to \Lambda_{k-1}$ and $d_k : \Lambda^{k-1} \to \Lambda^k$, we define the following relations

$$\begin{aligned} f_x &= d_k e, \qquad f_x \in \Lambda^k, e \in \Lambda^{k-1} \\ f &= -\partial_k e_x, \qquad e_x \in \Lambda_k, f \in \Lambda_{k-1} \end{aligned} \tag{26}$$

It is immediately checked that this defines a Dirac structure $\mathcal{D} \subset \Lambda^k \times \Lambda_k \times \Lambda^{k-1} \times \Lambda_{k-1}$. This allows us to define a dissipative port-Hamiltonian dynamics by imposing the following constitutive relations. First we associate to every k-cell an energy storage, leading to

$$\dot{x} = -f_x, \quad e_x = \frac{\partial H}{\partial x}(x), \quad x \in \Lambda^k \tag{27}$$

with $H(x)$ the total energy storage $H(x)$, and $x \in \Lambda^k$ the total vector of energy variables. Furthermore, we associate to every $(k-1)$-cell a (linear) resistive relation, leading to

$$e = -Rf, \quad R = R^T \geq 0 \tag{28}$$

Substituted in (26) this yields the relaxation dynamics

$$\dot{x} = -d_k e = d_k\, R\, f = -d_k\, R\, \partial_k \frac{\partial H}{\partial x}(x), \quad x \in \Lambda^k \tag{29}$$

with the property that

$$\frac{dH}{dt} = -(\partial_k \frac{\partial H}{\partial x}(x))^T\, R\, \partial_k \frac{\partial H}{\partial x}(x) = -f^T\, R\, f \leq 0 \tag{30}$$

For an *open* complex with external $(k-1)$-cells and external 'currents' $\bar{\Lambda} \subset \Lambda_{k-1}$ the definition is modified as follows. Instead of (26) we consider

$$\begin{aligned} f_x &= d_k \begin{bmatrix} e \\ e_b \end{bmatrix}, \qquad f_x \in \Lambda^k, \begin{bmatrix} e \\ e_b \end{bmatrix} \in \Lambda^{k-1}, e_b \in \bar{\Lambda}^{k-1} \\ \begin{bmatrix} f \\ f_b \end{bmatrix} &= -\partial_k e_x, \qquad e_x \in \Lambda_k, \begin{bmatrix} f \\ f_b \end{bmatrix} \in \Lambda_{k-1}, f_b \in \bar{\Lambda}_{k-1} \end{aligned} \tag{31}$$

with f_b, e_b corresponding to the *external* $(k-1)$-cells, and f, e corresponding to the *internal* cells. Imposing the same storage relations (27) and resistive relations (28) we arrive at

$$\begin{aligned} \dot{x} &= -d_k^r\, R\, \partial_k^r \tfrac{\partial H}{\partial x}(x) - d_k^b e_b \\ f_b &= -\partial_k^b \tfrac{\partial H}{\partial x}(x) \end{aligned} \tag{32}$$

where we have split d_k as $d_k = \begin{bmatrix} d_k^r & d_k^b \end{bmatrix}$ and $\partial_k = \begin{bmatrix} \partial_k^r \\ \partial_k^b \end{bmatrix}$ (according to the division of the $(k-1)$-cells into internal cells corresponding to resistive behavior and external cells corresponding to interconnection). This defines a port-Hamiltonian system with inputs e_b and outputs f_b.

A. Example: Heat transfer on a 2-complex

The above formulation of systems of conservation laws and dissipative port-Hamiltonian systems on k-complexes will be illustrated with the model of heat transfer in a 2-dimensional medium (for instance a plate). Instead of first considering the pde-model and then discretizing, we

will directly consider the dynamics on a 2-complex (as for instance arising from the triangulation of the 2-dimensional spatial domain). We assume the medium to be undeformable (hence mechanical work is neglected) and that there is no mass transfer.

We will write the heat transfer in terms of the conservation of internal energy. (An alternative description [7] is the representation in terms of the entropy, arising from the entropy balance equation [8], [2].) Let us identify the physical variables as chains and cochains of the given 2-complex. First, the internal energy u of the 2-complex corresponds to a 2-cochain, thus $u \in \Lambda^2$ (with every component of u denoting the energy of the corresponding 2-cell).

The thermodynamic properties are defined by Gibbs' relation, and generated by the *entropy function* $s = s(u) \in C^\infty\left(\Lambda^2\right)$ as thermodynamic potential. Since we consider transformations which are isochore and without mass transfer, Gibbs' relation reduces to the definition of the vector of intensive variables e_u which is (entropy-)conjugated to the vector of extensive variables u by

$$e_u = \frac{\partial s}{\partial u}(u) \tag{33}$$

(In fact, the components of the vector e_u are equal to the reciprocal of the temperature in each 2-cell.) Being a gradient vector it follows that $e_u \in \Lambda_2$.

Furthermore, the heat conduction is given by the *heat flux* $e \in \Lambda^1$, describing the heat flux through every 1-cell (edge). This flux arises from thermal non-equilibrium, defined by the fact that the temperature is varying over the 2-cells (or equivalently, in our formulation, by the fact that the components of the vector $e_u \in \Lambda_2$ are different from each other). Its conjugate vector of variables is the *thermodynamical driving force* vector, denoted by $f \in \Lambda_1$ and given as the vector of the differences of the reciprocals of the temperatures of the 2-cells with common edges (1-cells), and thus defined by

$$f = -\partial_2 e_u \tag{34}$$

By the classical Fourier's law expressed here in terms of the vector e_u and the thermodynamic generating force $f \in \Lambda_1$, the flux of entropy due to thermal non-equilibrium is expressed as

$$e = R(e_u)\, f, \tag{35}$$

with $R(e_u) = R^T(e_u) \geq 0$ a semi-positive linear mapping $\Lambda_1 \to \Lambda^1$ depending on the heat conduction coefficients. (Note the sign-difference with (28).) Finally the energy conservation law is expressed by

$$\frac{du}{dt} = -d_2 e \tag{36}$$

Hence the resulting system is a dissipative port-Hamiltonian system (of relaxation type) defined on the 2-complex, with vector of state variables x given by the internal energy vector u, and Hamiltonian $s(u)$. Note that by (30) the entropy $s(u)$ satisfies

$$\frac{ds}{dt} = (\partial_2 \frac{\partial s}{\partial u}(u))^T\, R(e_u)\, \partial_2 \frac{\partial s}{\partial u}(u) = f^T\, R(e_u)\, f \geq 0 \tag{37}$$

expressing the fact that the entropy $s(u)$ is monotonously *increasing*. (Note again the sign-difference with the treatment above, where the Hamiltonian H was decreasing.)

The exchange of heat through the boundary of the system can be incorporated as above, cf. (31, 32), by splitting the edges (1-cells) into internal edges with the resistive relation (35) and external (boundary) edges.

V. Conclusions

A basic framework has been laid down for the formulation of open physical systems on k-complexes, generalizing the graph-theoretic formulation of electrical circuit dynamics. This has been illustrated by the example of heat transfer on a 2-complex. This simple example already shows how one can directly define a finite-dimensional (port-Hamiltonian) dynamics, capturing the physical meaning of the involved variables and retaining the conservation laws, without the need to formulate the dynamics as a set of pde's (and possibly to discretize the pde's later on).

In future work we will apply and extend the framework to different classes of dynamical systems on k-complexes corresponding to different physical settings. Furthermore, by the extension to open k-complexes we will derive finite-dimensional boundary-control models. Finally, it seems of interest to extend the recently intensively studied theory of networks of systems to dynamical systems connected by higher-dimensional complexes.

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