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On Structured Robustness Analysis for Feedback Interconnections of Unstable Systems

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Abstract— **The purpose of this paper is to broaden the scope of integral-quadratic-constraint based *structured* robustness analysis in a way that accommodates feedback interconnections of *unstable* linear time-invariant systems. This is achieved by exploring the use of Vinnicombe's ν-gap metric as a measure of distance. Various standard robustness analysis problems are revisited within the context of the main result.**

Index Terms— **Feedback interconnections, robustness analysis, structured uncertainty, integral quadratic constraints (IQCs), ν-gap metric**

NOTATION

The symbols $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, respectively. $\mathbb{F}^{m\times q}$ denotes an m-row by q-column matrix with entries in $\mathbb{F}$ (e.g. $\mathbb{R}$ or $\mathbb{C}$). $\bar{\sigma}(X)$ and $\underline{\sigma}(X)$ respectively denote the maximum and minimum singular values of $X \in \mathbb{F}^{m\times q}$. A superscript T denotes matrix transpose, whereas $*$ denotes complex conjugate transpose. The determinant of a matrix $X \in \mathbb{F}^{m\times m}$ is denoted $\det(X)$, and when this is non-zero the inverse of X is denoted X^{-1}.

$\mathcal{R}^{m\times q}$ denotes the *proper* real rational transfer functions. All systems in this paper are considered to be multiplication operators with frequency domain symbols in $\mathcal{R}$; these correspond to linear shift-invariant systems in the time-domain. Given an $H \in \mathcal{R}^{m\times q}$, the conjugate transfer function $H^{\sim} \in \mathcal{R}^{q\times m}$ is defined by $H^{\sim}(s) := H(-s)^T$ (a.e.), so that $H^{\sim}(j\omega) = (H(j\omega))^*$ (a.e.). For notational convenience the input-output dimensions may be suppressed, as below. $\mathcal{RL}_\infty$ is the space of transfer functions $H \in \mathcal{R}$ that satisfy $\|H\|_\infty := \sup_{\omega\in\mathbb{R}\cup\infty} \bar{\sigma}(H(j\omega)) < \infty$. $\mathcal{RH}_\infty$ is the space of transfer functions $H \in \mathcal{RL}_\infty$ that are analytic (i.e. have no poles) in the open right-half plane $\mathbb{C}_+$. For $H \in \mathcal{RH}_\infty$, $\|H\|_\infty = \sup_{s\in\mathbb{C}_+} \bar{\sigma}(H(s))$.

The frequency-domain signal space $\mathcal{L}_2$ denotes the collection of functions $f : j\mathbb{R} \to \mathbb{C}^m$ (a.e.) for which $\|f\|_2 := \int_{-\infty}^{\infty} f(j\omega)^* f(j\omega)d\omega < \infty$. The corresponding inner product is denoted $\langle\cdot,\cdot\rangle_{\mathcal{L}_2}$. $\mathcal{L}_2$ is isometrically isomorphic (via the Fourier transform) to the space of finite-energy signals defined over the doubly-infinite time axis. The signal space $\mathcal{H}_2$ is the collection of $f \in \mathcal{L}_2$ that can be continued analytically into $\mathbb{C}_+$, with $\int_{-\infty}^{\infty} f(\sigma + j\omega)^* f(\sigma + j\omega)d\omega$ bounded uniformly for $\sigma > 0$. The corresponding inner product is denoted $\langle\cdot,\cdot\rangle_{\mathcal{H}_2}$. $\mathcal{H}_2$ is isometrically isomorphic (via the Fourier transform) to the space of finite-energy signals defined over only positive time.

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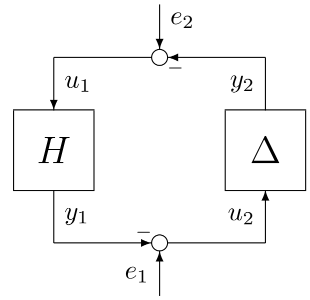

Fig. 1. Standard feedback interconnection

I. INTRODUCTION

Consider the feedback interconnection shown in Figure 1, where the two linear time-invariant systems involved have frequency-domain transfer functions $H, \Delta \in \mathcal{R}$. The closed-loop is said to be well-posed and stable if given any $e = (e_1, e_2) \in \mathcal{H}_2 \times \mathcal{H}_2$ there exists a unique $u = (u_1, u_2) \in \mathcal{H}_2 \times \mathcal{H}_2$ such that

$$\begin{aligned} u_2 &= -Hu_1 + e_1 \\ u_1 &= -\Delta u_2 + e_2 \end{aligned} \tag{1}$$

and

$$\|u\|_2 \le c\|e\|_2, \tag{2}$$

for some $c > 0$. Note that stability of the closed-loop is equivalent to the condition

$$\begin{pmatrix} H & I \\ I & \Delta \end{pmatrix}^{-1} \in \mathcal{RH}_\infty,$$

which is in turn equivalent to the condition

$$[H, \Delta] := \begin{pmatrix} H \\ I \end{pmatrix} (I - \Delta H)^{-1} \begin{pmatrix} -\Delta & I \end{pmatrix} \in \mathcal{RH}_\infty. \tag{3}$$

A condition for stability of the feedback interconnection of a given $H \in \mathcal{R}$ and any transfer function Δ in a given uncertainty set $\boldsymbol{\Delta} \subset \mathcal{R}$, is developed below. This is achieved in a way that permits exploitation of known *structure* on H and $\boldsymbol{\Delta}$, which does not appear to be immediately possible with the well-known gap-metric robustness results of [1], [2], while still accommodating *unstable* components in the feedback interconnection, by contrast with the standard integral quadratic constraint (IQC) analysis framework of [3], from where many key ingredients are taken. In short, the main purpose of this paper is to enrich the scope of IQC based structured robustness analysis for feedback interconnections of unstable systems by exploring the use of Vinnicombe's ν-gap metric as a distance measure [2]. The IQC and gap frameworks have previously been combined in general settings [4], [5], using general forms of the gap metric. Here the discussion is restricted to the simplest setting which allows easy use of the more convenient ν-gap. Several key ideas in robustness analysis, see e.g. [6], [7], [8], [9], [10], are revisited in this context.

It is instructive to first review some well-known characterisations of closed-loop stability in terms of coprime factorisations and graph symbols. Recall that any real rational transfer function admits normalised right and left coprime factorisations over $\mathcal{RH}_\infty$. In particular, given $H \in \mathcal{R}$, there exist (by construction – see [6]) transfer functions $N, M, \tilde{M}, \tilde{N}, X, \tilde{X}, Y, \tilde{Y} \in \mathcal{RH}_\infty$ such that

$$\begin{pmatrix} \tilde{X} & \tilde{Y} \\ -\tilde{M} & \tilde{N} \end{pmatrix} \begin{pmatrix} N & -X \\ M & Y \end{pmatrix} = I,$$

$M^\sim M + N^\sim N = I$, $\tilde{M}\tilde{M}^\sim + \tilde{N}\tilde{N}^\sim = I$ and

$$H = NM^{-1} = \tilde{M}^{-1}\tilde{N}.$$

Letting

$$G := \begin{pmatrix} N \\ M \end{pmatrix} \in \mathcal{RH}_\infty \quad \text{and} \quad \tilde{G} := \begin{pmatrix} -\tilde{M} & \tilde{N} \end{pmatrix} \in \mathcal{RH}_\infty,$$

it follows that [11]: $G^\sim G = I$; $\tilde{G}\tilde{G}^\sim = I$; and

$$\mathrm{gr}(H) = G\mathcal{H}_2 = \{w \in \mathcal{H}_2 \ : \ \tilde{G}w = 0\},$$

where the $\mathcal{H}_2$-graph of multiplication by H is defined by

$$\mathrm{gr}(H) := \left\{ \begin{pmatrix} y_1 \\ u_1 \end{pmatrix} \ : \ u_1 \in \mathcal{H}_2, \ y_1 = Hu_1 \in \mathcal{H}_2 \right\}.$$

The stable transfer functions G and $\tilde{G}$ are correspondingly called normalised right and left graph symbols for H. Now given normalised right and left coprime factorisations of a $\Delta = UV^{-1} = \tilde{V}^{-1}\tilde{U} \in \mathcal{R}$, let

$$\Gamma := \begin{pmatrix} V \\ U \end{pmatrix} \quad \text{and} \quad \tilde{\Gamma} := \begin{pmatrix} \tilde{U} & -\tilde{V} \end{pmatrix},$$

which are correspondingly normalised right and left (inverse) graph symbols for multiplication by Δ on $\mathcal{H}_2$.

Lemma 1: Given the notation just introduced the following are equivalent [11]:

1) The feedback interconnection of H and Δ is stable;
2) $[H, \Delta] \in \mathcal{RH}_\infty$;
3) $(\tilde{G}\Gamma)^{-1} \in \mathcal{RH}_\infty$;
4) $(\tilde{\Gamma}G)^{-1} \in \mathcal{RH}_\infty$.

Note that condition 3) can be expressed $\underline{\sigma}(\tilde{G}\Gamma)(j\omega) \neq 0 \ \forall \ \omega \in \mathbb{R} \cup \infty$ and $\mathrm{ind}(\tilde{G}\Gamma) = 0$, where the index of a transfer function $X, X^{-1} \in \mathcal{RL}_\infty$ is defined by

$$\mathrm{ind}(X) := \mathrm{wno}(\det(X))$$

and the winding-number $\mathrm{wno}(x)$ denotes the the net increase in the argument of the scalar transfer function $x(j\omega)$, as ω decreases from $+\infty$ to $-\infty$, choosing a continuous branch of the argument enclosing the open right-half plane of $\mathbb{C}$. Similarly, condition 4) can be expressed as $\underline{\sigma}(\tilde{\Gamma}G)(j\omega) \neq 0 \ \forall \ \omega \in \mathbb{R} \cup \infty$ and $\mathrm{ind}(\tilde{\Gamma}G) = 0$.

II. Gap-metric based robustness analysis

In the well-known gap-metric based robustness analysis of [1], [2], the uncertainty set $\mathbf{\Delta}$ is effectively taken to be a ball in $\mathcal{R}$ defined in terms of the gap or ν-gap metric, with there being no obvious way of exploiting any structure the systems H and Δ might have. In particular, the following is known to hold for the ν-gap metric, which is defined by

$$\delta_\nu(\Delta_0, \Delta_1) := \begin{cases} \|\tilde{\Gamma}_0\Gamma_1\|_\infty & \text{if } \underline{\sigma}(\Gamma_0^\sim\Gamma_1)(j\omega) \neq 0 \\ & \forall \, \omega \in \mathbb{R} \cup \infty \\ & \text{and } \mathrm{ind}(\Gamma_0^\sim\Gamma_1) = 0 \\ 1 & \text{otherwise} \end{cases},$$

where in keeping with the notation introduced above Γ_i and $\tilde{\Gamma}_i$ are normalised right and left (inverse) graph symbols for $\Delta_i \in \mathcal{R}$, $i = 0, 1$ [2]:

Proposition 2: Given $H, \Delta_0 \in \mathcal{R}$, with $[H, \Delta_0] \in \mathcal{RH}_\infty$, let $b(H, \Delta_0) := 1/\|[H, \Delta_0]\|_\infty$ (see (3) above). Then the following are equivalent:

1) $b(H, \Delta_0) > \, (\geq) \, \beta$;
2) $[H, \Delta_1] \in \mathcal{RH}_\infty$ for all $\Delta_1 \in \mathcal{R}$ satisfying $\delta_\nu(\Delta_0, \Delta_1) \leq \, (<) \, \beta$.

It is interesting to note that, under certain conditions, the conditions $b(H, \Delta_0) > \beta$ and $\delta_\nu(\Delta_0, \Delta_1) \leq \beta$ are related to complementary IQCs. Specifically (see [4], [12] for details):

- when $[H, \Delta_0] \in \mathcal{RL}_\infty \supset \mathcal{RH}_\infty$, the condition

$$b(H, \Delta_0) = \inf_{\omega \in \mathbb{R}} \underline{\sigma}(\tilde{\Gamma}_0 G)(j\omega) > \, (\geq) \, \beta$$

is equivalent to the existence of an $\epsilon > (\geq) \, 0$ such

$$\langle w, \Psi w \rangle_{\mathcal{L}_2} := \int_{-\infty}^{\infty} w(j\omega)^* \Psi(j\omega) w(j\omega) \, d\omega \leq \epsilon \langle w, w \rangle_{\mathcal{L}_2} \tag{4}$$

for all $w \in G\mathcal{L}_2$ (i.e. the $\mathcal{L}_2$-graph of multiplication by H), where

$$\Psi = \Psi^\sim := \begin{pmatrix} \beta I & \tilde{\Gamma}_0^\sim \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \beta I \\ \tilde{\Gamma}_0 \end{pmatrix} \in \mathcal{RL}_\infty; \tag{5}$$

- and when $\underline{\sigma}(\Gamma_0^\sim\Gamma_1)(j\omega) \neq 0 \, \forall \omega \in \mathbb{R} \cup \infty$ and $\mathrm{ind}(\Gamma_0^\sim\Gamma_1) = 0$, the condition

$$\delta_\nu(\Delta_0, \Delta_1) = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(\tilde{\Gamma}_0\Gamma_1)(j\omega) \leq \, (<) \, \beta$$

is equivalent to the existence of an $\epsilon \geq (>) \, 0$ such that

$$\langle v, \Psi v \rangle_{\mathcal{L}_2} \geq \epsilon \langle v, v \rangle_{\mathcal{L}_2} \tag{6}$$

for all $v \in \Gamma_1\mathcal{L}_2$ (i.e. the (inverse) $\mathcal{L}_2$-graph of multiplication by Δ_1).

Observe that the conditions (4) and (6) are complementary IQCs; that is, the quadratic forms on the left and right-hand sides are identical and the inequality constraints are complementary. While, under the restriction that the components of the feedback interconnection are stable, it is known how to exploit structure within the IQC framework of [3], it is not clear that use of the quadratic form with the particular Ψ in (5) would always be sufficiently flexible to achieve this. In Section III, an IQC based robust stability result, which is similar to Proposition 2 in that it also accommodates unstable components in the closed-loop, is established in a way that permits exploitation of known structure.

III. IQC BASED ROBUSTNESS ANALYSIS

In this section an IQC robustness analysis result is presented for feedback interconnections that may involve unstable components, followed by a brief discussion of how structure can be exploited.

Definition 1 (Integral Quadratic Constraints): Let $\Psi = \Psi^{\sim} \in \mathcal{L}_\infty$. Given an $\Delta \in \mathcal{R}$, it is said that $\Delta \in \mathrm{IQC}(\Psi)$ if

$$\sigma_\Psi(v) := \langle v, \Psi v\rangle_{\mathcal{L}_2} \geq 0 \quad \forall\, v \in \Gamma\mathcal{H}_2,$$

where Γ is a (not necessarily normalised) right (inverse) graph symbol for Δ. The IQC is said to be strict (denoted $\Delta \in \mathrm{SIQC}(\Psi)$) if there exists an $\epsilon > 0$ such that

$$\sigma_\Psi(v) \geq \epsilon\|v\|_2^2 \quad \forall\, v \in \Gamma\mathcal{H}_2.$$

Likewise, given a $H \in \mathcal{R}$, the complementary condition $H \in \mathrm{IQC}^c(\Psi)$ is said to hold if $\sigma_\Psi(w) \leq 0\,\forall\, w \in G\mathcal{H}_2$, where G is a (not necessarily normalised) right graph symbol for H, and hold strictly if $\sigma_\Psi(w) \leq -\epsilon\|w\|_2^2\,\forall\, w \in G\mathcal{H}_2$.

The following uncertainty set will be considered.

Definition 2 (Uncertainty Set): Let $\boldsymbol{\Delta}$ denote a subset of $\mathcal{R}$ that is connected in the topology induced by the ν-gap metric (i.e. the graph topology) in the sense that for any $\eta \in (0,1)$ and $\Delta_a, \Delta_b \in \boldsymbol{\Delta}$, there exists $\Delta_k \in \boldsymbol{\Delta}$, for $k = 0, \ldots, N$ and some integer $N > 0$, such that $\Delta_0 = \Delta_a$, $\Delta_N = \Delta_b$ and $\delta(\Delta_k, \Delta_{k+1}) \leq \eta\,\forall\, k = 0, \ldots, N-1$.

Proposition 3 (Main Result): Given $H \in \mathcal{R}$, suppose there exists a $\Delta_0 \in \boldsymbol{\Delta}$ such that the feedback interconnection $[H, \Delta_0] \in \mathcal{RH}_\infty$ (i.e. it is stable). If there exists a $\Psi = \Psi^{\sim} \in \mathcal{L}_\infty$ such that

(i) $H \in \mathrm{IQC}^c(\Psi)$, and
(ii) $\Delta \in \mathrm{SIQC}(\Psi)\,\forall\,\Delta \in \boldsymbol{\Delta}$,

then the closed-loop $[H, \Delta] \in \mathcal{RH}_\infty$ (i.e. it is stable) for all $\Delta \in \boldsymbol{\Delta}$.

Proof: A proof is provided in Section VII ■

Remark 1: One may replace (i) and (ii) in Proposition 3 by the alternative conditions

(I) $H \in \mathrm{SIQC}^c(\Psi)$, and
(II) $\Delta \in \mathrm{IQC}(\Psi)\,\forall\,\Delta \in \boldsymbol{\Delta}$,

It is important to observe that if $\Delta \in \mathrm{SIQC}(\Psi_i)$, $i = 1, \cdots, n$, then $\Delta \in \mathrm{SIQC}(x_1\Psi_1 + \cdots + x_n\Psi_n)$ for any scalar $x_i \geq 0$, $i = 1, \cdots, n$. Furthermore, when Δ is diagonally structured; i.e., $\Delta = \mathrm{diag}(\Delta_1, \cdots, \Delta_n)$, and $\Delta_i \in \mathrm{SIQC}(\Upsilon_i)$, $i = 1, \cdots, n$, then an IQC for Δ can be easily obtained by composing Υ_i appropriately. More specifically, if we introduce a block partition consistent with the input output dimensions of Δ_i as

$$\Upsilon_i = \begin{bmatrix} \Upsilon_{i,(11)} & \Upsilon_{i,(12)} \\ \Upsilon_{i,(12)}^{\sim} & \Upsilon_{i,(22)} \end{bmatrix}$$

and let the diagonal augmentation operator Υ be defined as

$$\mathrm{daug}(\Upsilon_1, \cdots, \Upsilon_n) = \left[\begin{array}{c|c} \mathrm{diag}(\Upsilon_{1,(11)}, \cdots, \Upsilon_{n,(11)}) & \mathrm{diag}(\Upsilon_{1,(12)}, \cdots, \Upsilon_{n,(12)}) \\ \hline \mathrm{diag}(\Upsilon_{1,(12)}^{\sim}, \cdots, \Upsilon_{n,(12)}^{\sim}) & \mathrm{diag}(\Upsilon_{1,(22)}, \cdots, \Upsilon_{n,(22)}) \end{array}\right]$$

then $\Delta \in \mathrm{SIQC}(\Upsilon)$. These features of IQC based analysis allows one to breakdown the task of characterizing Δ into smaller (and often easier) tasks of characterizing the elementary building blocks of Δ, which provides great flexibility in exploiting the structure of Δ for stability analysis. Once the IQCs for these building blocks are available, stability analysis for the interconnected system is then a rather straightforward matter of finding a single aggregate Ψ such that condition $H \in \mathrm{IQC}^c(\Psi)$ holds.

IV. VERIFICATION OF THE STABILITY CONDITION

Suppose a structural characterisation of the uncertainty set $\boldsymbol{\Delta}$ is obtained in terms of a set $\Psi_{\boldsymbol{\Delta}}$, i.e. each $\Psi \in \Psi_{\boldsymbol{\Delta}}$ has the property that $\Delta \in \mathrm{IQC}(\Psi)$, $\forall\Delta \in \boldsymbol{\Delta}$. Then by Proposition 3, the interconnection $[H, \Delta]$ is robustly stable if there exists a $\Psi \in \Psi_{\boldsymbol{\Delta}}$ such that $H \in \mathrm{SIQC}^c(\Psi)$.

There are several equivalent ways to verify $H \in \mathrm{SIQC}^c(\Psi)$ ($H \in \mathrm{IQC}^c(\Psi)$): $\exists\epsilon > 0$ ($\exists\epsilon \geq 0$) such that

(a) $G(j\omega)^*\Psi(j\omega)G(j\omega) \leq -\epsilon I,\ \forall\omega \in \mathbb{R}$
(b)

$$\left((I + H^{\sim}H)^{-\frac{1}{2}} \begin{bmatrix} H \\ I \end{bmatrix}^{\sim} \Psi \begin{bmatrix} H \\ I \end{bmatrix} (I + H^{\sim}H)^{-\frac{1}{2}}\right)(j\omega) \leq -\epsilon I, \quad \forall\omega \in \mathbb{R}$$

(c) If $H \in \mathcal{RL}_\infty$ then (b) simplifies to

$$\begin{bmatrix} H(j\omega) \\ I \end{bmatrix}^* \Psi(j\omega) \begin{bmatrix} H(j\omega) \\ I \end{bmatrix} \leq -\epsilon' I,\ \forall\omega \in \mathbb{R}$$

where $\epsilon' = \epsilon \sup_{\omega\in\mathcal{R}} \bar{\sigma}(I + H(j\omega)^*H(j\omega))$. The case with a $H \in \mathcal{R}$ with poles on the imaginary axis requires some care.

The condition $\Delta \in \mathrm{IQC}(\Psi)$ can be verified analogously.

The next result is a version of the dualization lemma in [8], [13]. The two complementary IQCs in Proposition 3 imply that the $\mathcal{L}_2$ graphs of H and Δ provide a direct sum decomposition of the signal space of the system in Figure 1. This in turn implies that the orthognal complements of the two $\mathcal{L}_2$ graphs satisfy a corresponding pair of complementary IQCs defined by the inverse multiplier. This can be beneficial in terms of verifying the IQC conditions.

Proposition 4: Suppose $\Psi = \Psi^{\sim} \in \mathcal{L}_\infty$ is invertible. Then the following are equivalent conditions. There exists $\epsilon > 0$ such that for all $\omega \in \mathbb{R}$.

(a) $(G^*\Psi G)(j\omega) \leq -\epsilon I$ and $(\Gamma^*\Psi\Gamma)(j\omega) \geq 0$,
(b) $(\tilde{G}\Psi^{-1}\tilde{G}^*)(j\omega) \geq 0$ and $(\tilde{\Gamma}\Psi^{-1}\tilde{\Gamma}^*)(j\omega) \leq -\epsilon I$

V. Examples

Example 1: Consider the case where Δ is a block-diagonal rational transfer function in

$$\boldsymbol{\Delta}(\mu) = \{\mathrm{diag}(\Delta_1, \ldots, \Delta_m) : \delta_\nu(\Delta_{k,0}, \Delta_k) \leq \mu, \ \forall k\}$$

where $\Delta_0 = \mathrm{diag}(\Delta_{1,0}, \ldots, \Delta_{m,0})$ is the nominal dynamics and $[H, \Delta_0]$ is known to be stable. This set is obviously connected in the ν-gap topology in the sense of Definition 2. The robust stability analysis problem is to find a bound on μ for which $[H, \Delta]$ is stable (with uniform norm bound) for all $\Delta \in \boldsymbol{\Delta}$. By proceeding along the lines of the previous discussion we know that each uncertainty satisfies $\Delta_k \in \mathrm{IQC}(\Psi_k)$ with $\Psi_k = \mu I - \tilde{\Gamma}_{k,0}^{\sim}\tilde{\Gamma}_{k,0}$. Combining these would in general lead to conservative analysis. Instead, we use scaling multipliers and define

$$\Psi_{\boldsymbol{\Delta}_k}(\mu) = \left\{ x(\mu I - \tilde{\Gamma}_{k,0}^{\sim}\tilde{\Gamma}_{k,0}) : x = x^{\sim} \geq 0 \right\} \subset \mathcal{RL}_\infty$$

Then every $\Delta \in \boldsymbol{\Delta}$ satisfies $\Delta \in \mathrm{IQC}(\Psi)$, $\forall \Psi \in \Psi_{\boldsymbol{\Delta}}(\mu)$, where

$$\Psi_{\boldsymbol{\Delta}}(\mu) = \{\mathrm{daug}(\Psi_1, \ldots, \Psi_m) : \Psi_k \in \Psi_{\boldsymbol{\Delta}_k}(\mu)\}.$$

The operator daug is defined in Section III. Given this IQC characterization the maximum robustness margin is then obtained as

$$\max\{\mu : (G^{\sim}\Psi G)(j\omega) \leq -\epsilon I : \Psi \in \Psi_{\boldsymbol{\Delta}}(\mu), \epsilon > 0\}.$$

Example 2: Consider a set of transfer functions $\boldsymbol{\Delta} \subset \mathcal{R}$. Suppose that a Δ_0 from $\boldsymbol{\Delta}$ can be stabilized by a proportional feedback of constant gain κ which belongs to the interval $[\alpha, \beta]$, $\alpha < \beta < 0$. That is, $[H, \Delta_0] \in \mathcal{RH}_\infty$, where $H := \kappa I$. Furthermore, suppose that, for any $\Delta \in \boldsymbol{\Delta}$,

$$\mathrm{ind}(I + \Delta^{\sim}\Delta_0) + \eta(\Delta_0) - \bar{\eta}(\Delta) = 0, \tag{7}$$

where $\eta(\Delta_0)$ is the number of open right half plane poles of Δ_0 and $\bar{\eta}(\Delta)$ is the number of closed right half plane poles of Δ. This means that the contour evaluation of $\Delta^{\sim}\Delta_0$ will encircle the critical -1 point $\bar{\eta}(\Delta) - \eta(\Delta_0)$ times in the positive direction. Condition (7) ensures that $\boldsymbol{\Delta}$ is connected in the ν-gap metric, provided $\det(I + \Delta^{\sim}\Delta_0)(j\omega) \neq 0 \, \forall\omega$.

That κ belongs to $[\alpha, \beta]$ implies that $H \in \mathrm{IQC}^c(\Psi)$ for any Ψ from Ψ_κ, where

$$\begin{aligned}\Psi_\kappa := \{\Psi = \Psi^{\sim} \in \mathcal{L}_\infty : \ & \Psi(j\omega) \text{ has the form} \\ & \begin{bmatrix} 2X & -(\alpha+\beta)X + Y \\ -(\alpha+\beta)X + Y^* & 2\alpha\beta X \end{bmatrix}; \\ & X = X^* \geq 0; \ Y + Y^* = 0\}.\end{aligned}$$

Following Proposition 3, we can conclude that H stabilizes every $\Delta \in \boldsymbol{\Delta}$ if we can find $\Psi \in \Psi_\kappa$ such that $\Delta \in \mathrm{SIQC}(\Psi)$ for all $\Delta \in \boldsymbol{\Delta}$. Convex duality theory can be used to show that such Ψ exists if and only if each $\Delta \in \boldsymbol{\Delta}$ satisfies

$$\mathrm{eig}(\Delta(j\omega)) \notin \left[\frac{1}{\beta}, \frac{1}{\alpha}\right], \quad \forall\omega \in \mathbb{R} \cup \infty \tag{8}$$

where $\mathrm{eig}(\cdot)$ denotes the eigenvalues of a matrix. The proof of this follows as in Example 2 of [14]. Condition (8) gives rise to a simple and low-complexity graphical test for stability of feedback interconnection of H and any Δ in $\boldsymbol{\Delta}$. The stability is robust for any constant κ in the interval $[\alpha, \beta]$.

VI. Robust Performance Analysis

The purpose of this section is to briefly review the basic ideas of robust performance analysis within the framework developed in the previous sections. Such analysis generally involves three different quadratic forms:

1) a performance constraint of the form

$$\sigma_p(y, e) = \langle (y, e), \Psi_p(y, e) \rangle_{\mathcal{L}_2} \leq 0, \ \forall e \in \mathcal{E}, \ \Delta \in \boldsymbol{\Delta}$$

where $y := (y_1, y_2) = (e_1 - u_2, e_2 - u_1)$ is the closed-loop response of the system (1) for a given $H \in \mathcal{R}$ and $\mathcal{E} \subset \mathcal{L}_2$ is a set of disturbance/noise signals;

2) a noise characterisation in the form of a convex cone $\Psi_{\mathcal{E}}$ such that

$$\sigma_n(e) = \langle e, \Psi_n e \rangle_{\mathcal{L}_2} \geq 0, \ \forall e \in \mathcal{E}, \ \Psi_n \in \Psi_{\mathcal{E}};$$

3) an uncertainty characterisation in the form of a convex cone $\Psi_{\boldsymbol{\Delta}}$ such that $\Delta \in \mathrm{IQC}(\Psi_\Delta)$, $\forall \Delta \in \boldsymbol{\Delta}$, $\Psi_\Delta \in \Psi_{\boldsymbol{\Delta}}$.

Characterisations 2) and 3) are denoted by $\mathcal{E} \in \mathrm{IQC}(\Psi_{\mathcal{E}})$ and $\boldsymbol{\Delta} \in \mathrm{IQC}(\Psi_{\boldsymbol{\Delta}})$, respectively, in the rest of this section.

A typical example of a performance constraint is the weighted induced gain constraint defined using

$$\Psi_p(j\omega) = \begin{bmatrix} W(j\omega) & 0 \\ 0 & -\gamma^2 I \end{bmatrix},$$

where $W = W^{\sim} \in \mathcal{RL}_\infty$ is positive semi-definite. An example of a signal set is the set of scalar signals that have white spectrum over the frequency range $-b \leq \omega \leq b$; i.e.

$$|\hat{e}(j\omega)|^2 = \begin{cases} \frac{\pi}{b}\|e\|_2^2, & \omega \in [-b, b] \\ 0, & |\omega| > b \end{cases}.$$

This set can be exactly characterized using the set of multipliers

$$\Psi_{\mathcal{E}} = \left\{ \Psi_n = \Psi_n^{\sim} \in \mathcal{RL}_\infty : \int_{-b}^{b} \mathrm{tr}(\Psi_n(j\omega)) d\omega \geq 0 \right\}$$

This and more general signal classes are discussed in [15].

Definition 3 (Robust performance): Given $H \in \mathcal{R}$, the feedback interconnection (1) exhibits robust performance if

(a) the closed-loop $[H, \Delta]$ is stable $\forall \Delta \in \boldsymbol{\Delta}$ and

(b) the closed-loop performance constraint $\sigma_p(y, e) \leq 0$ holds for all input-output pairs (y, e) (where $y = (e_1 - u_2, e_2 - u_1)$) whenever $e \in \mathcal{E}$ and $\Delta \in \boldsymbol{\Delta}$.

Let

$$M_1 := \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad M_2 := \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

and for given Ψ_p, $\Psi_\Delta \in \Psi_{\boldsymbol{\Delta}}$, and $\Psi_n \in \Psi_{\mathcal{E}}$, define

$$\Psi := \begin{bmatrix} M_1 G & M_2 \\ 0 & I \end{bmatrix}^{\sim} \Psi_p \begin{bmatrix} M_1 G & M_2 \\ 0 & I \end{bmatrix} + \begin{bmatrix} G^{\sim}\Psi_\Delta G & -G^{\sim}\Psi_\Delta \\ -\Psi_\Delta G & \Psi_\Delta + \Psi_n \end{bmatrix},$$

where G is a normalised right graph symbol for H.

Proposition 5: Given $H \in \mathcal{R}$ and $\Delta_0 \in \mathbf{\Delta} \subset \mathcal{R}$, suppose that $[H, \Delta_0] \in \mathcal{RH}_\infty$, $\mathcal{E} \in \mathrm{IQC}(\Psi_\mathcal{E})$ and $\mathbf{\Delta} \in \mathrm{IQC}(\Psi_\mathbf{\Delta})$. If there exists an $\epsilon > 0$, $\Psi_\Delta \in \Psi_\mathbf{\Delta}$ and $\Psi_n \in \Psi_\mathcal{E}$ such that

(i)

$$\Psi(j\omega) + \begin{bmatrix} \epsilon I & 0 \\ 0 & 0 \end{bmatrix} \leq 0, \ \forall \omega \in \mathbb{R} \quad \text{and}$$

(ii)

$$\begin{bmatrix} M_1 G(j\omega) \\ 0 \end{bmatrix}^* \Psi_p(j\omega) \begin{bmatrix} M_1 G(j\omega) \\ 0 \end{bmatrix} \geq 0, \quad \forall \omega \in \mathbb{R},$$

then the feedback interconnection (1) satisfies the robust performance specification in Definition 3.

Proof: Condition (a) in Definition 3 holds if

$$\sigma_p(M_1 w + M_2 e, e) \leq 0, \quad \forall w \in G\mathcal{L}_2, e \in \mathcal{L}_2$$
$$s.t. \begin{cases} \sigma_{\Psi_\Delta}(e - w) \geq 0 \\ \sigma_{\Psi_n}(e) \geq 0 \end{cases}$$

By S-procedure relaxation (see [16]), this is equivalent to the existence of $\tau_1, \tau_2 \geq 0$ such that

$$\sigma_p(M_1 G\nu + M_2 e, e) + \tau_1 \sigma_{\Psi_\Delta}(e - G\nu) + \tau_2 \sigma_{\Psi_n}(e) \leq 0$$

for all $\nu, e \in \mathcal{L}_2$. Since $\Psi_\mathbf{\Delta}$ and $\Psi_\mathcal{E}$ are cones we can without loss of generality assume $\tau_1 = \tau_2 = 1$. This inequality is implied by condition (i) in the statement of the proposition. Condition (ii) implies that $(G^\sim \Psi_\Delta G)(j\omega) \leq -\epsilon I$, $\forall \omega \in \mathbb{R}$, which in turn implies robust stability according to Proposition 3. ■

VII. Proofs of Propositions 3 and 4

Proofs of Propositions 3 and 4 are developed in the following subsections. In particular, Section VII-A gathers some useful identities. Given these and using ideas from [2] and [3], proofs are then presented for Propositions 3 and 4 in Sections VII-B and VII-C, respectively.

A. Useful identities

This section introduces some additional notation and gathers a few preliminary useful properties of normalised graph symbols and winding-numbers. The first property derives from normalisation. Given *normalised* right and left (inverse) graph symbols $\Gamma \in \mathcal{RH}_\infty$ and $\tilde{\Gamma} \in \mathcal{RH}_\infty$, respectively, of a transfer function matrix $\Delta \in \mathcal{R}$, the following identities hold by virtue of $\tilde{\Gamma}\Gamma = 0$, $\tilde{\Gamma}\tilde{\Gamma}^\sim = I$ and $\Gamma^\sim \Gamma = I$:

$$\begin{pmatrix} \tilde{\Gamma} \\ \Gamma^\sim \end{pmatrix} \begin{pmatrix} \tilde{\Gamma}^\sim & \Gamma \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \text{ and } \tilde{\Gamma}^\sim \tilde{\Gamma} + \Gamma \Gamma^\sim = I. \quad (9)$$

Like in [2], the identity (9) is used at various points in the proof of the main result. In particular, given normalised right and left (inverse) graph symbols $\Gamma_k \in \mathcal{RH}_\infty$ and $\tilde{\Gamma}_k \in \mathcal{RH}_\infty$, respectively, for $\Delta_k \in \mathcal{R}$ and $k = 0, \ldots, N$, it follows that $\Gamma_{k+1}^\sim \tilde{\Gamma}_k^\sim \tilde{\Gamma}_k \Gamma_{k+1} + \Gamma_{k+1}^\sim \Gamma_k \Gamma_k^\sim \Gamma_{k+1} = I$, by which

$$\underline{\gamma}(\Gamma_k^\sim \Gamma_{k+1}) = \sqrt{1 - \bar{\gamma}(\tilde{\Gamma}_k \Gamma_{k+1})^2}, \quad (10)$$

where

$$\underline{\gamma}(X) := \inf_{\omega \in \mathbb{R} \cup \infty} \underline{\sigma}(X)(j\omega) \text{ and } \bar{\gamma}(X) := \sup_{\omega \in \mathbb{R} \cup \infty} \bar{\sigma}(X)(j\omega).$$

Before proceeding to the proof of Proposition 3, it is also instructive to recall the following winding-number/index identities:

Lemma 6 (See pg. 16 of [11]): For $X, X^{-1}, Y \in \mathcal{RL}_\infty$

(i) $\mathrm{ind}(XY) = \mathrm{ind}(X) + \mathrm{ind}(Y)$,
(ii) $\mathrm{ind}(X^*) = -\mathrm{ind}(X)$,
(iii) $\mathrm{ind}(X^{-1}) = -\mathrm{ind}(X)$ and
(iv) if $\underline{\gamma}(X) > \bar{\gamma}(Y)$, then $\mathrm{ind}(X + Y) = \mathrm{ind}(X)$.

B. Proof of Proposition 3

Suppose that a $\Psi = \Psi^\sim \in \mathcal{RL}_\infty$ exists such that $H \in \mathrm{IQC}^c(\Psi)$ and $\Delta \in \mathrm{SIQC}(\Psi)$ for all $\Delta \in \mathbf{\Delta}$. With $\hat{\Psi} := 2\Psi - \epsilon I$ for an appropriate $\epsilon > 0$, these two conditions become

1) $\langle w, \hat{\Psi} w \rangle_{\mathcal{L}_2} \leq -\epsilon \|w\|_2^2 \, \forall \, w \in G\mathcal{H}_2$
2) For each $\Delta \in \mathbf{\Delta}$, $\langle v, \hat{\Psi} v \rangle_{\mathcal{L}_2} \geq \epsilon \|v\|_2^2 \, \forall \, v \in \Gamma \mathcal{H}_2$,

where $G \in \mathcal{RH}_\infty$ and $\Gamma \in \mathcal{RH}_\infty$ denote normalised right graph symbols for H and Δ, respectively. Using this notation, for any $(w, v) \in G\mathcal{H}_2 \times \Gamma\mathcal{H}_2$ and any $\Delta \in \mathbf{\Delta}$, it follows that

$$\begin{aligned} \epsilon(\|v\|_2^2 + \|w\|_2^2) & \\ &\leq \langle v, \hat{\Psi} v \rangle_{\mathcal{L}_2} - \langle w, \hat{\Psi} w \rangle_{\mathcal{L}_2} \\ &= \langle w + v, \hat{\Psi}(w + v) \rangle_{\mathcal{L}_2} - 2\mathrm{Re}\langle w, \hat{\Psi}(w + v) \rangle_{\mathcal{L}_2} \\ &\leq \|\hat{\Psi}\|_\infty \|w + v\|_2^2 + 2\|\hat{\Psi}\|_\infty \|w\|_2 \|w + v\|_2 \\ &\leq \|\hat{\Psi}\|_\infty \|w + v\|_2^2 + \frac{2\|\hat{\Psi}\|_\infty^2 \|w + v\|_2^2}{\epsilon} + \frac{\epsilon}{2}\|w\|_2^2 \end{aligned}$$

where the last inequality holds because $2xy \leq x^2 + y^2$ for any real numbers x and y. This inequality then implies

$$(1 + \frac{2}{\epsilon}\|\Psi\|_\infty)\|\Psi\|_\infty \|w + v\|_2^2 \geq \frac{\epsilon}{2}(\|v\|_2^2 + \|w\|_2^2).$$

Correspondingly, for all $(w, v) \in G\mathcal{H}_2 \times \Gamma\mathcal{H}_2$,

$$\|w + v\|_2^2 \geq \delta(\epsilon)^2 (\|v\|_2^2 + \|w\|_2^2) \quad (11)$$

with $\delta(\epsilon) := \sqrt{\frac{\epsilon^2}{2\|\hat{\Psi}\|_\infty(\epsilon + 2\|\hat{\Psi}\|_\infty)}}$. Since $(w, v) \in G\mathcal{H}_2 \times \Gamma\mathcal{H}_2$, this in turn implies

$$\begin{aligned} \sigma(q) := \|Gq_1 + \Gamma q_2\|_2^2 = \left\langle q, \begin{pmatrix} G^\sim \\ \Gamma^\sim \end{pmatrix} \begin{pmatrix} G & \Gamma \end{pmatrix} q \right\rangle_{\mathcal{L}_2} \quad (12) \\ \geq \delta(\epsilon)^2 \langle q, q \rangle_{\mathcal{H}_2} \end{aligned}$$

for all $q = (q_1, q_2) \in \mathcal{H}_2 \times \mathcal{H}_2$. Note that the quadratic forms involved here are 'shift-invariant' on $\mathcal{L}_2$. As such, (12) implies

$$\left\langle q, \begin{pmatrix} G^\sim \\ \Gamma^\sim \end{pmatrix} \begin{pmatrix} G & \Gamma \end{pmatrix} q \right\rangle_{\mathcal{L}_2} \geq \delta(\epsilon)^2 \langle q, q \rangle_{\mathcal{L}_2} \quad (13)$$

for all $q = (q_1, q_2) \in \mathcal{L}_2 \times \mathcal{L}_2$, since $\sigma(e^{-s\tau} q) = \sigma(q) \, \forall \, \tau < 0$, $q \in \mathcal{H}_2 \times \mathcal{H}_2$, $\cup_{\tau > -\infty} e^{-s\tau} \mathcal{H}_2$ is dense in $\mathcal{L}_2$ and $\sigma(\cdot)$ is continuous on $\mathcal{L}_2$.

Now from (13) it follows that, for all $q = (q_1, q_2) \in \mathcal{L}_2 \times \mathcal{L}_2$ satisfying $\|q_1\|_2 = \|q_2\|_2 = 1$,

$$\begin{aligned} & 2 + 2\,\mathrm{Re}\langle q_1, G^{\sim}\Gamma q_2\rangle \geq 2\delta(\epsilon)^2 \\ \Leftrightarrow\ & 1 - \sup_{\|q_2\|_2=1} \|G^{\sim}\Gamma q_2\|_2^2 \geq \delta(\epsilon)^2 \\ \Leftrightarrow\ & \inf_{\|q_2\|_2=1} \|\tilde{G}\Gamma q_2\|_2^2 \geq \delta(\epsilon)^2, \end{aligned}$$

where the equivalence of the first and second expressions holds because $q_1 = -G^{\sim}\Gamma q_2$ minimises $\langle q_1, G^{\sim}\Gamma q_2\rangle$ over the unit vectors in $\mathcal{L}_2$, and the final equivalence holds in a similar way to (10), using the fact that

$$\Gamma^{\sim}\Gamma = \Gamma^{\sim}(GG^{\sim} + \tilde{G}^{\sim}\tilde{G})\Gamma = I.$$

Note that the following uniform bound has been obtained:

$$\underline{\gamma}(\tilde{G}\Gamma) = \inf_{\|q_2\|_2=1} \|\tilde{G}\Gamma q_2\|_2 \geq \delta(\epsilon) \neq 0\,\forall\,\Delta \in \mathbf{\Delta}. \quad (14)$$

In view of the remarks following Lemma 1 it only remains to show that $\mathrm{ind}(\tilde{G}\Gamma) = 0$, since then $(\tilde{G}\Gamma)^{-1} \in \mathcal{RH}_\infty$, and thus, $[H, \Delta] \in \mathcal{RH}_\infty$ for all $\Delta \in \mathbf{\Delta}$. The remainder of this section is dedicated to establishing this fact by exploiting the hypothesis that the uncertainty set $\mathbf{\Delta}$ is connected in the graph topology.

By hypothesis there exists $\Delta_0 \in \mathbf{\Delta}$ such that $[H, \Delta_0] \in \mathcal{RH}_\infty$ and, for any $\eta \in (0, 1)$ and $\Delta \in \mathbf{\Delta}$, there exists an $N > 0$ and $\Delta_k \in \mathbf{\Delta}$, $k = 1, \ldots, N$, such that $\delta_\nu(\Delta_k, \Delta_{k+1}) \leq \eta$ and $\Delta_N = \Delta$. Let Γ_k and $\tilde{\Gamma}_k$ denote normalised right and left (inverse) graph symbols for Δ_k, $k = 0, \ldots, N$. The proof can proceed by induction, as described below.

$[H, \Delta_0] \in \mathcal{RH}_\infty$ by hypothesis. Suppose $[H, \Delta_k] \in \mathcal{RH}_\infty$, by which $\underline{\gamma}(\tilde{G}\Gamma_k) > 0$ (i.e. $\underline{\sigma}(\tilde{G}\Gamma_k)(j\omega) \neq 0\,\forall\,\omega \in \mathbb{R} \cup \infty$) and $\mathrm{ind}(\tilde{G}\Gamma_k) = 0$. Using (9) it follows that

$$\tilde{G}\Gamma_{k+1} = \tilde{G}\Gamma_k\Gamma_k^{\sim}\Gamma_{k+1} + \tilde{G}\tilde{\Gamma}_k^{\sim}\tilde{\Gamma}_k\Gamma_{k+1}. \quad (15)$$

Consider the first term on the right-hand side (15). From (14) and (10),

$$\begin{aligned} \underline{\gamma}(\tilde{G}\Gamma_k\Gamma_k^{\sim}\Gamma_{k+1}) &\geq \underline{\gamma}(\tilde{G}\Gamma_k)\underline{\gamma}(\Gamma_k^{\sim}\Gamma_{k+1}) \\ &\geq \delta(\epsilon)\sqrt{1 - \overline{\gamma}(\tilde{\Gamma}_k\Gamma_{k+1})^2} \\ &\geq \delta(\epsilon)\sqrt{1 - \delta_\nu(\Delta_k, \Delta_{k+1})^2}, \end{aligned}$$

where the last inequality holds by the definition of the ν-gap metric and since $\eta < 1$, by which $\underline{\sigma}(\Gamma_k^{\sim}\Gamma_{k+1})(j\omega) \neq 0\,\forall\,\omega \in \mathbb{R} \cup \infty$, $\mathrm{ind}(\Gamma_k^{\sim}\Gamma_{k+1}) = 0$, and hence, $\delta_\nu(\Delta_k, \Delta_{k+1}) = \overline{\gamma}(\tilde{\Gamma}_k\Gamma_{k+1})$. Moreover, using Lemma 6

$$\mathrm{ind}(\tilde{G}\Gamma_k\Gamma_k^{\sim}\Gamma_{k+1}) = \mathrm{ind}(\tilde{G}\Gamma_k) + \mathrm{ind}(\Gamma_k^{\sim}\Gamma_{k+1}) = 0. \quad (16)$$

Now consider the second term on the right-hand side of (15). Since the graph symbols are normalised, it follows that

$$\overline{\gamma}(\tilde{G}\tilde{\Gamma}_k^{\sim}\tilde{\Gamma}_k\Gamma_{k+1}) \leq \overline{\gamma}(\tilde{\Gamma}_k\Gamma_{k+1}) = \delta_\nu(\Delta_k, \Delta_{k+1}).$$

Thus, for $\eta < \delta(\epsilon) \Big/ \sqrt{1 + \delta(\epsilon)^2}$ the first term in (15) dominates the second and so by Lemma 6

$$\mathrm{ind}(\tilde{G}\Gamma_{k+1}) = \mathrm{ind}(\tilde{G}\Gamma_k\Gamma_k^{\sim}\Gamma_{k+1}) = 0,$$

where (16) has been used. This completes the proof.

C. Proof for Proposition 4

Let $\hat{\Psi} = \Psi + \nu I$, where $0 < \nu < \epsilon$. Then the two conditions in (a) can be stated as $\langle w, \hat{\Psi}w\rangle_{\mathcal{L}_2} \leq -\epsilon_1\|w\|_2^2\,\forall\,w \in G\mathcal{L}_2$ and $\langle v, \hat{\Psi}v\rangle_{\mathcal{L}_2} \geq \epsilon_2\|v\|_2^2$, $\forall\,v \in \Gamma\mathcal{L}_2$, where $\epsilon_1 = \epsilon - \nu$ and $\epsilon_2 = \nu$ (the same conditions hold on $\mathcal{H}_2$). Using the arguments in the proof of Proposition 3 it follows that (a) implies (13) in Section VII. This in turn implies the direct sum decomposition $\mathcal{L}_2 \times \mathcal{L}_2 = G\mathcal{L}_2 \oplus \Gamma\mathcal{L}_2$. The equivalence $(a) \Leftrightarrow (b)$ will follow since $\hat{\Psi}^{-1}\tilde{G}^{\sim}\mathcal{L}_2 = \Gamma\mathcal{L}_2$ and $\hat{\Psi}^{-1}\tilde{\Gamma}^{\sim}\mathcal{L}_2 = G\mathcal{L}_2$, which will be proven next. The inclusion $\hat{\Psi}^{-1}\tilde{G}^{\sim}\mathcal{L}_2 \subset \Gamma\mathcal{L}_2$ follows since if $\hat{\Psi}^{-1}\tilde{G}^{\sim}\mathcal{L}_2 \cap G\mathcal{L}_2 \neq 0$ then $\hat{\Psi}^{-1}\tilde{G}^{\sim}v_0 = Gw_0$ for some nonzero $v_0, w_0 \in \mathcal{L}_2$. This would imply that

$$-\epsilon_1\|w_0\|_2^2 \geq \langle Gw_0, \hat{\Psi}Gw_0\rangle_{\mathcal{L}_2} = \langle \hat{\Psi}^{-1}\tilde{G}^{\sim}v_0, \hat{\Psi}Gw_0\rangle_{\mathcal{L}_2} = 0$$

which is a contradiction. The inclusion $\hat{\Psi}^{-1}\tilde{\Gamma}^{\sim}\mathcal{L}_2 \subset G\mathcal{L}_2$ is proven similarly. Notice further that $(G\mathcal{L}_2)^{\perp} = \tilde{G}^{\sim}\mathcal{L}_2$ and $(\Gamma\mathcal{L}_2)^{\perp} = \tilde{\Gamma}^{\sim}\mathcal{L}_2$, which implies that $\tilde{G}^{\sim}\mathcal{L}_2 \oplus \tilde{\Gamma}^{\sim}\mathcal{L}_2 = \mathcal{L}_2 \times \mathcal{L}_2$. Hence, we have shown $\hat{\Psi}^{-1}\tilde{G}^{\sim}\mathcal{L}_2 = \Gamma\mathcal{L}_2$ and $\hat{\Psi}^{-1}\tilde{\Gamma}^{\sim}\mathcal{L}_2 = G\mathcal{L}_2$. This implies $\tilde{G}\hat{\Psi}^{-1}\tilde{G}^{\sim} = \Gamma^{\sim}\hat{\Psi}\Gamma \geq \epsilon_2 I$ and $\tilde{\Gamma}\hat{\Psi}^{-1}\tilde{\Gamma}^{\sim} = G^{\sim}\hat{\Psi}G \leq -\epsilon_1 I$. As $\nu \to 0$ we get the inequalities in (a) and (b).

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