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Global Inverse Optimal Controller with Guaranteed Convergence Rate for Input-affine Nonlinear Systems

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***Abstract*— Dilated homogeneous systems are local canonical forms of nonlinear control systems. In this paper, we propose a global inverse optimal controller with guaranteed convergence rate by implementing the local homogeneity. First, we clearly describe assumptions, and then design a global inverse optimal controller achieving local homogeneity for input-affine local-homogeneous nonlinear systems by using local-homogeneous control Lyapunov functions. The proposed controller guarantees convergence rate thanks to the local homogeneity. Finally, we discuss what systems it is available for, and confirm the effectiveness of the proposed controller by computer simulation.**

I. Introduction

Control Lyapunov function based controller design attracts much attention in nonlinear control theory. In the previous works [2][5], a global stabilizing controller was proposed for input-affine nonlinear systems with control Lyapunov functions. Then, the controller was modified to satisfy input constraints [3][4]. Moreover, an inverse optimal control problem has been already solved [2][5]. However, these controllers do not guarantee convergence rate. This may result in slow convergence phenomena.

Homogeneous systems appear naturally as local approximation to nonlinear systems [8][9]. In [1], homogeneous (inverse optimal) stabilizing controllers were provided for input-affine homogeneous systems with homogeneous control Lyapunov functions. The homogeneous degree of the system determines convergence rate [7]. However, homogeneous controllers generally do not achieve global stability for non-homogeneous systems.

It is still an interesting problem whether we can design a global stabilizing controller with guaranteed convergence rate for non-homogeneous systems. For the problem, we propose a global inverse optimal controller with guaranteed convergence rate by utilizing the local homogeneity.

In Section II, we introduce definitions and previous results, and in Section III, we show the main result of this paper. First, we clearly describe assumptions, and then design a global inverse optimal controller achieving local homogeneity for input-affine local-homogeneous nonlinear systems by using local-homogeneous control Lyapunov functions. The proposed controller guarantees convergence rate thanks to the local homogeneity. In Section IV, we summarize the previous results obtained in [1]-[5], and prove the main result. In Section V, we discuss what systems satisfy our assumptions, and in Section VI, confirm the effectiveness of the proposed controller by computer simulation.

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II. Preliminary

We consider the following input-affine nonlinear system:

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

where $x \in \mathbb{R}^n$ is a state vector, $u \in \mathbb{R}^m$ is an input vector, $f(x)$ and $g(x)$ are continuous mappings, and $f(0) = 0$. Let $g_i(x)$ and $g^j(x)$ denote the i-th row vector and the j-th column vector of $g(x)$, respectively.

Definition 1 (control Lyapunov function) *[11] A C^1 proper positive-definite function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a control Lyapunov function (clf) for system (1) if*

$$\inf_{u \in \mathbb{R}^m} \{L_f V + L_g V \cdot u\} < 0, \quad \forall x \in \mathbb{R}^n \backslash \{0\}, \tag{2}$$

where $L_f V := \partial V/\partial x \cdot f(x)$ and $L_g V := \partial V/\partial x \cdot g(x)$. □

Definition 2 (small control property) *[11] A control Lyapunov function $V(x)$ for system (1) is said to satisfy the small control property (scp) if for any $\varepsilon > 0$, there is $\delta > 0$ such that*

$$0 \neq \|x\| < \delta \implies \exists \|u\| < \varepsilon \;\; s.t. \;\; L_f V + L_g V \cdot u < 0.$$

□

Theorem 1 *[11] System (1) is globally asymptotically stabilizable by a controller that attains continuity except at the origin if and only if a control Lyapunov function exists. System (1) is globally asymptotically stabilizable by a continuous controller if and only if a control Lyapunov function with the small control property exists.* □

Definition 3 (dilation) *[9] A mapping*

$$\Delta_\varepsilon^r x = (\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n)^T, \quad \forall \varepsilon > 0, \; \forall x \in \mathbb{R}^n \backslash \{0\}$$

is said to be a dilation on $\mathbb{R}^n$, where $r = (r_1, \ldots, r_n)^T$ and $0 < r_i < \infty$ $(i = 1, \ldots, n)$. □

Definition 4 (homogeneous function) *[9] A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous of degree $k \in \mathbb{R}$ with respect to the dilation $\Delta_\varepsilon^r x$ if*

$$V(\Delta_\varepsilon^r x) = \varepsilon^k V(x).$$

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□

Definition 5 (homogeneous system) *[9] System (1) is said to be homogeneous of degree $\tau \in \mathbb{R}$ with respect to the dilations $\Delta_\varepsilon^r x$ and $\Delta_\varepsilon^s u$ if*

$$f(\Delta_\varepsilon^r x) + g(\Delta_\varepsilon^r x)\Delta_\varepsilon^s u = \varepsilon^\tau \Delta_\varepsilon^r \{f(x) + g(x)u\}.$$

□

Definition 6 (homogeneous approximation) *[9] An homogeneous function $V_h(x)$ is said to be homogeneous approximation of $V(x)$ if there exists $V_o(x)$ such that*

$$V(x) = V_h(x) + V_o(x) \tag{3}$$

and

$$\lim_{\varepsilon\to 0} \frac{V_o(\Delta_\varepsilon^r x)}{\varepsilon^k} = 0 \tag{4}$$

uniformly on $S^{n-1} := \{x \in \mathbb{R}^n | \; \|x\|_2 = 1\}$.

An homogeneous system

$$\dot{x} = f_h(x) + g_h(x)u \tag{5}$$

is said to be homogeneous approximation of (1) if there exist $f_o(x)$ and $g_o(x)$ such that

$$f(x) + g(x)u = f_h(x) + g_h(x)u + f_o(x) + g_o(x)u \tag{6}$$

and

$$\lim_{\varepsilon\to 0} \frac{f_{o,i}(\Delta_\varepsilon^r x) + g_{o,i}(\Delta_\varepsilon^r x)\Delta_\varepsilon^s u}{\varepsilon^{\tau+r_i}} = 0, \quad \forall i = 1,\dots,n \tag{7}$$

uniformly on S^{n+m-1}. □

Theorem 2 *[10] We consider the following asymptotically stable homogeneous system of degree τ with respect to $\Delta_\varepsilon^r x$:*

$$\dot{x} = f(x), \tag{8}$$

where $x \in \mathbb{R}^n$ is a state vector, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping, and $f(0) = 0$. Let $k > 0$ be a constant and $p > 0$ an integer satisfying

$$k - p \cdot \max_{1\le i\le m} r_i > 0.$$

Then, there exists an homogeneous Lyapunov function of degree k which is C^∞ on $\mathbb{R}^n \backslash \{0\}$ and C^p at the origin. □

Definition 7 (exponential stability) *[7] Let $\|\cdot\|_{\{r,q\}}$ be any homogeneous norm. The origin of system (8) is said to be exponentially stable if there exist a neighborhood U of the origin and constants $M, D > 0$ such that for each $x_0 \in U\backslash\{0\}$, the solution $x(t)$ with $x(0) = x_0$ is defined on $[0,\infty)$ and*

$$\|x(t)\| \le M e^{-Dt} \|x(0)\|_{\{r,q\}}, \qquad \forall t \ge 0.$$

□

Definition 8 (finite-time stability) *[7] The origin of system (8) is said to be finite-time stable if it is stable and there exist a neighborhood U of the origin and a function $T : U\backslash\{0\} \to \mathbb{R}_{>0}$ such that for each $x_0 \in U\backslash\{0\}$, the solution $x(t)$ with $x(0) = x_0$ is defined on $[0, T(x))$, $x(t) \in U\backslash\{0\}$, $\forall t \in [0, T(x))$, and $\lim_{t\to T(x)} x(t) = 0$.* □

Lemma 1 *[7] We consider asymptotically stable homogeneous system (8) of degree τ.*

1) *If $\tau > 0$, $x \to 0$ as $t \to \infty$ for all $x \in \mathbb{R}^n$.*
2) *If $\tau = 0$, the origin is exponentially stable.*
3) *If $\tau < 0$, the origin is finite-time stable.*

□

Proposition 1 *[1] Homogeneous control Lyapunov functions for input-affine homogeneous systems always satisfy the small control property.* □

Lemma 2 *[9] Let $V(x)$ be an homogeneous control Lyapunov function for homogeneous system (1). Then, for each continuous function $\lambda : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$, $V(x)$ is also an homogeneous clf for the following system:*

$$\dot{x} = f(x) - \lambda(x)\nu(x) + g(x)u, \tag{9}$$

where $\nu(x) = (r_1 x_1, \dots, r_n x_n)^T$. □

III. Main result

We assume the following:

Assumption 1
1) *System (1) has homogeneous approximation (5) of degree τ with respect to $\Delta_\varepsilon^r x$ and $\Delta_\varepsilon^s u$.*
2) *$V(x)$ is a clf for system (1) such that the homogeneous approximation $V_h(x)$ of degree k with respect to $\Delta_\varepsilon^r x$ is a clf for system (5).*

□

Then, we obtain the following lemma:

Lemma 3 *Under Assumption 1, we can reconstruct a clf $\bar{V}(x)$ for system (1) such that the homogeneous approximation $\bar{V}_h(x)$ of degree $\bar{k}$ with respect to $\Delta_\varepsilon^r x$ is a clf for system (5) satisfying*

$$\bar{k} + \tau - \max_{1\le j\le m} s_j > 0. \tag{10}$$

Proof: If Assumption 1 is satisfied, $\bar{V}(x) := V^{\frac{\bar{k}}{k}}(x)$ ($\bar{k} \ge k$) is also a clf for system (1) because

$$\dot{\bar{V}}(x) = \frac{\bar{k}}{k} V^{\frac{\bar{k}}{k}-1}(x)\dot{V}(x).$$

Moreover, the homogeneous approximation $\bar{V}_h(x)$ of degree $\bar{k}$ with respect to $\Delta_\varepsilon^r x$ is a clf for system (5) because

$$\bar{V}_h(x) = \lim_{\varepsilon\to 0} \frac{\bar{V}(\Delta_\varepsilon^r x)}{\varepsilon^{\bar{k}}} = V_h^{\frac{\bar{k}}{k}}(x)$$

$$\dot{\bar{V}}_h(x) = \frac{\bar{k}}{k} V_h^{\frac{\bar{k}}{k}-1}(x)\dot{V}_h(x).$$

Since $\bar{k}$ can be chosen as large as condition (10) is satisfied, we obtain the lemma. ■

By Lemma 3, the following additional condition:

$$k+\tau-\max_{1\le j\le m} s_j>0 \tag{11}$$

does not narrow the class of system (1). Under Assumption 1 and condition (11), we obtain the following global inverse optimal controller with guaranteed convergence rate:

Theorem 3 (Main result) *We suppose Assumption 1 and condition (11). Let $c>0$ be a constant. Then, the following input globally asymptotically stabilizes the origin:*

$$u_j=-\frac{1}{R(x)}|L_{g^j}V|^{\frac{s_j}{\tau+k-s_j}}\operatorname{sgn}(L_{g^j}V) \quad (j=1,\dots,m), \tag{12}$$

where

$$R(x)=\begin{cases}\dfrac{2}{P_a+|P_a|+c}\cdot\dfrac{\tau+k-\max s_j}{\tau+k} & (L_gV\neq 0)\\ \dfrac{2}{c}\cdot\dfrac{\tau+k-\max s_j}{\tau+k} & (L_gV=0)\end{cases} \tag{13}$$

$$P_a(x)=\frac{L_fV}{\sum_{j=1}^{m}|L_{g^j}V|^{\frac{\tau+k}{\tau+k-s_j}}}. \tag{14}$$

Moreover, input (12) minimizes the following cost function and achieves a sector margin $\left(\frac{\tau+k-\max s_j}{\tau+k},\infty\right)$:

$$J=\int_0^\infty\left\{\ell(x)+\sum_{j=1}^{m}\frac{s_j}{\tau+k}R^{\frac{\tau+k-s_j}{s_j}}(x)|u_j|^{\frac{\tau+k}{s_j}}\right\}dt, \tag{15}$$

where

$$\ell(x)=\sum_{j=1}^{m}\frac{\tau+k-s_j}{\tau+k}\cdot\frac{1}{R(x)}|L_{g^j}V|^{\frac{\tau+k}{\tau+k-s_j}}-L_fV. \tag{16}$$

Each $u_j(x)$ is continuous and has local homogeneous approximation of degree s_j. Furthermore, the following are true:

1) *If $\tau>0$, $x\to 0$ as $t\to\infty$ for all $x\in\mathbb{R}^n$.*
2) *If $\tau=0$, the origin becomes exponentially stable.*
3) *If $\tau<0$, the origin becomes finite-time stable.*

□

The proof of Theorem 3 is given in the next section. If we do not adhere to the inverse optimality, we can liberally adjust a sector margin by employing another controller $\bar{u}=\gamma u$. For example, if we choose

$$u_j=\begin{cases}-\dfrac{P_a+|P_a|+c}{2}\cdot|L_{g^j}V|^{\frac{s_j}{\tau+k-s_j}}\operatorname{sgn}(L_{g^j}V) & (L_gV\neq 0)\\ 0 & (L_gV=0)\end{cases} \quad (j=1,\dots,m)$$

instead of (12), it achieves a sector margin $(1,\infty)$.

IV. Proof of the Main Result

A. Generalization of clf-based controller

To prove Theorem 3, we summarize the previous results obtained in [1]-[5] for the following input form:

$$u_j=-\frac{1}{R(x)}|L_{g^j}V|^{a_j}\operatorname{sgn}(L_{g^j}V) \quad (j=1,\dots,m). \tag{17}$$

First, we collect stabilizing controllers [1],[3]-[5] as the following:

Theorem 4 *Let $V(x)$ be a clf for system (1), and $a_j:\mathbb{R}^n\to\mathbb{R}_{>0}$ and $c:\mathbb{R}^n\to\mathbb{R}_{>0}$ continuous functions satisfying*

$$c|L_{g^j}V|^{a_j}\to 0 \quad \text{as} \quad L_gV\to 0 \quad (j=1,\dots,m). \tag{18}$$

Then, the following input globally asymptotically stabilizes the origin:

$$u_j=\begin{cases}-\dfrac{P_a+|P_a|+c}{2}\cdot|L_{g^j}V|^{a_j}\operatorname{sgn}(L_{g^j}V) & (L_gV\neq 0)\\ 0 & (L_gV=0)\end{cases}, \quad (j=1,\dots,m) \tag{19}$$

where

$$P_a(x)=\frac{L_fV}{\sum_{j=1}^{m}|L_{g^j}V|^{a_j+1}}. \tag{20}$$

Moreover, input (19) is continuous on $\mathbb{R}^m\backslash\{0\}$, and is also continuous at the origin if

$$P_a|L_{g^j}V|^{a_j}\to 0 \quad \text{as} \quad x\to 0 \quad (j=1,\dots,m). \tag{21}$$

□

Proof: Since $P_a(x)$ is continuous on $\{x\in\mathbb{R}^n|\ L_gV\neq 0\}$, (19) is also continuous on $\{x\in\mathbb{R}^n|\ L_gV\neq 0\}$. By (2), $L_fV<0$ in a small neighborhood of $x\in\{x\in\mathbb{R}^n|\ L_gV=0\ \wedge\ x\neq 0\}$. If $L_gV\neq 0$ and $L_fV<0$,

$$u_j=-\frac{c}{2}\cdot|L_{g^j}V|^{a_j}\operatorname{sgn}(L_{g^j}V). \tag{22}$$

By (18) and (22), (19) is continuous except at the origin. If condition (21) is satisfied, it is obvious that (19) is continuous at the origin.

If $L_gV=0$, $\dot{V}(x)=L_fV<0$ for all $x\in\mathbb{R}^n\backslash\{0\}$. If $L_gV\neq 0$ and $P_a(x)\le 0$, $\dot{V}(x)<0$ is clear. If $L_gV\neq 0$ and $P_a(x)>0$, (19) brings

$$\dot{V}(x)=-\frac{c}{2}\sum_{j=1}^{m}|L_{g^j}V|^{a_j+1}<0.$$

Since $\dot{V}(x)<0$ for all $x\in\mathbb{R}^n\backslash\{0\}$, input (19) globally asymptotically stabilizes the origin. ■

Remark 1 *By setting $a_j=1$, (19) coincides with Sontag's stabilizing controller [5]. By setting $a_j=\frac{1}{k-1}$ or*

$a_j = \frac{1}{\hat{k}(x)-1}$, the directional vector of (19) corresponds with stabilizing controllers [4][3]. If system (1) and $V(x)$ are homogeneous and $a_j = \frac{s_j}{\tau+k-s_j}$, (19) is identified with homogeneous stabilizing controller [1].

By [1],[3]-[5], there exist a_j $(j = 1, \dots, m)$ satisfying condition (21) if and only if $V(x)$ has the scp. □

Then, we unify the results [1][2][5] on inverse optimal problems. A cost function and a sector margin achieved by (17) are derived as follows:

Lemma 4 *Let $V(x)$ be a clf for system (1), $a_j : \mathbb{R}^n \to \mathbb{R}_{>0}$ $(j = 1, \dots, m)$ continuous functions, and $\ell(x)$ a function defined by*

$$\ell(x) = \sum_{j=1}^{m} \frac{1}{a_j+1} \cdot \frac{1}{R(x)} |L_{g^j} V|^{a_j+1} - L_f V, \tag{23}$$

where $R : \mathbb{R}^n \to \mathbb{R}_{>0}$ is a positive-valued function that is continuous on $\mathbb{R}^n \backslash \{0\}$ and $\ell(x) \geq 0$ $(\forall x \in \mathbb{R}^n)$. Then, input (17) globally asymptotically stabilizes the origin and minimizes the cost function

$$J = \int_0^\infty \left\{ \ell(x) + \sum_{j=1}^{m} \frac{a_j}{a_j+1} R^{\frac{1}{a_j}}(x) |u_j|^{\frac{a_j+1}{a_j}} \right\} dt. \tag{24}$$

Moreover, it guarantees a sector margin $\left(\frac{1}{\min a_j+1}, \infty\right)$. Input (17) is continuous on $\mathbb{R}^n \backslash \{0\}$. □

Proof: Since $R(x)$ is continuous except at the origin, input (17) is also continuous except at the origin.

By (17) and (23), $\ell(x)$ can be rewritten to

$$\ell(x) \leq -\dot{V}\left(x,\ \frac{1}{\min a_j+1} \cdot u\right).$$

We define

$$\gamma_j(x) = \frac{1}{R(x)} |L_{g^j} V|^{a_j} \operatorname{sgn}(L_{g^j} V),$$

and let $\phi_j(\gamma_j)$ be a sector nonlinearity in $\left(\frac{1}{\min a_j+1}, \infty\right)$. Then, the input

$$\hat{u}_j = -\phi_j(\gamma_j) \qquad (j = 1, \dots, m)$$

guarantees

$$\dot{V}(x, \hat{u}) \leq \dot{V}\left(x,\ \frac{1}{\min a_j+1} \cdot u\right) \leq -\ell(x).$$

Since $\ell(x) \geq 0$, it achieves at least a sector margin $\left(\frac{1}{\min a_j+1}, \infty\right)$. Hence, input (17) asymptotically stabilizes the origin, and $V(x(\infty)) = 0$. Then, cost function (24) can be rewritten to

$$J = \int_0^\infty \left\{ L_f V + L_g V \cdot u + \ell(x) + \sum_{j=1}^{m} \frac{a_j}{a_j+1} R^{\frac{1}{a_j}}(x) |u_j|^{\frac{a_j+1}{a_j}} \right\} dt + V(x(0)).$$

We define

$$K(x,u) = L_f V + L_g V \cdot u + \ell(x) + \sum_{j=1}^{m} \frac{a_j}{a_j+1} R^{\frac{1}{a_j}}(x) |u_j|^{\frac{a_j+1}{a_j}}, \tag{25}$$

and let input $\bar{u}$ which minimizes $K(x,u)$ for each x. The discontinuity of $R(x)$ at the origin does not cause any problems (See [5].) Input $\bar{u}$ is uniquely determined because $K(x,u)$ is a strictly convex function in u for fixed x. Hence, input $\bar{u}$ minimizes $K(x,u)$ if and only if $\partial K / \partial u(x, \bar{u}) = 0$. Differentiating both sides of (25) with respect to u_j, we achieve

$$\frac{\partial K}{\partial u_j}(x,u) = L_{g^j} V + R^{\frac{1}{a_j}}(x) |u_j|^{\frac{1}{a_j}} \operatorname{sgn}(u_j).$$

Input $\bar{u}$ satisfying $\partial K / \partial u(x, \bar{u}) = 0$ coincides with (17), and results in $K(x, \bar{u}) = 0$. Hence, input (17) minimizes cost function (24) and $\min J = V(x(0))$. ■

The following corollary gives the same form of inputs and sector margins as controllers of [1][2][5]:

Corollary 1 *Let $V(x)$ be a clf for system (1), $a_j : \mathbb{R}^n \to \mathbb{R}_{>0}$ $(j = 1, \dots, m)$ continuous functions, $R : \mathbb{R}^n \to \mathbb{R}_{>0}$ a positive-valued function that is continuous on $\mathbb{R}^n \backslash \{0\}$, and*

$$\gamma \geq \max a_j + 1 \tag{26}$$

a constant. We assume that (17) is a globally asymptotically stabilizing controller. Then, the input

$$\bar{u} = \gamma u \tag{27}$$

globally asymptotically stabilizes the origin. Moreover, input (27) minimizes the following cost function and guarantees a sector margin $(\frac{1}{\gamma}, \infty)$:

$$J = \int_0^\infty \left\{ \ell(x) + \sum_{j=1}^{m} \frac{a_j}{a_j+1} \left(\frac{R(x)}{\gamma}\right)^{\frac{1}{a_j}} |u_j|^{\frac{a_j+1}{a_j}} \right\} dt, \tag{28}$$

where

$$\ell(x) = \sum_{j=1}^{m} \frac{1}{a_j+1} \cdot \frac{\gamma}{R(x)} |L_{g^j} V|^{a_j+1} - L_f V. \tag{29}$$

Input (27) is continuous on $\mathbb{R}^n \backslash \{0\}$. □

Proof: Since (17) is a globally asymptotically stabilizing controller and $\gamma > 1$, (27) is also a globally asymptotically stabilizing controller and achieves at least a sector margin $(\frac{1}{\gamma}, \infty)$. By (17) and (29), $\ell(x)$ can be rewritten to

$$\ell(x) \geq -\dot{V}\left(x,\ \frac{1}{\max a_j+1} \cdot \gamma u\right). \tag{30}$$

By (26) and (30), we obtain $\ell(x) \geq 0$. The rest of the proof is the same as Lemma 4. ■

The above discussion is limited to abstract structure of inverse optimal controllers. We summarize concrete inverse optimal controllers [1][2][5] as the following:

Theorem 5 *Let $V(x)$ be a clf for system (1), $a_j : \mathbb{R}^n \to \mathbb{R}_{>0}$ and $c : \mathbb{R}^n \to \mathbb{R}_{>0}$ continuous functions satisfying (18), $P_a(x)$ a function defined by (20), and $\ell(x)$ a function defined by (23). We choose $R(x)$ as*

$$R(x) = \begin{cases} \dfrac{2}{P_a + |P_a| + c} \cdot \dfrac{1}{\max a_j + 1} & (L_g V \neq 0) \\ \dfrac{2}{c} \cdot \dfrac{1}{\max a_j + 1} & (L_g V = 0) \end{cases}. \quad (31)$$

Then, input (17) globally asymptotically stabilizes the origin and minimizes cost function (24). Moreover, it achieves a sector margin $\left(\frac{1}{\max a_j + 1}, \infty\right)$. Input (17) is continuous on $\mathbb{R}^n \backslash \{0\}$, and is also continuous at the origin if condition (21) is satisfied. □

Proof: By (31), $R(x) > 0$ in $\mathbb{R}^n$. Since $P_a(x)$ is continuous on $\{x \in \mathbb{R}^n | \ L_g V \neq 0\}$, $R(x)$ is also continuous on $\{x \in \mathbb{R}^n | \ L_g V \neq 0\}$. If $L_g V \neq 0$ and $L_f V < 0$, $R(x) = \frac{2}{c} \cdot \frac{1}{\max a_j + 1}$. Hence, $R(x)$ is continuous on $\mathbb{R}^n \backslash \{0\}$.

If $L_g V = 0$, $\ell(x) = -L_f V \geq 0$ by (2) and (23). If $L_g V \neq 0$ and $P_a(x) \leq 0$, $\ell(x) > 0$ by (20), (23) and (31). If $L_g V \neq 0$ and $P_a(x) > 0$,

$$\ell(x) \geq \frac{c}{2} \sum_{j=1}^{m} |L_{g^j} V|^{a_j + 1} > 0$$

by (20), (23) and (31). Therefore, $\ell(x) \geq 0$ in $\mathbb{R}^n$.

Note that all conditions in Lemma 4 are satisfied, and input (17) globally asymptotically stabilizes the origin and minimizes cost function (24). Moreover, input (17) is continuous except at the origin. If condition (21) is satisfied, it is clear that input (17) is continuous at the origin.

We define

$$\gamma_j(x) = \frac{1}{R(x)} |L_{g^j} V|^{a_j} \operatorname{sgn}(L_{g^j} V),$$

and let $\phi_j(\gamma_j)$ be a sector nonlinearity in $\left(\frac{1}{\max a_j + 1}, \infty\right)$. If $L_g V \neq 0$ and $P_a(x) > 0$, the input

$$\hat{u}_j = -\phi_j(\gamma_j) \qquad (j = 1, \ldots, m)$$

gives

$$\dot{V}(x, \hat{u}) = L_f V - L_g V \cdot \phi(\gamma) < 0.$$

Therefore, it achieves a sector margin $\left(\frac{1}{\max a_j + 1}, \infty\right)$. ■

B. Proof of Theorem 3

By Lemma 1 and Theorem 5, Theorem 3 is successfully proved as the following:

Proof: Substitute

$$a_j = \frac{s_j}{\tau + k - s_j} \qquad (j = 1, \ldots, m)$$

in Theorem 5, and we obtain the following facts:

1) Input (12) globally asymptotically stabilizes the origin.
2) Input (12) minimizes cost function (15).
3) Input (12) achieves a sector margin $\left(\frac{\tau + k - \max s_j}{\tau + k}, \infty\right)$.
4) Each $u_j(x)$ is continuous except at the origin.

By Assumption 1, $V(x)$ and (1) can be rewritten to (3) and (6). Differentiating both sides of (4) with respect to x_i,

$$\lim_{\varepsilon \to 0} \frac{\dfrac{\partial V_o}{\partial x_i}(\Delta_\varepsilon^r x)}{\varepsilon^{k - r_i}} = 0. \quad (32)$$

By (3), (6), (14) and (32),

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{L_f V(\Delta_\varepsilon^r x)}{\varepsilon^{\tau + k}} = L_{f_h} V_h(x) \\ &\lim_{\varepsilon \to 0} \frac{L_{g^j} V(\Delta_\varepsilon^r x)}{\varepsilon^{\tau + k - s_j}} = L_{g_h^j} V_h(x) \\ &\lim_{\varepsilon \to 0} P_a(\Delta_\varepsilon^r x) = \frac{L_{f_h} V_h(x)}{\sum_{j=1}^{m} \left| L_{g_h^j} V_h(x) \right|^{\frac{\tau + k}{\tau + k - s_j}}}. \end{aligned} \quad (33)$$

By (12), (13) and (33), $u_j(x)$ and $P_a |L_{g^j} V|^{a_j}$ have local homogeneous approximation of degree $s_j > 0$. Hence, condition (21) is satisfied, and $u_j(x)$ is also continuous at the origin. The convergence rate is proved by Lemma 1. ■

V. Discussion

In this section, we introduce nonlinear systems with special structures satisfying Assumption 1. We consider system (1) satisfying the following assumption:

Assumption 2 (input homogeneous transformation)

1) *There exists a continuous input*

$$u = h(x) + v \quad (34)$$

such that the resulting system

$$\dot{x} = f(x) + g(x)h(x) + g(x)v \quad (35)$$

becomes homogeneous of degree τ with respect to $\Delta_\varepsilon^r x$ and $\Delta_\varepsilon^s v$.

2) *There exists an asymptotically stabilizing controller $v(x)$ such that each $v_j(x)$ is continuous and homogeneous of degree s_j.*
3) $$\lim_{\varepsilon \to 0} \frac{g_i(\Delta_\varepsilon^r x) h(\Delta_\varepsilon^r x)}{\varepsilon^{\tau + r_i}} = 0, \quad \forall i = 1, \ldots, n$$

uniformly on S^{n-1}.

□

The last condition implies that system (1) has homogeneous approximation. By the first two conditions in Assumption 2 and Theorem 2, the closed-loop system has an homogeneous Lyapunov function $V(x)$. Since $V(x)$ becomes a control Lyapunov function for systems (1) and (35), Assumption 1 is always satisfied under Assumption 2.

On the other hand, notice that system (9) in Lemma 2 satisfies Assumption 1.

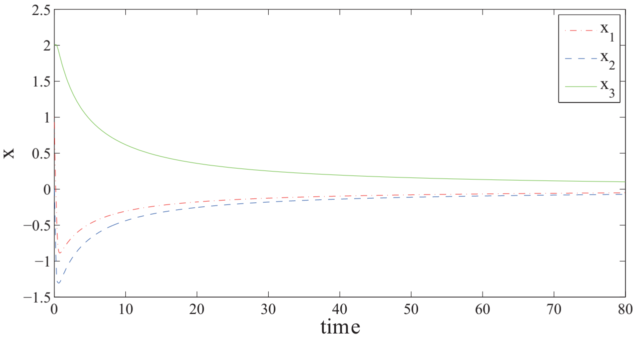

Fig. 1. Response of state with controller (37)

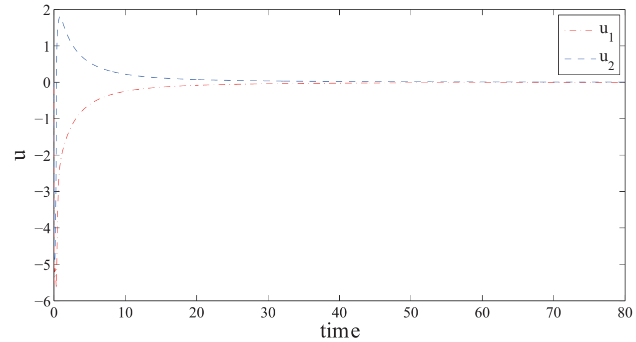

Fig. 2. Change in input with controller (37)

VI. EXAMPLE

We consider the following satellite system [6]:

$$\begin{aligned}\dot{x}_1 &= J_1 x_2 x_3 + u_1\\ \dot{x}_2 &= J_2 x_3 x_1 + u_2\\ \dot{x}_3 &= J_3 x_1 x_2.\end{aligned} \tag{36}$$

Since system (36) is homogeneous of degree $\tau = 1$ with respect to $r = (1,1,1)^T$ and $s = (2,2)^T$, exponential or finite-time stability can not be achieved by a controller $u(x)$ satisfying $u(\Delta_\varepsilon^1 x) = \varepsilon^2 u(x)$ (See Lemma 1.) For example, we choose the following homogeneous clf of degree $k = 4$:

$$V(x) = x_1^4 + x_1|x_3|^3 + x_2^4 + x_2 x_3^3 + 2x_3^4.$$

By [1], we obtain the following homogeneous inverse optimal controller with a sector margin $\left(\frac{3}{5}, \infty\right)$:

$$u_j = \begin{cases} -\dfrac{5}{6}(P_h + |P_h| + c)|L_{g^j}V|^{\frac{2}{3}}\operatorname{sgn}(L_{g^j}V) \\ \qquad \left(\left|4x_1^3 + |x_3|^3\right| + \left|4x_2^3 + x_3^3\right| \neq 0\right) \\ 0 \qquad \left(\left|4x_1^3 + |x_3|^3\right| + \left|4x_2^3 + x_3^3\right| = 0\right)\end{cases}, \tag{37}$$

where

$$\begin{aligned}\dot{V}(x,u) &= \left(4x_1^3 + |x_3|^3\right)(J_1 x_2 x_3 + u_1)\\ &\quad + \left(4x_2^3 + x_3^3\right)(J_2 x_3 x_1 + u_2)\\ &\quad + \left(3x_1 x_3^2 \operatorname{sgn}(x_3) + 3x_2 x_3^2 + 8x_3^3\right) J_3 x_1 x_2\\ P_h(x) &= \frac{L_f V}{|4x_1^3 + |x_3|^3|^{\frac{5}{3}} + |4x_2^3 + x_3^3|^{\frac{5}{3}}}.\end{aligned}$$

Figures 1 and 2 show responses for $(J_1, J_2, J_3) = \left(-1, 1, -\frac{1}{3}\right)$, $c = 1$ and $x(0) = (1,0,2)^T$. Notice that states converge to zero very slowly. Hence, we design another controller.

System (36) has the following approximation:

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= J_3 x_1 x_2,\end{aligned} \tag{38}$$

which is homogeneous of degree $\tau = 0$ with respect to $r = (1,1,2)^T$ and $s = (1,1)^T$. We choose the following homogeneous clf of degree $k = 4$:

$$V(x) = x_1^4 + x_1|x_3|^{\frac{3}{2}} + x_2^4 + x_2|x_3|^{\frac{3}{2}}\operatorname{sgn}(x_3) + 2x_3^2.$$

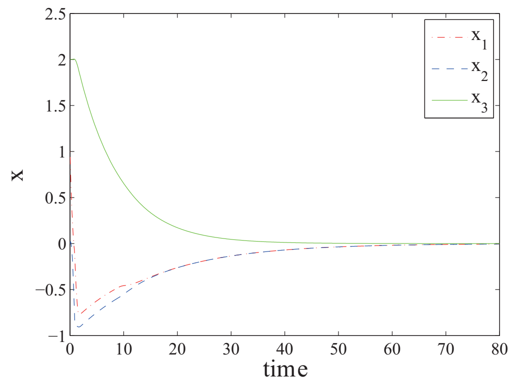

Fig. 3. Response of state with controller (39)

Then, Assumption 1 and condition (11) are satisfied. By Theorem 3, we obtain the following inverse optimal controller with a sector margin $\left(\frac{3}{4}, \infty\right)$:

$$u_j = \begin{cases} -\dfrac{2}{3}(P_a + |P_a| + c)|L_{g^j}V|^{\frac{1}{3}}\operatorname{sgn}(L_{g^j}V) \\ \qquad \left(\left|4x_1^3 + |x_3|^{\frac{3}{2}}\right| + \left|4x_2^3 + |x_3|^{\frac{3}{2}}\operatorname{sgn}(x_3)\right| \neq 0\right) \\ 0 \quad \left(\left|4x_1^3 + |x_3|^{\frac{3}{2}}\right| + \left|4x_2^3 + |x_3|^{\frac{3}{2}}\operatorname{sgn}(x_3)\right| = 0\right)\end{cases}, \tag{39}$$

where

$$\begin{aligned}\dot{V}(x,u) &= \left(4x_1^3 + |x_3|^{\frac{3}{2}}\right)(J_1 x_2 x_3 + u_1)\\ &\quad + \left(4x_2^3 + |x_3|^{\frac{3}{2}}\operatorname{sgn}(x_3)\right)(J_2 x_3 x_1 + u_2)\\ &\quad + \left(\frac{3}{2}x_1|x_3|^{\frac{1}{2}}\operatorname{sgn}(x_3) + \frac{3}{2}x_2|x_3|^{\frac{1}{2}} + 4x_3\right) J_3 x_1 x_2\\ P_a(x) &= \frac{L_f V}{\left|4x_1^3 + |x_3|^{\frac{3}{2}}\right|^{\frac{4}{3}} + \left|4x_2^3 + |x_3|^{\frac{3}{2}}\operatorname{sgn}(x_3)\right|^{\frac{4}{3}}}.\end{aligned}$$

The origin of the closed system becomes exponentially stable by Theorem 3. Figures 3 and 4 show responses for $(J_1, J_2, J_3) = \left(-1, 1, -\frac{1}{3}\right)$, $c = 1$ and $x(0) = (1,0,2)^T$. We can confirm that states converge to zero faster than the case of Figs. 1 and 2.

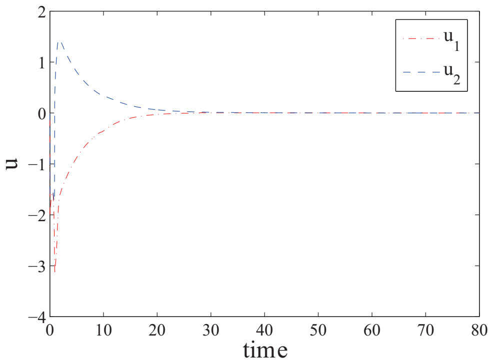

Fig. 4. Change in input with controller (39)

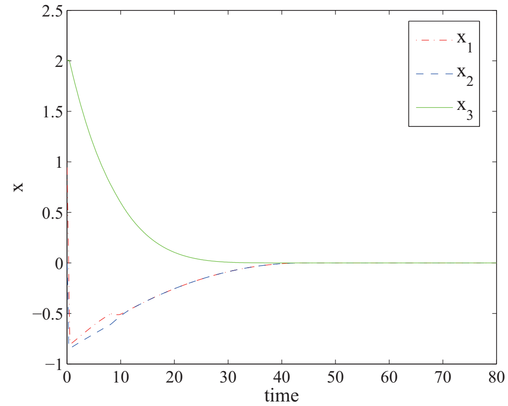

Fig. 5. Response of state with controller (40)

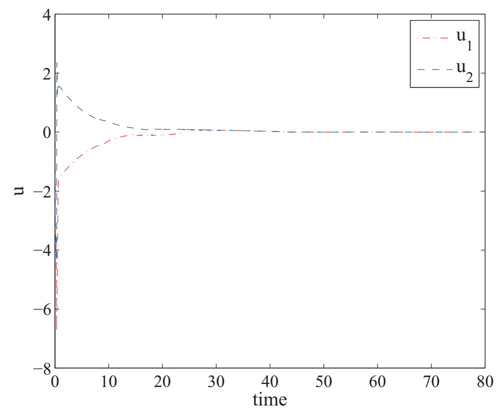

Fig. 6. Change in input with controller (40)

System (38) is also homogeneous approximation of degree $\tau = -1$ with respect to $r = (2,2,5)^T$ and $s = (1,1)^T$. We choose the following homogeneous clf of degree $k = 8$:

$$V(x) = x_1^4 + x_1|x_3|^{\frac{6}{5}} + x_2^4 + x_2|x_3|^{\frac{6}{5}}\operatorname{sgn}(x_3) + 2|x_3|^{\frac{8}{5}}.$$

Then, Assumption 1 and condition (11) are satisfied. By Theorem 3, we obtain the following inverse optimal controller with a sector margin $\left(\frac{6}{7}, \infty\right)$:

$$u_j = \begin{cases} -\dfrac{7}{12}(P_a + |P_a| + c)|L_{g^j}V|^{\frac{1}{6}}\operatorname{sgn}(L_{g^j}V) \\ \quad \left(\left|4x_1^3 + |x_3|^{\frac{6}{5}}\right| + \left|4x_2^3 + |x_3|^{\frac{6}{5}}\operatorname{sgn}(x_3)\right| \neq 0\right) \\ 0 \quad \left(\left|4x_1^3 + |x_3|^{\frac{6}{5}}\right| + \left|4x_2^3 + |x_3|^{\frac{6}{5}}\operatorname{sgn}(x_3)\right| = 0\right) \end{cases}, \tag{40}$$

where

$$\dot{V}(x,u) = \left(4x_1^3 + |x_3|^{\frac{6}{5}}\right)(J_1x_2x_3 + u_1)$$
$$+ \left(4x_2^3 + |x_3|^{\frac{6}{5}}\operatorname{sgn}(x_3)\right)(J_2x_3x_1 + u_2)$$
$$+ \left(\frac{6}{5}x_1|x_3|^{\frac{1}{5}}\operatorname{sgn}(x_3) + \frac{6}{5}x_2|x_3|^{\frac{1}{5}} + \frac{16}{5}|x_3|^{\frac{3}{5}}\operatorname{sgn}(x_3)\right)J_3x_1x_2$$
$$P_a(x) = \frac{L_fV}{\left|4x_1^3 + |x_3|^{\frac{6}{5}}\right|^{\frac{7}{6}} + \left|4x_2^3 + |x_3|^{\frac{6}{5}}\operatorname{sgn}(x_3)\right|^{\frac{7}{6}}}.$$

The origin of the closed system becomes exponentially stable by Theorem 3. Figures 5 and 6 show responses for $(J_1, J_2, J_3) = \left(-1, 1, -\frac{1}{3}\right)$, $c = 1$ and $x(0) = (1,0,2)^T$. We can confirm that states converge to zero in a finite time.

Remark 2 *As in the above example, local homogeneous degrees of nonlinear systems are not determined uniquely. So, we need to choose local homogeneous degrees to fit the purpose.*

□

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