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# Nonlinear Feedback Laws for Practical Stabilization of Systems with Input and State Constraints*

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*Abstract*— **This paper presents a method for designing nonlinear state feedback laws for systems with input and state constraints. The objective is to achieve practical stabilization with large stability region and strong disturbance rejection. Two invariant sets will be constructed within the state constraints: the outer one for stabilization and the inner one for asymptotic disturbance rejection. The nonlinear feedback law is designed such that all trajectories starting from the outer invariant set will enter the inner invariant set and stay there. Both invariant sets will be constructed by using the convex hull function, a recently introduced non-quadratic Lyapunov function. Since the invariant sets are convex hull of ellipsoids, they are able to incorporate the input and state constraints more effectively than simple ellipsoids, thus promising larger stability region within state constraint and stronger disturbance rejection capability. Since the convex hull functions are constructed from quadratic functions, the optimization problems can be treated with LMI-based method. Numerical examples demonstrate the effectiveness of the design methods.**

**Keywords: Constrained control, input saturation, practical stabilization, Lyapunov functions.**

## I. Introduction

Consider the following linear system,

$$\begin{aligned} \dot{x} &= Ax + Bu + Tw, \\ y &= Cx, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the control input, $w \in \mathbb{R}^p$ the disturbance and $y \in \mathbb{R}^q$ the output. Let $G$ be a $r \times n$ matrix. Denote $G_i$ as the $i$th row of $G$. Assume that the system is subject to the following constraints:

*Input constraints:* $|u_i| \leq 1$ for $i = 1, 2, \cdots, m$.

*State constraints:* $|G_i x| \leq 1$ for $i = 1, 2, \cdots, r$.

Note that output constraints can be considered as a special case of state constraints. Denote the state constraint set as

$$X_c = \{x \in \mathbb{R}^n : |G_i x| \leq 1, \ \ i = 1, 2, \cdots, r.\}. \tag{2}$$

Assume that the disturbance is bounded by $w(t)^{\mathrm{T}} w(t) \leq 1$ for all $t$. This type of $w$ is called unit peak disturbances [1].

Typical design objectives for the above system under constraints include achieving large stability region within $X_c$ and strong disturbance rejection performance (see, e.g., [2], [6], [4], [5], [11], [23], [24]). In our previous works [15], [16], [17], we considered various analysis and design problems in the absence of state constraint, i.e., the case where $X_c = \mathbb{R}^n$. In [15], the null controllable region, the largest stability region that can be achieved, was characterized; in [16], systems with two anti-stable open-loop poles were considered and semi-global stabilization strategies (which means that the stability region can be made to include any prescribed compact set within the null controllable region) were proposed; In [17], stability and disturbance rejection problems were addressed for more general systems by using Lyapunov functions. Since quadratic Lyapunov functions were used for analysis, the original problems were transformed into optimization problems with linear matrix inequality (LMI) constraints, and the resulting feedback laws were saturated linear, i,e., $u = \mathrm{sat}(Fx)$.

In practical systems, not only the control inputs are subject to hard bound, many other physical quantities, such as the voltage, the velocity, the displacement and the temperature, must be kept within a strict limit during the operation of a device. These quantities may represent certain output or state and their limits can be generally imposed as the state constraints, $|G_i x| \leq 1$ for $i = 1, 2, \cdots, r$, by properly scaling the matrix $G$.

Systems with both input and state constraints have been extensively studied in the literature (see, e.g., [2], [8], [11], [19], [23], [24] and the reference therein.) While some of the works addressed the systems under the model predictive control framework [19], [23], others attempted to design feedback laws by using Lyapunov functions and invariant sets [8], [10], [11]. In [10], quadratic Lyapunov functions were used for constructing invariant ellipsoid within state constraint set. In [11], some algorithms were proposed for finding the maximal set within the state constraint that can be made invariant by the constrained input, for discrete-time systems. If the initial condition belongs to this maximal set, the state constraint can be satisfied with the constrained input for all time. Although theoretically this maximal set can be approximated, its structure becomes more complicated as the number of steps increases and the control algorithm becomes harder, if not impossible, to implement.

In this paper, we would like to address some feedback design problems for the continuous-time system (1) under both input and state constraints, by using non-quadratic Lyapunov functions. Design objectives include 1) Constructing a large invariant set within the state constraint set so that all trajectories starting from within this set will satisfy the state constraint, for all possible disturbances. This invariant set is called a practical stability region; 2) Achieving a small asymptotic output bound in the presence of disturbances with

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a guaranteed practical stability region; and 3) Achieving a large practical stability region with a guaranteed asymptotic output bound.

The key point of the Lyapunov approach is to construct invariant sets using level sets of Lyapunov functions. When quadratic Lyapunov functions are used, the level sets are ellipsoids and the design problems can be formulated into optimization problems with LMI constraints by using the procedures in [1]. Since the input and state constraints usually take the form of polytopes, it seems natural to use polytopic invariant set as in [11]. However, it is known that the construction of polytopic invariant set can be very complicated and time consuming. With the recent efforts in the construction of nonquadratic Lyapunov functions, some numerically tractable Lyapunov functions have been developed (see e.g.,[3], [9], [14], [20]). Most of the Lyapunov functions in these works pertain to or are composed from several quadratic functions and thus lead to optimization problems with matrix inequality constraints, generally a mixture of LMIs and bilinear matrix inequalities (BMIs). Although the optimization problems are generally non-convex, suboptimal solutions can be obtained with LMI-based algorithms.

In this work, we would like to use the convex-hull (of quadratics) function as in [9], [14]. Since the level sets of a convex hull function is the convex hull of ellipsoids, it has the potential for better incorporating the structure of input and state constraints, thus leading to larger stability region and smaller asymptotic output bound. It is shown in [13] that the convex hull function will yield non-conservative bilinear matrix inequality (BMI) conditions for stability/stabilization of linear differential inclusions, and for constrained control system, it will yield a stability region as close as possible to the maximal achievable one.

This paper is organized as follows. Section II describes the design problems and gives a brief review of the convex hull function and some matrix conditions for set invariance under persistent disturbance. Section III develops design methods for enlarging practical stability region and for reducing asymptotic output bound. Motivated by a numerical example, this section also develops a method for achieving small asymptotic output bound with a guaranteed practical stability region. Section IV concludes the paper.

**Notation**

- For $x \in \mathbb{R}^n$, $|x|_\infty := \max_i |x_i|$, $|x|_2 := (x^{\mathrm{T}}x)^{\frac{1}{2}}$.
- $I[k_1, k_2]$: for integers $k_1 < k_2$, $I[k_1, k_2] := \{k_1, k_1 + 1, \cdots, k_2\}$.
- $\mathrm{co}\, S$: The convex hull of a set $S$.
- $\mathcal{E}(P)$: for $P \in \mathbb{R}^{n\times n}$, $\mathcal{E}(P) := \{x \in \mathbb{R}^n : \ x^{\mathrm{T}}Px \le 1\}$.
- $L_V$: 1-level set of a function $V$, $L_V := \{x \in \mathbb{R}^n : V(x) \le 1\}$.
- $\mathcal{L}(H) := \{x \in \mathbb{R}^n : \ |Hx|_\infty \le 1\}$.

About the relationship between $\mathcal{E}(P)$ and $\mathcal{L}(H)$, we have,

$$\mathcal{E}(P) \subseteq \mathcal{L}(H) \Longleftrightarrow H_\ell P^{-1} H_\ell^{\mathrm{T}} \le 1 \quad \forall\, \ell \in I[1, r]. \tag{3}$$

## II. Problem statement and preliminaries

### A. Problem statement

Consider system (1) with a nonlinear state feedback $u = f(x)$. Let $X_c$ be the state constraint set given by (2). For the closed-loop system

$$\begin{aligned} \dot{x} &= Ax + Bf(x) + Tw, \\ y &= Cx, \end{aligned} \tag{4}$$

we use the following sets and quantities to measure its stability and performance:

*Stability region:* Suppose $w = 0$. A set $X_s \subset X_c$ is called a stability region if, for every $x(0) \in X_s$, we have $x(t) \in X_c$ for all $t$ and $x(t) \to 0$ as $t \to \infty$.

*Practical stability region:* Suppose $w^{\mathrm{T}}(t)w(t) \le 1$. A set $X_{ps} \subset X_c$ is called a practical stability region if there exists a bounded set $\mathcal{D} \subset X_c$, such that for every $x(0) \in X_{ps}$ and all possible $w(t)$, we have $x(t) \in X_c$ for all $t > 0$ and there exists a $t_0$ such that $x(t) \in \mathcal{D}$ for all $t > t_0$.

*Asymptotic output bound:* Suppose $w^{\mathrm{T}}(t)w(t) \le 1$. Let $X_{ps} \subset X_c$ be a practical stability region. A number $\gamma > 0$ is called an asymptotic output bound for $X_{ps}$ if, for every $x(0) \in X_{ps}$ and all possible $w(t)$, $\limsup_{t\to\infty} |y(t)|_2 \le \gamma$.

The objective of this work is to design a feedback law $u = f(x)$ so that the closed-loop system has,

- a large stability region $X_s$;
- a large practical stability region $X_{ps}$;
- a small asymptotic output bound $\gamma$ for a guaranteed practical stability region $X_{ps}$, or,
- a large practical stability region $X_{ps}$ with a guaranteed asymptotic output bound $\gamma$.

### B. The convex hull function

In what follows, we give a brief review of the definition and some properties of the convex hull function that will be necessary for the development of the main results. The convex hull function is constructed from a family of positive definite matrices. Given $Q_j \in \mathbb{R}^{n\times n}, Q_j = Q_j^{\mathrm{T}} > 0, j \in I[1, J]$. Let

$$\Gamma := \left\{\gamma \in \mathbb{R}^J : \ \ \gamma_1 + \gamma_2 + \cdots + \gamma_J = 1, \gamma_j \ge 0\right\},$$

the convex hull function is defined as

$$V_c(x) := \min_{\gamma\in\Gamma} x^{\mathrm{T}} \left(\sum_{j=1}^{J} \gamma_j Q_j\right)^{-1} x. \tag{5}$$

For simplicity, we say that $V_c$ is composed from $Q_j$'s. This function was first used in [14] to study constrained control systems, where it was called the composite quadratic function. It was later called convex hull function, or convex hull of quadratics, in [9], since it is the convex hull (by the definition of [22]) of the family of quadratics $x^T Q_k^{-1} x$. If we define the 1-level set of $V_c$ as

$$L_{V_c} := \left\{x \in \mathbb{R}^n : \ \ V_c(x) \le 1\right\},$$

and denote the 1-level set of the quadratic function $x^{\mathrm{T}}Px$ as

$$\mathcal{E}(P) := \left\{x \in \mathbb{R}^n : \ \ x^{\mathrm{T}}Px \le 1\right\},$$

then

$$L_{V_c} = \left\{ \sum_{j=1}^{J} \gamma_j x_j : \ x_j \in \mathcal{E}(Q_j^{-1}), \gamma \in \Gamma \right\},$$

which means that $L_{V_c}$ is the convex hull of the family of ellipsoids, $\mathcal{E}(Q_j^{-1}), j \in I[1, J]$.

It is evident that $V_c$ is homogeneous of degree 2, i.e., $V_c(\alpha x) = \alpha^2 V_c(x)$. Also established in [9], [14] is that $V_c$ is convex and continuously differentiable.

For a compact convex set $S$, a point $x$ on the boundary of $S$ (denoted as $\partial S$) is called an extreme point if it cannot be represented as the convex combination of any other points in $S$. A compact convex set is completely determined by its extreme points. Since $L_{V_c}$ is the convex hull of $\mathcal{E}(Q_j^{-1})$, $j \in I[1, J]$, an extreme point must be on the boundaries of both $L_{V_c}$ and $\mathcal{E}(Q_j^{-1})$ for some $j \in I[1, J]$. Denote

$$E_k := \partial L_{V_c} \cap \partial \mathcal{E}(Q_k^{-1}) = \left\{ x \in \mathbb{R}^n : V_c(x) = x^{\mathrm{T}} Q_k^{-1} x = 1 \right\}.$$

Then $\bigcup_{k=1}^{J} E_k$ contains all the extreme points of $L_{V_c}$. The exact description of $E_k$ is given as follows.

*Lemma 1:* For each $k \in I[1, J]$,

$$E_k = \{ x \in \partial L_{V_c} : x^{\mathrm{T}} Q_k^{-1} (Q_j - Q_k) Q_k^{-1} x \leq 0, j \in I[1, J] \}. \tag{6}$$

For $x \in \mathbb{R}^n$, define

$$\gamma^*(x) := \arg\min_{\gamma \in \Gamma} x^{\mathrm{T}} \left( \sum_{j=1}^{J} \gamma_j Q_j \right)^{-1} x. \tag{7}$$

The function $\gamma^*(x)$ can be computed by solving a simple LMI problem obtained via Schur complements [14].

### C. Matrix conditions for invariance of level set

By Lyapunov approach, the stability and performance are usually characterized via invariant level sets of Lyapunov functions. Thus it is important to obtain a numerically tractable condition on set invariance. In [12], we derived a matrix condition for the controlled invariance of level sets for linear differential inclusions (LDIs). In [13], it was shown that, in the absence of disturbance, this matrix condition is necessary and sufficient for robust stabilization of LDIs. This condition trivially applies to the open-loop system (1) in the absence of input and state constraints.

*Theorem 1:* Consider $V_c$ composed from $Q_k \in \mathbb{R}^{n \times n}$, $Q_k = Q_k^{\mathrm{T}} > 0, k \in I[1, J]$. Suppose that there exist $Y_k \in \mathbb{R}^{m \times n}, k \in I[1, J]$, $\lambda_{jk} \geq 0$, $j, k \in I[1, J]$ and $\beta > 0$ such that

$$\begin{bmatrix} M_k + \beta Q_k & T \\ T^{\mathrm{T}} & -\beta I \end{bmatrix} \leq 0 \quad \forall k, \tag{8}$$

where

$$M_k = Q_k A^{\mathrm{T}} + A Q_k + Y_k^{\mathrm{T}} B^{\mathrm{T}} + B Y_k - \sum_{j=1}^{J} \lambda_{jk} (Q_j - Q_k). \tag{9}$$

For $x \in \mathbb{R}^n$, let $\gamma^*(x)$ be defined as in (7) and let

$$Y(\gamma^*) = \sum_{k=1}^{J} \gamma_k^* Y_k, \quad Q(\gamma^*) = \sum_{k=1}^{J} \gamma_k^* Q_k, \tag{10}$$

$$F(\gamma^*) = Y(\gamma^*) Q(\gamma^*)^{-1}. \tag{11}$$

Define $f(x) = F(\gamma^*(x))x$. Then $L_{V_c}$ is an invariant set for the linear closed-loop system (4), which means that all trajectories starting from $L_{V_c}$ will stay inside for any possible disturbance satisfying $w(t)^{\mathrm{T}} w(t) \leq 1, \forall t \geq 0$. Moreover, For all $x_0 \in \mathbb{R}^n$ and all possible disturbances, $x(t)$ will converge to $L_{V_c}$. $\diamond$

The main idea behind the theorem is that, under the matrix condition (8) and the control law constructed from (10)-(11), we have

$$\dot{V}_c = (\partial V_c(x))^{\mathrm{T}} (Ax + Bf(x) + Tw) \leq -\beta V_c(x) + \beta w^{\mathrm{T}} w, \tag{12}$$

for all $x \in \mathbb{R}^n, w \in \mathbb{R}^p$, where $\partial V_c(x)$ denotes the partial derivative. If $w^{\mathrm{T}} w \leq 1$ and $V_c(x) = 1$, we have $\dot{V}_c \leq 0$ and $V_c$ is nonincreasing. Hence $L_{V_c}$ is an invariant set. If $V_c(x) > 1$, then $\dot{V}_c$ is strictly decreasing. Hence any trajectory starting from outside of $L_{V_c}$ will converge to $L_{V_c}$.

In the absence of disturbances, i.e., $T$=0, (8) reduces to

$$M_k + \beta Q_k \leq 0 \quad \forall k.$$

This ensures that $\dot{V}_c \leq -\beta V_c(x)$ for all $x$ and that every trajectory converges to the origin.

The purpose of this paper is to apply the above result to constrained control systems for several design objectives. Two key points to be addressed are: which level sets satisfy the state constraints, and which control laws satisfy the input constraints? By imposing these additional restrictions appropriately, we should be able to search for the optimal (or suboptimal) convex hull function that would yield a large stability region, a small asymptotic output bound, or a trade off between these objectives.

## III. Design for constrained control systems

### A. Enlarging practical stability region

We consider the problem of enlarging the practical stability region inside the state constraint set $X_c$. The problem of enlarging the stability region is a special case of the former one with $T = 0$. We will use $L_{V_c}$ of a certain convex hull function to construct the practical stability region.

Under the matrix condition in Theorem 1 and the given feedback law, $L_{V_c}$ is an invariant set in the presence of disturbance. Since $L_{V_c}$ is bounded, to make it a practical stability region, we need to have

$$L_{V_c} \subset X_c,$$

and to satisfy the input constraint, we need to ensure that

$$|f(x)|_\infty \leq 1 \quad \forall\, x \in L_{V_c}.$$

There might be infinitely many $V_c$'s satisfying these restrictions. To choose the "optimal" one such that $L_{V_c}$ is large in some sense, we need a measure of the size of a set so that we can formulate an optimization problem. Here we borrow the measure from [17].

Given a set of reference points $x_\kappa, \kappa = 1, 2, \cdots, K$, the "inner" size of $L_{V_c}$ is measured by

$$\alpha_{\rm in} := \max\{\alpha : \alpha x_\kappa \in L_{V_c}, \kappa \in I[1, K]\}. \quad (13)$$

Using this measure of inner size, the problem of enlarging the practical stability region can be formulated as,

$$\sup_{Q_k, Y_k, \lambda_{jk}, \beta} \alpha \quad (14)$$

$$\text{s.t.} \quad \alpha x_\kappa \in L_{V_c}, \quad \kappa \in I[1, K], \quad (15)$$

$$L_{V_c} \subset X_c, \quad (16)$$

$$|f(x)|_\infty \le 1 \quad \forall\, x \in L_{V_c}, \quad (17)$$

$$\begin{bmatrix} M_k + \beta Q_k & T \\ T^{\rm T} & -\beta I \end{bmatrix} \le 0 \quad \forall k, \quad (18)$$

$$Q_k > 0, \;\; \beta > 0, \;\; \lambda_{jk} \ge 0.$$

To solve the above problem with LMI-based method, we need to transform the constraints (15) to (17) into matrix inequalities.

*Claim 1:* The constraints (15) to (17) are equivalent to the existence of $\gamma_\kappa \in \Gamma, \kappa \in I[1, K]$ such that

$$\begin{bmatrix} 1 & \alpha x_\kappa^{\rm T} \\ \alpha x_\kappa & \sum_{j=1}^J \gamma_{\kappa j} Q_j \end{bmatrix} \ge 0, \quad \kappa \in I[1, K], \quad (19)$$

$$G_s Q_j G_s^{\rm T} \le 1, \qquad s \in I[1, r], j \in I[1, J], \quad (20)$$

$$\begin{bmatrix} 1 & Y_{k\ell} \\ Y_{k\ell}^{\rm T} & Q_k \end{bmatrix} \ge 0, \quad k \in I[1, J], \ell \in I[1, m], \quad (21)$$

where $G_s$ is the $s$th row of $G$ and $Y_{k\ell}$ is the $\ell$th row of $Y_k$.

From Claim 1, the objective of maximizing the practical stability region within state and input constraint can be formulated as the following optimization problem:

$$\sup_{Q_j > 0, \lambda_{jk} \ge 0, \beta > 0, \gamma_\kappa \in \Gamma} \alpha \quad \text{s.t.} \;\; (18), (19), (20), (21). \quad (22)$$

Inequalities (18)-(19) contain bilinear terms, and (20)-(21) contain all linear terms. As in our other works involving convex hull functions, we combined the path-following algorithm and the direct iterative algorithm for the optimization problem. Numerical examples have been conducted which show the effectiveness of the algorithm and the advantage of nonlinear feedback control over linear feedback control for constrained control systems (see Example 1 in Section III-C).

### B. Reducing asymptotic output bound for $x(0) = 0$

Assume zero initial condition for the state: $x(0) = 0$. Suppose that $L_{V_c}$ is an invariant set for all $w$ such that $|w|_2 \le 1$. Then $x(t) \in L_{V_c}$ for all $t \ge 0$. Let

$$\delta_b = \max\{|Cx|_2 : \;\; x \in L_{V_c}\}, \quad (23)$$

then $\delta_b$ is an upper bound for the output $y(t)$. To minimize this output bound, we may construct an invariant set $L_{V_c}$ within $X_c$ with a minimal $\delta_b$. In what follows, we use matrix inequalities to characterize $\delta_b$.

From (23), $\delta_b$ is the minimal $\delta$ such that $x^{\rm T} C^{\rm T} C x \le \delta^2$ for all $x \in L_{V_c}$, i.e.,

$$\delta_b = \min\{\delta : \;\; L_{V_c} \subset \delta \mathcal{E}(C^{\rm T} C)\}.$$

Since $L_{V_c}$ is the convex hull of $\mathcal{E}(Q_j^{-1})$, $j \in I[1, J]$, and $\mathcal{E}(C^{\rm T} C)$ is a convex set, $L_{V_c} \subset \delta \mathcal{E}(C^{\rm T} C)$ if and only if $\mathcal{E}(Q_j^{-1}) \subset \delta \mathcal{E}(C^{\rm T} C)$ for all $j$, i.e., $C^{\rm T} C \le \delta^2 Q_j^{-1}$. By Schur complements, these inequalities are equivalent to

$$C Q_j C^{\rm T} \le \delta^2 I, \quad j \in I[1, J], \quad (24)$$

and $\delta_b$ is the minimal $\delta$ satisfying (24).

Similarly to the problem of enlarging the practical stability region, we also need to ensure that $|f(x)|_\infty \le 1$ for all $x \in L_{V_c}$ and $L_{V_c} \subset X_c$. After incorporating all the constraints, the problem of minimizing the output bound for $x(0) = 0$ can be formulated as

$$\inf_{Q_j > 0, \lambda_{jk} \ge 0, \beta > 0} \delta \quad \text{s.t.} \;\; (18), (24), (20), (21). \quad (25)$$

### C. Small asymptotic output bound with guaranteed practical stability region

The feedback law from solving (22) that achieves a large practical stability region may produce an asymptotic output bound not sufficiently small. On the other hand, the feedback law from solving (25) for small output bound may not be able to generate a satisfactory practical stability region. This is demonstrated through the following example.

*Example 1:* Consider the balance beam test rig in [18]. Under a certain current assignment strategy, the system is described as

$$\dot{x} = Ax + Bu + 0.25Bw, \quad y = Cx, \quad (26)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.211 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ -0.5258 \end{bmatrix}, \; C = [1 \;\; 0].$$

We added the disturbance term to incorporate unknown external force. The input constraint is $|u| \le 1$ and the state constraint is $|x_1| \le 0.004$. The disturbance is bounded by $|w| \le 1$. In [18], a saturated linear state feedback was designed to produce a contractively invariant ellipsoid within the state constraint set.

First, we would like to design a state feedback to achieve a large practical stability region. If quadratic Lyapunov function is used, a practical stability region is obtained as an ellipsoid, see the dashed boundary in Fig. 1. If the convex hull of three quadratics is used, a practical stability region is obtained as the convex hull of three ellipsoids, see the outer boundary in Fig 1. The matrices $Q_i$'s for the convex hull function and the matrices $Y_i$'s for the feedback law are given as follows:

$$Q_1 = \begin{bmatrix} 1.6000 & -2.0071 \\ -2.0071 & 201.4514 \end{bmatrix} \times 10^{-5}; \quad (27)$$

$$Q_2 = \begin{bmatrix} 1.5159 & -20.0736 \\ -20.0736 & 362.9777 \end{bmatrix} \times 10^{-5}; \quad (28)$$

$$Q_3 = \begin{bmatrix} 1.6000 & -9.4538 \\ -9.4538 & 327.4302 \end{bmatrix} \times 10^{-5}; \quad (29)$$

$$Y_1 = [3.7282 \;\; 11.4842] \times 10^{-3}; \quad (30)$$

$$Y_2 = [-1.9737 \;\; 49.5580] \times 10^{-3}; \quad (31)$$

$$Y_3 = [0.7836 \;\; 46.4727] \times 10^{-3}. \quad (32)$$

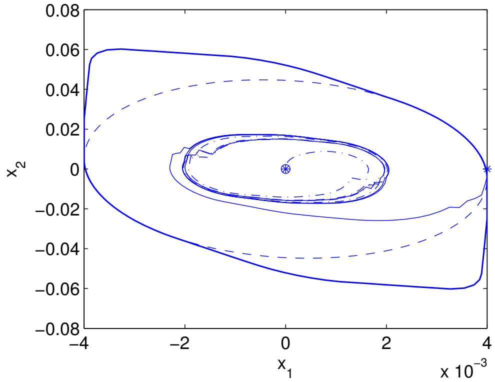


Fig. 1. Large stability region but poor disturbance rejection

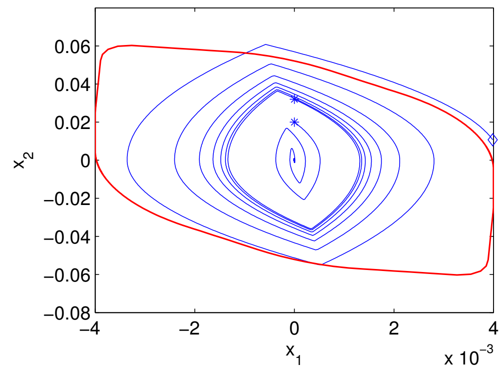


Fig. 2. Strong disturbance rejection but small stability region

Under the feedback law $u = f(x)$ constructed in Theorem 1 and a certain disturbance signal, a trajectory starting from $x(0) = (0.004, 0)$ is plotted in the same figure (solid curve). Another trajectory starting from the origin is also generated (dash-dotted curve). It should be noted that for each of the trajectories, the disturbance at each time instant is chosen such that the time derivative of $V_c$ is maximized. This can be considered as the worst "disturbance" with respect to the Lyapunov function $V_c$. As can be seen from the trajectories, the asymptotic output bound is about $2 \times 10^{-3}$ for both case.

Next, we attempt to minimize the asymptotic output bound. If we solve the optimization problem (25) with $J = 1$ (quadratic Lyapunov function), the output bound for $x(0) = 0$ can be made as small as possible by increasing $\beta$. As a result, the state feedback gain would be too high. For instance, with $\beta = 100$, the output bound is $1.4851 \times 10^{-7}$ and the linear state feedback gain is $F = [3.9113 \quad 0.0041] \times 10^6$. However, with the saturated state feedback law $u = \text{sat}(Fx)$, the practical stability region may not be large enough. We simulated the closed-loop system with a feedback law resulting from $\beta = 10$, which is $u = \text{sat}([3.9924 \quad 0.0399] \times 10^4 x)$. The asymptotic output bound for trajectories starting from $x(0) = 0$ is $1.4709 \times 10^{-5}$. In Fig. 2, we plotted two trajectories under this feedback law with initial conditions marked by "*". The trajectory starting from $(0.002, 0)$ goes near the origin but the trajectory starting from $(0.0032, 0)$ diverges and hit the righthand side boundary of the state constraint at the point marked with a diamond. This means that the balance beam hits the stator with a positive velocity. Thus the point $(0.0032, 0)$ and all points on the trajectory starting from it do not belong to the practical stability region.

The above example motivates us to combine the objective for large practical stability region and that for small asymptotic output bound. Can we ensure a certain practical stability region while reducing the asymptotic output bound as much as possible?

Suppose that we have two invariant sets $L_{V_{c1}}$, $L_{V_{c0}}$ with $L_{V_{c0}} \subset L_{V_{c1}}$, and $L_{V_{c0}}$ is considerably smaller than $L_{V_{c1}}$. Is it possible to construct a feedback law such that every trajectory starting from $L_{V_{c1}}$ will converge to the smaller invariant set $L_{V_{c0}}$? The following theorem says that under certain condition, this can be done.

*Theorem 2:* Consider $V_{c1}$ composed from $\hat{Q}_k \in \mathbb{R}^{n\times n}$, and $V_{c0}$ composed from $\bar{Q}_k \in \mathbb{R}^{n\times n}$, $\hat{Q}_k > 0, k \in I[1, J]$, $\bar{Q}_k > 0, k \in I[1, J]$. Suppose $\bar{Q}_k < \hat{Q}_k$ for each $k$. Suppose that there exist $\hat{Y}_k, \bar{Y}_k \in \mathbb{R}^{m\times n}, k \in I[1, J]$, $\lambda_{jk} \geq 0$, $j, k \in I[1, J]$ and $\beta > 0$ such that

$$\begin{bmatrix} 1 & \hat{Y}_{k\ell} \\ \hat{Y}_{k\ell}^{\mathrm{T}} & \hat{Q}_k \end{bmatrix} \geq 0, \quad k \in I[1, J], \ell \in I[1, m] \quad (33)$$

$$\begin{bmatrix} 1 & \bar{Y}_{k\ell} \\ \bar{Y}_{k\ell}^{\mathrm{T}} & \bar{Q}_k \end{bmatrix} \geq 0, \quad k \in I[1, J], \ell \in I[1, m] \quad (34)$$

$$\begin{bmatrix} \hat{M}_k + \beta\hat{Q}_k & T \\ T^{\mathrm{T}} & -\beta I \end{bmatrix} < 0 \quad \forall k, \quad (35)$$

$$\begin{bmatrix} \bar{M}_k + \beta\bar{Q}_k & T \\ T^{\mathrm{T}} & -\beta I \end{bmatrix} \leq 0 \quad \forall k, \quad (36)$$

where

$$\hat{M}_k = \hat{Q}_k A^{\mathrm{T}} + A\hat{Q}_k + \hat{Y}_k^{\mathrm{T}} B^{\mathrm{T}} + B\hat{Y}_k - \sum_{j=1}^{J} \lambda_{jk}(\hat{Q}_j - \hat{Q}_k)$$

$$\bar{M}_k = \bar{Q}_k A^{\mathrm{T}} + A\bar{Q}_k + \bar{Y}_k^{\mathrm{T}} B^{\mathrm{T}} + B\bar{Y}_k - \sum_{j=1}^{J} \lambda_{jk}(\bar{Q}_j - \bar{Q}_k)$$

Then a nonlinear state feedback satisfying the input constraint can be constructed such that $L_{V_{c1}}$ is an invariant set and for every $x(0) \in L_{V_{c1}}$, all possible trajectories under $|w(t)|_2 \leq 1$ will converge to $L_{V_{c0}}$. ⋄

By Theorem 1, both the level sets $L_{V_{c1}}$ and $L_{V_{c0}}$ can be made invariant with certain feedback laws satisfying the input constraints. But they are not just any pair of level sets satisfying the condition of Theorem 1. The matrices $\hat{Q}_j$'s and $\bar{Q}_j$'s need to satisfy the matrix inequalities with the same parameters $\beta$ and $\lambda_{jk}$'s.

If we want to ensure a certain practical stability region while reducing the asymptotic output bound as much as possible, we may first solve problem (22) for a large practical stability region, then use the resulting $\beta$ and $\lambda_{jk}$'s to minimize $\delta$ in problem (25). On the other hand, if we want to ensure a certain asymptotic output bound while enlarging the practical stability region, we may solve problem (25) first for a small asymptotic output bound then use the resulting $\beta$ and $\lambda_{jk}$'s to maximize $\alpha$ in (22).

**Example 1 continued.** We use Theorem 2 to design a feedback law to keep the large practical stability region in Fig. 1 while the asymptotic output bound is reduced. We pick $\hat{Q}_k = Q_k, \hat{Y}_k = Y_k$ where $Q_k$'s and $Y_j$'s are from (27)-(32), with corresponding $\lambda_{jk}$ and $\beta$. Then we solve the optimization problem (25) for minimal $\delta$ by fixing $\lambda_{jk}$'s and $\beta$, under the restriction that $\bar{Q}_k < \hat{Q}_k$. Let the optimal matrices to the resulting LMI problem be $\bar{Q}_k$'s and $\bar{Y}_k$'s. As a result, we obtain

$$\bar{Q}_1 = \bar{Q}_2 = \bar{Q}_3 = \begin{bmatrix} 0.5538 & -0.6924 \\ -0.6924 & 691.7638 \end{bmatrix} \times 10^{-7};$$

$$\bar{Y}_1 = \bar{Y}_2 = \bar{Y}_3 = [1.3129 \;\; 67.3419] \times 10^{-4}.$$

With these $\hat{Q}_k, \bar{Q}_k, \hat{Y}_k, \bar{Y}_k$, a feedback law can be constructed. Fig. 3 shows the two invariant sets and a trajectory starting from the boundary of the outer invariant set. The trajectory converges to the inner invariant set. At every time instant, the disturbance is chosen as the "worst one" which maximizes $\partial V_{cp}^T(x)(Ax + Bf(x) + Tw)$. The simulation confirms that the practical stability region is retained while the effect of disturbance is much suppressed.

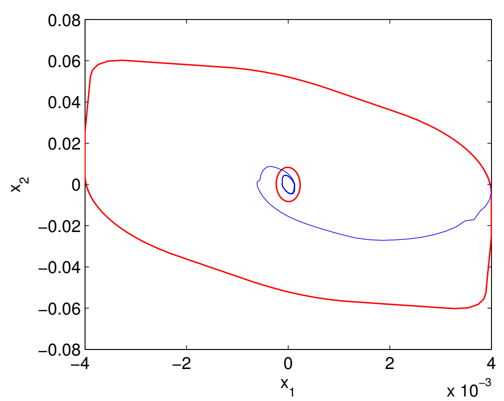


Fig. 3. Two invariant sets for both stability and disturbance rejection

## IV. Conclusions

We developed LMI-based methods for the construction of nonlinear feedback laws for linear systems with input and state constraints. The convex hull quadratic Lyapunov functions are used to guide the design for achieving a few objectives of stabilization and disturbance rejection. The advantages of nonlinear feedback over linear feedback has been demonstrated through some numerical examples.

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