Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Analytical Approximation Method for the Center Manifold in the Nonlinear Output Regulation Problem

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Abstract— **In nonlinear output regulation problems, it is necessary to solve the so-called regulator equations consisting of a partial differential equation and an algebraic equation. It is known that, for the hyperbolic zero dynamics case, solving the regulator equations is equivalent to calculating a center manifold for zero dynamics of the system. The present paper proposes a successive approximation method for obtaining center manifolds and shows its effectiveness by applying it for an inverted pendulum example.**

I. INTRODUCTION

The output regulation (alternatively, servomechanism) problem is one of central problems in control theory. This problem deals with asymptotic tracking of prescribed reference signals and/or asymptotic rejection of undesired disturbances in the output of a dynamical system when these signals and/or disturbances are generated by an autonomous exosystem.

For linear systems the output regulation problem was completely solved in the 1970s in the works of B. A. Francis, W. M. Wonham, E. J. Davison, and others [1], [2], [3]. This research resulted in the well-known internal model principle and clarified that the solvability of the linear output regulation problem is related to the solvability of two linear matrix equations, so-called *the regulator equations*.

In 1990 A. Isidori and C. I. Byrnes obtained a necessary and sufficient condition for the solvability of local nonlinear output regulation problems [4], which consists of a set of a partial differential equation and an algebraic equation which is called *the nonlinear regulator equations*. Also, it was shown that the nonlinear regulator equations are closely related to a center manifold of an extended system and that the internal model principle in the linear theory can be generalized with the notion of embedding. However, practical applications of output regulation are still difficult since no method is available to obtain exact solutions of the regulator equations. Analytical approximation methods based on the Taylor expansion are studied in [5], [6], [7], [8], [9], however, they often require solving equations of large size, especially when control systems are of high dimension or higher order approximations are necessary for precise output regulation. As a certain counterpart, numerical approximation methods based on neural networks are studied in [10], [11]. On the other hand, numerical methods based on the successive approximation of the differential part of the regulator equation by the finite-element method while trying to minimize a functional expressing the error of its algebraical part is proposed in [12].

In this paper, we modify, by using a successive approximation method, the proof of the Center Manifold Theorem which employs the Contraction Mapping Theorem. This successive approximation does not require solving any equations and seems to be suitable for computer calculations. This method is applicable for nonlinear output regulation problem for the hyperbolic zero dynamics case in which the regulator equations reduce to the center manifold equation for zero dynamics. Furthermore, it can be seen that the calculations of the proposed algorithm are all algebraic when the nonlinearities are polynomial. By using an inverted pendulum example, we demonstrate how this method works. We believe that even apart from the significance in the controller design for output regulation problems, this approximation method is of importance in dynamical systems theory since center manifolds play important roles for such as bifurcation theory (see, e.g., [13]).

II. THE NONLINEAR OUTPUT REGULATION PROBLEM

For the sake of completeness, we review here some facts about the nonlinear output regulation problem. For a detailed description consult [14], [15], [16], [17].

A. System Equations and Basic Assumptions

We consider the problem of output regulation for nonlinear systems modeled by equations of the form

$$\begin{aligned} \dot{x} &= f(x,u,w) \\ e &= h_r(x,u,w) \\ y &= h_m(x,u,w), \end{aligned} \tag{1}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, exogenous signal $w \in W \subset \mathbb{R}^q$, regulated output $e \in \mathbb{R}^{p_r}$, and measured output $y \in \mathbb{R}^{p_m}$. The exogenous signal w, which can be viewed as a disturbance or as a reference signal, is generated by an external autonomous system, which is called an $exosystem$,

$$\dot{w} = s(w). \tag{2}$$

It is assumed that the functions $f(x,u,w)$, $h_r(x,u,w)$, $h_m(x,u,w)$ and $s(w)$ are C^k (for some large k) of

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their arguments, and also that $f(0,0,0)=0$, $h_r(0,0,0)=0$, $h_m(0,0,0)=0$ and $s(0)=0$.

In the context of the output regulation problem, plant (1) and exosytem (2) satisfy the following assumptions.

Assumption 1 (Neutral Stability) *The equilibrium $w = 0$ of the exosytem (2) is stable (in the ordinary sense of Lyapunov), and there exists an open neighborhood $W_0 \subset W$ of the point $w = 0$ in which every point is Poisson stable.*

Assumption 2 (Linear Stabilizability) *The pair*

$$\left(\tfrac{\partial f}{\partial x}(0,0,0), \tfrac{\partial f}{\partial u}(0,0,0)\right)$$

is stabilizable and the pair

$$\left(\left[\tfrac{\partial h_m}{\partial x}(0,0,0) \quad \tfrac{\partial h_m}{\partial w}(0,0,0)\right], \begin{bmatrix} \tfrac{\partial f}{\partial x}(0,0,0) & \tfrac{\partial f}{\partial w}(0,0,0) \\ 0 & \tfrac{\partial s}{\partial w}(0) \end{bmatrix}\right)$$

is detectable.

B. The Nonlinear Local Output Regulation Problem

We consider a controller modeled by equations of the form

$$\begin{aligned} \dot{\xi} &= \eta(\xi, y) \\ u &= \theta(\xi), \end{aligned} \tag{3}$$

with state $\xi \in \mathbb{R}^r$, in which $\eta(\xi, y)$ and $\theta(\xi)$ are C^k functions of their arguments satisfying $\eta(0,0)=0$, $\theta(0)=0$. The problem of local output regulation is to design the controller (3) such that the closed-loop system

$$\begin{aligned} \dot{x} &= f(x, \theta(\xi), w) \\ \dot{\xi} &= \eta(\xi, h_m(x, \theta(\xi), w)) \\ \dot{w} &= s(w) \\ e &= h_r(x, \theta(\xi), w), \end{aligned} \tag{4}$$

satisfies the following two properties.

Property 1 (Local Internal Stability) *For $w(t) \equiv 0$ the closed-loop system (4) has an asymptotically stable linearization at the origin.*

Property 2 (Local Asymptotic Output Zeroing) *For every solution of the closed-loop system (4) starting close enough to the origin $(x, \xi, w) = (0,0,0)$ regulated output satisfies*

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} h_r(x(t), \theta(\xi(t)), w(t)) = 0.$$

C. Solvability of the Local Output Regulation Problem

The following result is well known as necessary and sufficient condition for the solvability of the local output regulation problem.

Theorem 1 *Consider the plant (1), with exosystem (2). Suppose that Assumptions 1 and 2 are satisfied. Then the problem of local output regulation is solvable if and only if there exist mappings $x = \pi(w)$ and $u = c(w)$, with $\pi(0) = 0$ and $c(0) = 0$, both defined in a neighborhood $W_0 \subset W$ of the origin, satisfying the conditions*

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), c(w), w) \\ 0 &= h_r(\pi(w), c(w), w). \end{aligned} \tag{5}$$

The set of equations (5), which determine the solvability of the local output regulation problem, are called the *regulator equations*. It is well known that the existence of solutions of the regulator equations is intimately related to the properties of the so-called *zero dynamics* of the nonlinear system (1) with (2). The zero dynamics of a given nonlinear system is essentially the collection of all the state trajectories which are compatible with the constraint that the output is identically zero for all time. In order to present a useful sufficient condition, the following assumption is needed.

Assumption 3 *There exists a locally maximal output zeroing manifold M_e for plant (1) with exosystem (2), which is characterized by*

$$M_e = \{\, (x, w) \in \Gamma_e \mid H_e(x, w) = 0 \,\},$$

where Γ_e is an open neighborhood of the origin of $\mathbb{R}^{n+q}$ and $H_e : \mathbb{R}^{n+q} \to \mathbb{R}^l$ for some integer l is a sufficiently smooth function satisfying $H_e(0,0) = 0$ and

$$\textit{rank}\ \frac{\partial H_e}{\partial x}(0,0) = l.$$

The following theorem provides a sufficient condition for the solvability of the regulator equations.

Theorem 2 *Suppose that Assumption 3 is satisfied. Then there exist smooth mappings $x = \pi(w)$ and $u = c(w)$, with $\pi(0) = 0$ and $c(0) = 0$, both defined in a neighborhood $W_0 \subset W$ of the origin, satisfying the regulator equations (5), provided that the zero dynamics of plant (1) with exosystem (2) have a hyperbolic equilibrium at the origin $(x, w) = (0, 0)$.*

The fact that M_e is an output zeroing manifold implies the existence of a locally defined sufficiently smooth feedback control $u_e(x, w)$ satisfying $u_e(0,0) = 0$ such that, under the control $u = u_e(x, w)$, M_e is an invariant manifold of the system, which is contained in the kernel of the mapping $h_r(x, u_e(x, w), w)$. More specifically, we can rewrite plant (1) with exosystem (2) as follows

$$\begin{aligned} \dot{x_1} &= f_1(x_1, x_2, u, w) \\ \dot{x_2} &= f_2(x_1, x_2, u, w) \\ \dot{w} &= s(w) \\ e &= h_r(x_1, x_2, u, w), \end{aligned} \tag{6}$$

then there exist $x_1 = \sigma(x_2, w)$ and $u = u_e(x, w)$ such that

$$\begin{aligned} \frac{\partial \sigma}{\partial x_2} f_2(\sigma(x_2, w), x_2, u_e(\sigma(x_2, w), x_2, w), w) + \frac{\partial \sigma}{\partial w} s(w) \\ = f_1(\sigma(x_2, w), x_2, u_e(\sigma(x_2, w), x_2, w), w) \\ 0 = h_r(\sigma(x_2, w), x_2, u_e(\sigma(x_2, w), x_2, w), w). \end{aligned}$$

Furthermore, the zero dynamics of the composite system (6) are governed by the following system

$$\dot{x}_2 = f_2(\sigma(x_2,w), x_2, u_e(\sigma(x_2,w), x_2, w), w) =: \delta(x_2, w),$$
$$\dot{w} = s(w).$$

Since the zero dynamics have a hyperbolic equilibrium at the origin, from the center manifold theorem there is a C^k center manifold $x_2 = \gamma(w)$ satisfying $\frac{\partial\gamma}{\partial w}s(w) = \delta(\gamma(w), w)$. By using this center manifold, we can design the controller for the local output regulation problem satisfying Properties 1 and 2.

We summarize that under Assumptions 1, 2 and 3, the local output regulation problem reduces to compute the center manifold for zero dynamics.

III. ANALYTICAL APPROXIMATION METHODS FOR CENTER MANIFOLDS

As we have seen in the previous section, the construction of the control laws for solving the output regulation problem for the hyperbolic zero dynamics case relies on the solution of the center manifold equation. For a practical computation in this case, we propose a successive approximation method to calculate the center manifold analytically, instead of the standard Taylor expansion method. The advantage of the proposed method is its recursiveness and integrability: there is no need to form or solve simultaneous equations like the Taylor expansion method and in the case of polynomial nonlinearities, the integrations required in the algorithm are all algebraic since the integrands appearing consist of exponential and trigonometric functions of t.

In this section we modify the proof of the Center Manifold Theorem, which is originally proven by using the Contraction Mapping Theorem (see, e.g., [18], [13] and [19]), by using the successive iteration method which is easy to carry out with computers.

We consider the system of the form

$$\begin{cases} \dot{x} = Ax + f(x,y,z) \\ \dot{y} = By + g(x,y,z) \\ \dot{z} = Cz + h(x,y,z), \end{cases} \tag{7}$$

where $(x,y,z)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^l$. We introduce following assumptions.

Assumption 4 *A is an $n\times n$ real matrix, which has eigenvalues with zero real parts, satisfying that for each $a > 0$ there exist a constant $C_1(a) > 0$ such that for all $t\in\mathbb{R}$ and $x\in\mathbb{R}^n$,*

$$\left|e^{At}x\right| \leqslant C_1(a)e^{a|t|}|x|.$$

B is an $m\times m$ real matrix, which has eigenvalues with negative real parts, satisfying that there exist a constant $b > 0$ and $C_2 > 0$ such that for all $t \geqslant 0$ and $y\in\mathbb{R}^m$,

$$\left|e^{Bt}y\right| \leqslant C_2e^{-bt}|y|.$$

C is an $l\times l$ real matrix, which has eigenvalues with positive real parts, satisfying that there exist a constant $c > 0$ and $C_3 > 0$ such that for all $t \leqslant 0$ and $z\in\mathbb{R}^l$,

$$\left|e^{Ct}z\right| \leqslant C_3e^{ct}|z|.$$

Assumption 5 *f:$\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^l\to\mathbb{R}^n$, g:$\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^l\to\mathbb{R}^m$, h:$\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^l\to\mathbb{R}^l$ are C^r functions ($r\geqslant 2$) satisfying that for $|x|\leqslant\varepsilon$, $|x'|\leqslant\varepsilon$, $|y|\leqslant\varepsilon$, $|y'|\leqslant\varepsilon$, $|z|\leqslant\varepsilon$, $|z'|\leqslant\varepsilon$ there exist continuous functions $K_1(\varepsilon)$, $K_2(\varepsilon)$, $K_3(\varepsilon)$ such that*

$$\begin{cases} |f(x,y,z)|\leqslant\varepsilon K_1(\varepsilon) \\ |g(x,y,z)|\leqslant\varepsilon K_2(\varepsilon) \\ |h(x,y,z)|\leqslant\varepsilon K_3(\varepsilon) \\ |f(x,y,z)-f(x',y',z')|\leqslant K_1(\varepsilon)(|x-x'|+|y-y'|+|z-z'|) \\ |g(x,y,z)-g(x',y',z')|\leqslant K_2(\varepsilon)(|x-x'|+|y-y'|+|z-z'|) \\ |h(x,y,z)-h(x',y',z')|\leqslant K_3(\varepsilon)(|x-x'|+|y-y'|+|z-z'|), \end{cases}$$

and $f(0,0,0)=0, g(0,0,0)=0, h(0,0,0)=0,$

$$\left(\tfrac{\partial f}{\partial x}(0,0,0), \tfrac{\partial f}{\partial y}(0,0,0), \tfrac{\partial f}{\partial z}(0,0,0)\right)=0,$$
$$\left(\tfrac{\partial g}{\partial x}(0,0,0), \tfrac{\partial g}{\partial y}(0,0,0), \tfrac{\partial g}{\partial z}(0,0,0)\right)=0,$$
$$\left(\tfrac{\partial h}{\partial x}(0,0,0), \tfrac{\partial h}{\partial y}(0,0,0), \tfrac{\partial h}{\partial z}(0,0,0)\right)=0,$$
$$K_1(0)=0, K_2(0)=0, K_3(0)=0.$$

Let us define the sequences $\{x_k(t,\xi)\}$, $\{v_k(\xi)\}$ and $\{w_k(\xi)\}$ by

$$\begin{cases} x_{k+1}(t,\xi) = e^{At}\xi \\ \qquad + \displaystyle\int_0^t e^{A(t-s)}f(x_k(s,\xi), v_k(x_k(s,\xi)), w_k(x_k(s,\xi)))ds \\ v_{k+1}(\xi) = \displaystyle\int_0^\infty e^{Bs}g(x_k(s,\xi), v_k(x_k(s,\xi)), w_k(x_k(s,\xi)))ds \\ w_{k+1}(\xi) = -\displaystyle\int_{-\infty}^0 e^{Cs}h(x_k(s,\xi), v_k(x_k(s,\xi)), w_k(x_k(s,\xi)))ds \end{cases} \tag{8}$$

for k=$0,1,2,\cdots$, and

$$\begin{cases} x_0(t,\xi) = e^{At}\xi \\ v_0(\xi) = 0 \\ w_0(\xi) = 0 \end{cases} \tag{9}$$

The following theorem states that the sequences $\{x_k(t,\xi)\}$, $\{v_k(\xi)\}$ and $\{w_k(\xi)\}$ are the approximating solutions to the exact solutions of (7) on the center manifold.

Theorem 3 *Under Assumptions 4 and 5, system (7) has a local center manifold $y = v(x)$, $z = w(x)$, and sequences $v_k(\xi)$, $w_k(\xi)$ in (8) are uniformly convergent to this center manifold as $k\to\infty$.*

Proof: First, let $\psi:\mathbb{R}^n\to[0,1]$ be a C^∞ function, which is called cut-off function, such that

$$\psi(x) = \begin{cases} 1, & |x|\leqslant 1 \\ 0, & |x|\geqslant 2. \end{cases}$$

For $\varepsilon>0$ define $F:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^l\to\mathbb{R}^n$, $G:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^l\to\mathbb{R}^m$, $H:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^l\to\mathbb{R}^l$ by

$$\begin{aligned}F(x,y,z)&=f(x\psi(x/\varepsilon),y,z)\\G(x,y,z)&=g(x\psi(x/\varepsilon),y,z)\\H(x,y,z)&=h(x\psi(x/\varepsilon),y,z).\end{aligned}$$

Note that $F(x,y,z)=f(x,y,z)$, $G(x,y,z)=g(x,y,z)$, $H(x,y,z)=h(x,y,z)$ for all $|x|\leqslant\varepsilon$. We can show that extended system of (7)

$$\left\{\begin{array}{l}\dot{x}=Ax+F(x,y,z)\\\dot{y}=By+G(x,y,z)\\\dot{z}=Cz+H(x,y,z)\end{array}\right.\tag{10}$$

has a center manifold for sufficiently small ε, then show that this manifold is a local center manifold for the original system (7) in a sufficiently small neighborhood of the origin (see, e.g., [13]).

Next, let us define the sequences $\{\bar{x}_k(t,\xi)\}$, $\{\bar{v}_k(\xi)\}$ and $\{\bar{w}_k(\xi)\}$ by

$$\begin{cases}\bar{x}_{k+1}(t,\xi)=e^{At}\xi\\\qquad+\displaystyle\int_0^t e^{A(t-s)}F(\bar{x}_k(s,\xi),\bar{v}_k(\bar{x}_k(s,\xi)),\bar{w}_k(\bar{x}_k(s,\xi)))ds\\\bar{v}_{k+1}(\xi)=\displaystyle\int_0^\infty e^{Bs}G(\bar{x}_k(s,\xi),\bar{v}_k(\bar{x}_k(s,\xi)),\bar{w}_k(\bar{x}_k(s,\xi)))ds\\\bar{w}_{k+1}(\xi)=-\displaystyle\int_{-\infty}^0 e^{Cs}H(\bar{x}_k(s,\xi),\bar{v}_k(\bar{x}_k(s,\xi)),\bar{w}_k(\bar{x}_k(s,\xi)))ds\end{cases}\tag{11}$$

for $(k=0,1,2,\cdots)$, and

$$\begin{cases}\bar{x}_0(t,\xi)=e^{At}\xi\\\bar{v}_0(\xi)=0\\\bar{w}_0(\xi)=0,\end{cases}\tag{12}$$

then we can show that $\bar{v}_k(\xi)$ and $\bar{w}_k(\xi)$ uniformly converge to the center manifold of (10) as $k\to\infty$. For simplicity we prove here the theorem for the following stable eigenvalue case, the proof of full system (7) is similar to this.

$$\left\{\begin{array}{l}\dot{x}=Ax+f(x,y)\\\dot{y}=By+g(x,y).\end{array}\right.$$

1) First, we show that the sequence $\{\bar{v}_k(\xi)\}$ is uniformly bounded, that is, there is a constant $\varepsilon>0$ such that for every $\xi\in\mathbb{R}^n$, $|\bar{v}_k(\xi)|\leqslant\varepsilon$, $(k=0,1,2,\cdots)$.
This will be shown by an induction.
2) Next, we show that the sequence $\{\bar{v}_k(\xi)\}$ satisfies a Lipschitz condition, that is, there is a constant $p_1>0$ such that for every $\xi,\xi'\in\mathbb{R}^n$, $|\bar{v}_k(\xi)-\bar{v}_k(\xi')|\leqslant p_1|\xi-\xi'|$, $(k=0,1,2,\cdots)$.
This will be also shown by an induction. If $k=0$ the statement is apparently true. Assume inductively that the statement is true up to k. Let $k+1$, then by using

$$\begin{aligned}&\left|\bar{x}_k(t,\xi)-\bar{x}_k(t,\xi')\right|\\&\leqslant C_1(a)\left|\xi-\xi'\right|e^{a|t|}\sum_{k=0}^{k}\frac{\left[C_1(a)K_1(\varepsilon)(1+p_1)|t|\right]^k}{k!},\end{aligned}$$

we obtain

$$|\bar{v}_{k+1}(\xi)-\bar{v}_{k+1}(\xi')|\leqslant\frac{C_1(a)C_2K_2(\varepsilon)(1+p_1)}{b-a-C_1(a)K_1(\varepsilon)(1+p_1)}\left|\xi-\xi'\right|.$$

Thus, it suffices to choose $a>0,\varepsilon>0$ such that

$$\left\{\begin{array}{l}b-a>0\\\gamma_2:=\dfrac{C_1(a)K_1(\varepsilon)(1+p_1)}{b-a}<1\\\gamma_3:=\dfrac{C_1(a)C_2K_2(\varepsilon)(1+p_1)}{b-a-C_1(a)K_1(\varepsilon)(1+p_1)}\leqslant p_1.\end{array}\right.$$

3) Next, we show that the sequence $\{\bar{v}_k(\xi)\}$ is uniformly convergent.
Since the set of Lipschitz functions is a complete space with supremum norm $\left\|\bar{v}_k\right\|=\sup_{\xi\in\mathbb{R}^n}\left|\bar{v}_k(\xi)\right|$, it suffices to show that $\left\|\bar{v}_k-\bar{v}_{k-1}\right\|\to0$ as $k\to\infty$. Let $\left\|\bar{v}_0-\bar{v}_{-1}\right\|=\varepsilon$, then inductively we get

$$\begin{aligned}&\left|\bar{x}_k(t,\xi)-\bar{x}_{k-1}(t,\xi)\right|\leqslant\frac{C_1(a)K_1(\varepsilon)}{a}e^{a|t|}\\&\quad\times\sum_{l=0}^{k-1}\frac{\left[C_1(a)K_1(\varepsilon)(1+p_1)|t|\right]^{k-1-l}}{(k-1-l)!}\left\|\bar{v}_l-\bar{v}_{l-1}\right\|.\end{aligned}\tag{13}$$

Using (13), we obtain

$$\left\|\bar{v}_{k+1}-\bar{v}_k\right\|\leqslant\frac{\beta_{k+1}}{\gamma_2}\leqslant\frac{\beta_1}{\gamma_2}{\gamma_4}^k,\tag{14}$$

where

$$\begin{aligned}\beta_k&=\frac{C_2K_2(\varepsilon)}{a}\sum_{l=0}^{k-1}\left\|\bar{v}_l-\bar{v}_{l-1}\right\|{\gamma_2}^{k-l},(k=1,2,3,\cdots)\\\gamma_4&:=\frac{C_1(a)K_1(\varepsilon)(1+p_1)}{b-a}+\frac{C_2K_2(\varepsilon)}{a}=\gamma_2+\frac{C_2K_2(\varepsilon)}{a},\\\beta_1&=\frac{\varepsilon C_1(a)C_2K_1(\varepsilon)K_2(\varepsilon)(1+p_1)}{a(b-a)}.\end{aligned}$$

Therefore, if $a>0,\varepsilon>0$ such that $\gamma_4<1$, then the sequence $\{\bar{v}_k(\xi)\}$ is uniformly convergent.
4) Next, we show that the sequence $\{\bar{x}_k(t,\xi)\}$ is pointwise convergent.
Let us define

$$S_k=\sum_{l=0}^{k-1}\frac{\{(b-a)\gamma_2|t|\}^l}{l!},\ (k=1,2,3,\cdots),$$

and using (13), (14) and formula of Abel's partial summation, we get

$$\begin{aligned}&\left|\bar{x}_{k+1}(t,\xi)-\bar{x}_k(t,\xi)\right|\\&\leqslant\frac{C_1(a)K_1(\varepsilon)}{a}e^{a|t|}\sum_{l=0}^{k}\frac{\left\{(b-a)\gamma_2|t|\right\}^{k-l}}{(k-l)!}\left\|\bar{v}_l-\bar{v}_{l-1}\right\|\\&=\frac{\beta_1}{\gamma_2}S_{k+1}{\gamma_4}^{k-2}(1+\gamma_4)\\&\leqslant\frac{C_1(a)K_1(\varepsilon)}{a}\frac{\beta_1}{\gamma_2}e^{\{a+(b-a)\gamma_2\}|t|}{\gamma_4}^{k-2}(1+\gamma_4)\end{aligned}$$

for $k=2,3,4,\cdots$. Thus, if $\gamma_4<1$ is satisfied, $\{\bar{x}_k(t,\xi)\}$ is pointwise convergent for bounded interval of t.

5) Finally, we suppose that $\{\bar{v}_k(\xi)\}$ is uniformly convergent to $\bar{v}(\xi)$ and $\{\bar{x}_k(t,\xi)\}$ is pointwise convergent to $\bar{x}(t,\xi)$, then we show that $\bar{x}(t,\xi)$ is a solution of

$$\dot{x} = Ax + F(x, h(x)) \tag{15}$$

with initial value $\bar{x}(0,\xi) = \xi$, and that $\bar{v}(\bar{x}(t,\xi))$ is a center manifold of (10).

We can apply Lebesgue's termwise integration theorem to (11), then we get

$$\bar{x}(t,\xi)=e^{At}\xi+\int_0^t e^{A(t-s)}F(\bar{x}(s,\xi),\bar{v}(\bar{x}(s,\xi)))ds$$
$$\bar{v}(\bar{x}(t,\xi))=e^{Bt}\bar{v}(\xi)+\int_0^t e^{B(t-s)}G(\bar{x}(s,\xi),\bar{v}(\bar{x}(s,\xi)))ds,$$

moreover, differentiating both sides of these equations by t, we obtain

$$\begin{aligned}&\dot{\bar{x}}(t,\xi) = A\bar{x}(t,\xi) + F(\bar{x}(t,\xi),\bar{v}(\bar{x}(t,\xi)))\\&\frac{\partial \bar{v}}{\partial x}(\bar{x}(t,\xi))\big[A\bar{x}(t,\xi) + F(\bar{x}(t,\xi),\bar{v}(\bar{x}(t,\xi)))\ \big]\\&\qquad = B\bar{v}(\bar{x}(t,\xi)) + G(\bar{x}(t,\xi),\bar{v}(\bar{x}(t,\xi))).\end{aligned}$$

Therefore, $\bar{x}(t,\xi)$ is a solution of (15) with $\bar{x}(0,\xi) = \xi$, and $\bar{v}(\bar{x}(t,\xi))$ is a center manifold of (10). ∎

IV. EXAMPLE

Let us consider an application of Theorem 3 to nonlinear output regulation problem. Consider the inverted pendulum on a cart system which is shown in Fig. 1. This system is a well-known unstable, nonminimum phase, hyperbolic zero dynamics, nonlinear system. Assume the pendulum is freely hinged to the cart, which is free to move on a horizontal plane, and the control available is a force applicable to the cart. The control problem is to design a state feedback controller for the system such that the position of the cart can asymptotically track a sinusoidal input. This problem has been well studied in [20].

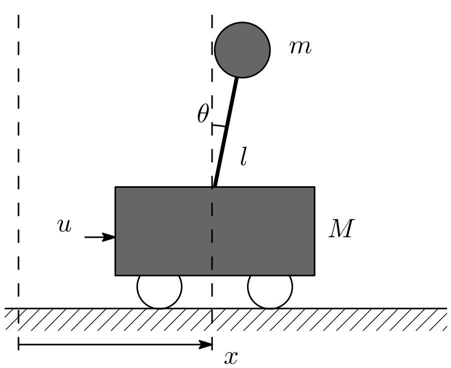

Fig. 1. Inverted pendulum on a cart.

The equations of motion for this system can be described by [20] (see also [16])

$$\begin{aligned}(M+m)\ddot{x} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + b\dot{x} = u,\\ ml^2\ddot{\theta} + ml\ddot{x}\cos\theta - mgl\sin\theta = 0,\end{aligned}$$

where M is the mass of the cart, m is the mass of the pendulum, l is the length of the pendulum, g is the gravitational acceleration, b is the coefficient of viscous friction for the motion of the cart, θ is the angle that the pendulum makes with vertical, x is the position of the cart, and u is the applied force. With the choice of the state variables $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \theta$, $x_4 = \dot{\theta}$, the state-space equations of the system are

$$\dot{x} = f(x) + g(x)u, \ \ y = h_m(x), \tag{16}$$

where,

$$x = \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\end{bmatrix},\ g(x) = \begin{bmatrix} 0\\ \frac{1}{M+m\sin^2 x_3}\\ 0\\ \frac{-\cos x_3}{l(M+m\sin^2 x_3)}\end{bmatrix},\ h_m(x) = x_1,$$

$$f(x) = \begin{bmatrix} x_2\\ \frac{mlx_4^2\sin x_3 - bx_2 - mg\cos x_3\sin x_3}{M+m\sin^2 x_3}\\ x_4\\ \frac{(M+m)g\sin x_3 + bx_2\cos x_3 - mlx_4^2\sin x_3\cos x_3}{l(M+m\sin^2 x_3)}\end{bmatrix}.$$

The reference input signal $y_d{=}A_m\sin\omega t$ can be considered as an output of the linear harmonic oscillator (exosystem)

$$\begin{aligned}&\dot{w_1} = \omega w_2,\ \ w_1(0) = 0,\\&\dot{w_2} = -\omega w_1,\ \ w_2(0) = A_m,\\&y_d = w_1.\end{aligned} \tag{17}$$

By using the feedback transformation

$$\begin{aligned}u = &-(M + m\sin^2 x_3)\omega^2 w_1\\&-(mlx_4^2\sin x_3 - bx_2 - mg\cos x_3\sin x_3),\end{aligned}$$

system (16) and the exosystem (17) take the normal form (see [16] for details), then the zero dynamics of the composite system become

$$\begin{aligned}&\dot{x_3} = x_4, &&\dot{w_1} = \omega w_2,\\&\dot{x_4} = \tfrac{\omega^2}{l}w_1\cos x_3 + \tfrac{g}{l}\sin x_3, &&\dot{w_2} = -\omega w_1,\end{aligned} \tag{18}$$

and thus, the zero dynamics of the inverted pendulum system has a hyperbolic equilibrium as the eigenvalues of the Jacobian matrix at origin are given by $\pm\sqrt{g/l}$. By Theorem 2, it follows that the solution of regulator equations associated with the inverted pendulum system exists, and the regulator equations reduce to the center manifold equation (partial differential equation part) associated with (18) as follows

$$\begin{aligned}&\frac{\partial\gamma_1}{\partial w_1}\omega w_2 - \frac{\partial\gamma_1}{\partial w_2}\omega w_1 = \gamma_2,\\&\frac{\partial\gamma_2}{\partial w_1}\omega w_2 - \frac{\partial\gamma_2}{\partial w_2}\omega w_1 = \frac{\omega^2}{l}w_1\cos\gamma_1 + \frac{g}{l}\sin\gamma_1.\end{aligned} \tag{19}$$

An approximate solution to (19) can be obtained by using Theorem 3. First we choose the matrix T such that the linear part of (18) is diagonalized. Next let $[\bar{x}^1\ \bar{x}^2\ \bar{y}\ \bar{z}]^T = T^{-1}[w_1\ \ w_2\ \ x_3\ \ x_4]^T$, then in the new coordinate (18) becomes

$$\begin{aligned}&\dot{\bar{x}}^1 = -\omega\bar{x}^2, &&\dot{\bar{y}} \ = -\sqrt{g/l}\ \bar{y} + \bar{g}(\bar{x},\bar{y},\bar{z}),\\&\dot{\bar{x}}^2 = \omega\bar{x}^1, &&\dot{\bar{z}} \ = \sqrt{g/l}\ \bar{z} + \bar{h}(\bar{x},\bar{y},\bar{z}),\end{aligned} \tag{20}$$

where $\bar{g}(\bar{x},\bar{y},\bar{z}) = \bar{h}(\bar{x},\bar{y},\bar{z}) = \frac{\omega^2\bar{x}^1}{2l}(\cos\Theta - 1) - \frac{g}{2l}(\sin\Theta - \Theta)$, $\Theta = \frac{\omega^2\bar{x}^1}{g+l\omega^2} + \frac{\sqrt{l}\bar{y}-\sqrt{l}\bar{z}}{\sqrt{g}}$. Now, Theorem 3 can be applied to (20), after the calculations of sequences (8), the sequences are transformed into the original coordinates by using T, then we obtain the approximate solution to (19). We remark that approximations $\sin x \sim x - x^3/6$ and $\cos x \sim 1 - x^2/2$ are employed here.

We now can solve the local output regulation problem for the system (16) and the exosystem (17) using the controller design method from [16]. First, we can obtain an approximation of the solution of the regulator equations as follows

$$\pi(w_1,w_2) = \left[\begin{array}{cccc} w_1 & \omega w_2 & \gamma_1(w_1,w_2) & \gamma_2(w_1,w_2) \end{array}\right]^T,$$
$$c(w_1,w_2) = -(M + m(\sin\gamma_1)^2)\omega^2 w_1 - \left(ml\gamma_2^2\sin\gamma_1 - b\omega w_2 - mg\cos\gamma_1\sin\gamma_1\right).$$

Next, choose a matrix K such that $\frac{\partial f(0)}{\partial x} + g(0)K$ is Hurwitz. Then the controller $u = c(w) + K(x - \pi(w))$ solves the problem. Let b =12.98 [N/(m/sec)] , M =1.378 [kg], l =0.325 [m], g =9.8 [m/sec^2], m =0.051 [kg], ω =1.8 [rad/sec], A_m=1, and let the eigenvalues of the matrix $\frac{\partial f(0)}{\partial x} + g(0)K$ be $[-0.763 \pm 2.27j, -1.13 \pm 0.745j]$, then $K = [0.48\ 13.70\ 19.11\ 1.93]$. The results of simulation are presented in Fig. 2 – Fig. 4. Fig. 2 shows the variable $x_1(t)$ and the external signal $w_1(t)$, Fig. 3 shows the regulated output $e(t)$, and Fig. 4 shows the control $u(t)$. It is seen that the nonlinear controller performs much better than the linear controller.

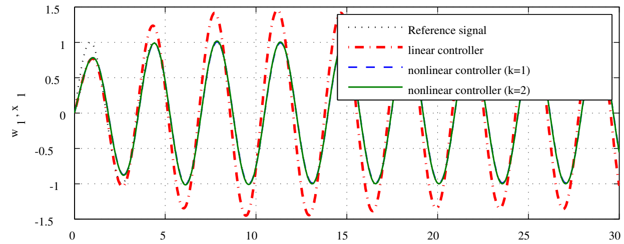

Fig. 2. Reference signal $w_1(t)$ and $x_1(t)$.

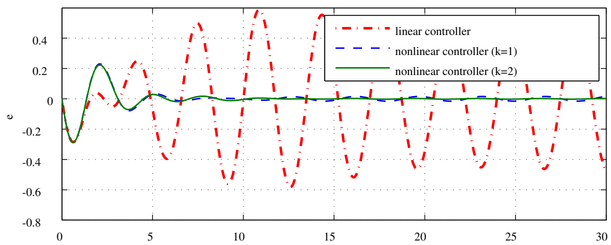

Fig. 3. Regulated output $e(t)$.

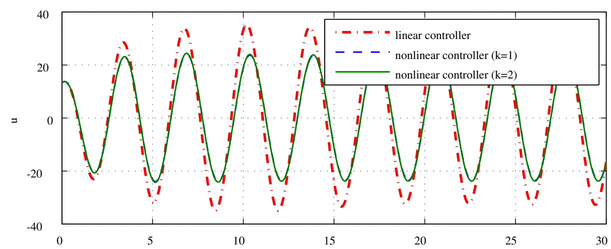

Fig. 4. Control $u(t)$.

V. CONCLUSIONS

In this paper, we proposed an successive approximation method for the computation of center manifolds. This method does not require solving any equations unlike the Taylor expansion method and seems to be suitable for computer implementation. The proposed algorithm computes approximate solutions recursively and, when nonlinearities are polynomial, the computations are all algebraic. We demonstrated how the proposed method works for the output regulation problem with hyperbolic zero dynamics using an inverted pendulum system. We would like to remark that the method may also be useful for bifurcation theory in dynamical system theory in which center manifolds play an important role.

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