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Stability Analysis of Dynamical Neural Networks with Uncertain Delays

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Abstract— **The problem of stability of dynamical neural networks with uncertain delays is studied, where uncertain delays are assumed to be constant. In this paper, a new approach is developed to establish delay-dependent sufficient conditions for asymptotic stability of delayed neural networks. The new approach is a combination of the discretized Lyapunov functional method and the free-weighting matrix technique. The established delay-dependent sufficient conditions are expressed by means of linear matrix inequalities, and thus are easily checkable. The new delay-dependent stability conditions are further illustrated by numerical results and are also compared with the existing results.**

I. INTRODUCTION

During the past decade, the research on neural networks has been an active area due to potential important applications of neural networks in such fields as control engineering, signal processing, pattern recognition, associative memories and so on. In these applications, the stability of the equilibrium plays a critical role. Given the fact that in biological and artificial neural systems there inevitably exist integration and communication delays which may cause oscillation and instability, considerable effort has been made on stability analysis of delayed neural networks (DNNs) over the past years. The stability results for delayed neural networks can be classified into two categories: delay-independent stability criteria and delay-dependent stability criteria. The former do not make use of information on the size of delays while the latter include such information. It is known that delay-dependent stability conditions are generally less conservative than delay-independent ones, especially when the size of the delay is small.

Among the commonly-used approaches for establishing delay-dependent stability conditions of delayed neural networks, Lyapunov function approaches [13]-[17] and Lyapunov functional approaches [6], [10]-[12], [18]-[20]. are two representative and yet different types of methodologies. Lyapunov function approaches impose no restriction on the derivative of the time-delay and usually give simple stability criteria, especially when the delayed neural networks have multiple time-varying delays. On the other hand, Lyapunov functional approaches normally require the information on the derivative of the time-delay and the obtained stability criteria are expressed in terms of linear matrix inequalities (LMIs). In general, the stability results obtained by the Lyapunov functional approach are less conservative than those obtained by the Lyapunov function approach because the Lyapunov functional approach makes use of more information about the system. It is well known that when using the Lyapunov functional approach, the choice of an appropriate Lyapunov functional is crucial for deriving less conservative stability criteria. In the delay-dependent stability analysis of linear time-delay systems, a model transformation technique is often used to construct Lyapunov functionals (see, e.g., [3]-[5]). In order to reduce the conservatism caused by the bounding techniques on cross product terms for the model transformation methods, the free-weighting matrix method was proposed in [7]-[9], in which the free-weighting matrices are introduced between the terms in the Leibniz formula and no bounding techniques on cross product terms are involved. Recently, this method has been applied to delay-dependent stability analysis for delayed neural networks [6], [9], [19]. Another important method for stability analysis of linear time-delay systems is the discretized Lyapunov functional method developed in [1]. The discretization based method appears to be very efficient since some examples show that the obtained results are close to the analytical ones.

The purpose of the present paper is to analyze stability of delayed neural networks with uncertain constant delays. The key idea is to integrate the discretized Lyapunov functional method with the free-weighting matrix technique. This new analytical method enables us to establish novel linear matrix inequality based delay-dependent sufficient conditions for asymptotic stability of neural networks with uncertain interval delay. As will be shown through numerical results, the delay-dependent stability criteria proposed in this paper are essentially less conservative than the existing ones.

II. Problem Formulation

In the sequel, if not explicitly stated, matrices are assumed to have compatible dimensions. The notation $M > (\geq, <, \leq)$ 0 is used to denote a positive-definite (positive-semidefinite, negative, negative-semidefinite) matrix. $\|\cdot\|$ denotes the Euclidean vector norm or the spectral norm of matrices.

In this paper, we study the delayed neural network which is described by the following delay-differential equation

$$\frac{du(t)}{dt} = -Cu(t) + Ag(u(t)) + Bg(u(t-\tau)) + I, \quad (1)$$

where

$$u = [u_1, u_2, \ldots, u_n]^T$$

is a real n-vector which denotes the state variables associated with the neurons,

$$C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$$

is a diagonal matrix representing self-feedback term, $A,\ B \in \mathbb{R}^{n\times n}$ are the connection weight matrix and the delayed connection weight matrix, respectively, and

$$g(u(t)) = [g_1(u_1(t)), g_2(u_2(t)), \ldots, g_n(u_n(t))]^T$$

denotes the neuron activation function. I is a real constant input n-vector. The delay τ is an uncertain constant satisfying

$$0 \leq h_1 \leq \tau \leq h_2, \quad (2)$$

where h_1 and h_2 are known constants. Set

$$r_0 = \frac{1}{2}(h_2 + h_1)$$

and

$$\delta = \frac{1}{2}(h_2 - h_1).$$

Throughout the paper, the following assumption is adopted.

(**H**) Each function g_i is continuous, and there exists scalar k_i such that for any $\alpha,\ \beta \in \mathbb{R}$,

$$0 \leq [g_i(\alpha) - g_i(\beta)](\alpha - \beta) \leq k_i(\alpha - \beta)^2.$$

Introduce the matrix

$$K = \mathrm{diag}(k_1, k_2, \ldots, k_n).$$

As usual, an n-vector

$$u^* = [u_1^*, u_2^*, \ldots, u_n^*]^T$$

is said to be an equilibrium point of system (1) if it satisfies

$$Cu^* = (A + B)g(u^*) + I.$$

In this paper, it is assumed that some conditions are satisfied so that the equilibrium point of delayed neural network (1) does exist. For notational convenience, the equilibrium point u^* of delayed neural network (1) will be shifted to the origin. Applying transformation

$$x(t) = u(t) - u^*$$

yields the following new description of system (1):

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t-\tau)), \qquad (3)$$

where

$$x = [x_1, x_2, \ldots, x_n]^T$$

is the state vector of the new system (3), and

$$f(x) = [f(x_1), f(x_2), \ldots, f_n(x_n)]^T$$

with

$$f_i(x_i) = g_i(x_i + u_i^*) - g_i(u_i^*),\ i = 1, 2, \ldots, n.$$

It is easy to see that by assumption (**H**), f satisfies the following condition:

(**H**$'$) Each function f_i is continuous with

$$f_i(0) = 0,$$

and for any $\alpha,\ \beta \in \mathbb{R}$,

$$0 \leq [f_i(\alpha) - f_i(\beta)](\alpha - \beta) \leq k_i(\alpha - \beta)^2.$$

The following Lemma is important for developing the main results in the next section.

Lemma 1: [21]: For any vectors $x, y \in \mathbb{R}^n$, matrix $0 < P \in \mathbb{R}^{n\times n}$,

$$2x^T y \leq x^T P^{-1} x + y^T P y.$$

III. Main Theoretical Results

In this section, we will establish asymptotic stability criteria for system (3) with uncertain constant time delay. As illustrated before, our idea is to use the discretized Lyapunov functional method in conjunction with the technique of introducing the free-weighting matrix between the terms of the Leibniz-Newton formula. Our main results are stated in the following theorem.

Theorem 1: Consider system (3) satisfying assumption (**H**$'$). For any given positive integer N, set

$$h = r_0/N.$$

The origin of delayed neural network (3) is asymptotically stable for any constant delay $\tau = r$ satisfying (2) if there exist $n \times n$ matrices $P > 0,\ P_1,\ P_2,\ U \geq 0,\ Z_1 > 0,\ Z_2 > 0,\ Z_3 \geq 0,$

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \geq 0,$$

$$D_i = \mathrm{diag}(d_{i1}, d_{i2}, \ldots, d_{in}) \geq 0,$$

$$M_{ij},\quad i = 1, 2,\quad j = 1, 2, 3, 4,$$

$$S_p > 0,\ Q_p,\ R_{pq} = R_{qp}^T,\ p = 0, 1, \ldots, N,$$
$$q = 0, 1, \ldots, N,$$

such that the following linear matrix inequalities hold:

$$\begin{bmatrix} \Xi_0 & D^s & D^a & \sqrt{r_0}M_1 & \sqrt{\delta}M_2 \\ * & -R_d - S_d & 0 & 0 & 0 \\ * & 0 & -3S_d & 0 & 0 \\ * & 0 & 0 & -Z_1 & 0 \\ * & 0 & 0 & 0 & -Z_2 \end{bmatrix} < 0, \quad (4)$$

and

$$\begin{bmatrix} P & \tilde{Q} \\ * & \tilde{R} + \tilde{S} \end{bmatrix} > 0, \qquad (5)$$

where

$$M_i^T = \left[M_{i1}^T\ M_{i2}^T\ M_{i3}^T\ M_{i4}^T\ 0\ 0\right],\quad i = 1, 2,$$

$$\Xi_0 = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & P_1^T A + D_1 K & P_1^T B \\ * & \Omega_{22} & \Omega_{23} & -M_{22} & P_2^T A + \Lambda & P_2^T B \\ * & * & \Omega_{33} & \Omega_{34} & 0 & 0 \\ * & * & * & \Omega_{44} & 0 & D_2 K \\ * & * & 0 & 0 & U - 2D_1 & 0 \\ * & * & 0 & * & 0 & -U - 2D_2 \end{bmatrix},$$

$$\Omega_{11} = -C^T P_1 - P_1^T C + M_{11} + M_{11}^T + Q_0 + Q_0^T + S_0 + Z_3,$$

$$\Omega_{12} = P - P_1^T - C^T P_2 + M_{12}^T,$$

$$\Omega_{13} = M_{13}^T - M_{11} - Q_N + M_{21},$$

$$\Omega_{14} = M_{14}^T - M_{21},$$

$$
\begin{aligned}
\Omega_{22} &= -P_2 - P_2^T + r_0 Z_1 + \delta Z_2, \\
\Omega_{23} &= -M_{12} + M_{22}, \\
\Omega_{33} &= -M_{13} - M_{13}^T - S_N + M_{23} + M_{23}^T, \\
\Omega_{34} &= -M_{23} - M_{14}^T + M_{24}^T, \\
\Omega_{44} &= -Z_3 - M_{24} - M_{24}^T, \\
\tilde{Q} &= [Q_0 \; Q_1 \; \dots \; Q_N], \\
\tilde{S} &= \operatorname{diag}\left(\frac{1}{h}S_0, \frac{1}{h}S_1, \dots, \frac{1}{h}S_N\right), \\
S_d &= \operatorname{diag}\left(S_0 - S_1, S_1 - S_2, \dots, S_{N-1} - S_N\right), \\
R_d &= \begin{bmatrix} R_{d11} & R_{d12} & \dots & R_{d1N} \\ R_{d21} & R_{d22} & \dots & R_{d2N} \\ \dots & \dots & \dots & \dots \\ R_{dN1} & R_{dN2} & \dots & R_{dNN} \end{bmatrix}, \\
R_{dpq} &= h(R_{p-1,q-1} - R_{pq}), \\
D^s &= [D_1^s \; D_2^s \; \dots \; D_N^s], \\
D^a &= [D_1^a \; D_2^a \; \dots \; D_N^a], \\
D_p^s &= \begin{bmatrix} \frac{h}{2}(R_{0,p-1} + R_{0p}) - (Q_{p-1} - Q_p) \\ \frac{h}{2}(Q_{p-1} + Q_p) \\ -\frac{h}{2}(R_{N,p-1} + R_{Np}) \\ 0 \\ 0 \end{bmatrix}, \\
D_p^a &= \begin{bmatrix} -\frac{h}{2}(R_{0,p-1} - R_{0p}) \\ -\frac{h}{2}(Q_{p-1} - Q_p) \\ \frac{h}{2}(R_{N,p-1} - R_{Np}) \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}
$$

Remark 1: Theorem 1 provides the linear matrix inequality based sufficient conditions to verify asymptotic stability of system (3) over a given delay interval $[h_1, h_2]$ when delay τ is constant. In practice, it appears useful to find the largest interval of constant delay $\tau = r$ over which system (3) is stable. In light of Theorem 1, one can use the following procedure to find the maximum delay interval of delayed neural network (3).

Step 1: Set $h_1 = h_2$ and apply Theorem 1 to find the maximum of h_2 denoted by $h_{2\max}$ such that linear matrix inequalities (4)-(5) are satisfied.

Step 2: Set $h_2 = h_{2\max}$ and apply Theorem 1 to find the minimum of h_1 denoted by $h_{1\min}$ such that linear matrix inequalities (4)-(5) hold.

Step 3: Set $h_2 = h_{1\min}$ and apply Theorem 1 to find the minimum of h_1 denoted by $h_{1\min}$ such that linear matrix inequalities (4)-(5) hold.

Step 4: If $h_2 = h_{1\min}$, then exit; if $h_2 > h_{1\min}$, go to step 3.

Utilizing the above procedure, we will be able to obtain an interval $[h_{1\min}, h_{2\max}]$ of constant delay $\tau = r$ over which system (3) is asymptotically stable.

IV. Numerical Example

In this section, we will use numerical results to illustrate the effectiveness of the newly obtained delay-dependent stability conditions.

Consider a delayed neural network (3) with uncertain constant delay $\tau = r$ and

$$C = \operatorname{diag}(1.2769, 0.6231, 0.9230, 0.4480),$$

$$K = \operatorname{diag}(0.1137, 0.1279, 0.7994, 0.2368),$$

$$A = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix}.$$

This example was studied in [11]. Theorem 1 is applied to calculate the maximum time delay $r_{\max}$ that the neural network can tolerate and still retain stability, with the interior-point algorithms [22] being used for solving the related linear matrix inequalities. The results for different N are listed in Table I. Moreover, using the procedure given in Remark 1, one can verify that this system is asymptotically stable over $[0, r_{\max}]$. For comparison purposes, the stability criteria in [11], [18], [10] are also applied to this example, and it

TABLE I
UPPER BOUND $r_{\max}$ FOR EXAMPLE 1

N	1	2	3	4
$r_{\max}$	4.2469	4.2659	4.2669	4.2671

is found that the resulting $r_{\max}$ is 1.4224, 2.7528, 3.5841, respectively. From this example, it can be seen that the newly obtained delay-dependent stability conditions are less conservative than those derived by the existing methods.

V. CONCLUSIONS

This paper has been devoted to the study of asymptotic stability for delayed neural networks with uncertain constant time-delay. The novelty lies in that the new analytical method has been proposed by integrating the discretized Lyapunov functional method with the free-weighting matrix technique, which paves the way for constructing the new Lyapunov functional. On this basis, the new delay-dependent sufficient conditions for asymptotic stability of delayed neural networks have been derived. These stability conditions, which are expressed in form of linear matrix inequalities, can be solved efficiently by using interior point algorithms for dealing with linear matrix inequalities. Finally, the numerical results have demonstrated that these new delay-dependent stability conditions are less conservative than the existing ones.

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REFERENCES

[1] K. Gu, "Discretized LMI set in the stability problem of linear time delay systems," *Int. J. Control*, vol. 68, pp. 923-934, 1997.

[2] K. Gu, V. Kharitonov, and J. Chen, *Stability of Time-Delay Systems*. Boston, MA: Birkhuser, 2003.

[3] E. Fridman, "New Lyapunov-Krasovskii functionals for stability of linear retard and neutral type systems," *Syst. Contr. Lett.,* vol. 43, pp. 309-319, 2001.

[4] E. Fridman and U. Shaked, "A descriptor system approach to H_∞ control of time-delay systems," *IEEE Trans. Autom. Control*, vol. 47, pp. 253-279, 2002.

[5] Y. S. Moon, P. Park, W. H. Kwon, and Y. S. Lee, "Delay-dependent robust stabilization of uncertain state-delayed systems," *Int. J. Control*, vol. 74, pp. 1447-1455, 2001.

[6] Y. He, Q.-G. Wang, and M. Wu, "LMI-based stability criteria for neural networks with multiple time-varying delays," *Physica D*, vol. 212, pp. 126-136, 2005.

[7] Y. He, M. Wu, J. H. She, and G. P. Liu, "Delay-dependent robust stability criteria for uncertain neutral systems with mixed delays," *Syst. Contr. Lett.*, vol. 51, no. 1, pp. 57-65, 2004.

[8] Y. He, Q.-G. Wang, L. Xie, and C. Lin, "Further improvement of free-weighting matrices technique for sytems with time-varying delay," *IEEE Trans. Autom. Control*, vol. 52, no. 2, pp. 293-299, 2007.

[9] Y. He, Q.-G. Wang, L. Xie, and C. Lin, "Delay-range-dependent stabililty for systems with time-varying delay," *Automatica*, vol. 43, pp. 371-376, 2007.

[10] Y. He, G. Liu, and D. Rees, "New delay-dependent stability criteria for neural networks with time-varying delay," *IEEE Trans. Neural Networks*, vol. 18, no. 1, pp. 310-314, 2007.

[11] S. Xu, J. Lam, D. W. C. Ho, and Y. Zou, "Novel global asymptotic stability criteria for delayed cellular neural networks," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 52, no. 6, pp. 349-353, 2005.

[12] S. Xu, J. Lam, and D. W. C. Ho, "A new LMI condition for delay-dependent asymptotic stability of delayed Hopfield neural networks," *IEEE Trans. Circuits Syst. II*, vol. 53, no. 3, pp. 230-234, 2006.

[13] H. Ye and A. N. Michel, "Robust stability of nonlinear time-delay systems with applications to neural networks," *IEEE Trans. Circuits Syst. I*, vol. 43, pp. 532-543, 1996.

[14] Z. Zeng and J. Wang, "Improved conditions for global exponential

stability of recurrent neural networks with time-varying delays,", *IEEE Trans. Neural Networks*, vol. 17, no. 3, pp. 623-635, 2006.

[15] W.-H. Chen, Z.-H. Guan, and X. Lu, "Delay-dependent exponential stability of neural networks with variable delays," *Phys. Lett. A*, vol. 326, pp. 355-363, 2004.

[16] W.-H. Chen and X. Lu, "New Delay-dependent exponential stability criteria for neural networks with variable delays," *Phys. Lett. A*, vol. 351, pp. 53-58, 2006.

[17] W.-H. Chen, X. Lu, and D.-Y. Liang, "Global exponential stability for discrete-time neural networks with variable delays," *Phys. Lett. A*, vol. 358, pp. 186-198, 2006.

[18] W.-H. Chen, X. Lu, Z.-H. Guan, and W. X. Zheng, "Delay-dependent exponential stability of neural networks with variable delay: an LMI approach," *IEEE Trans. Circuits Syst. II*, vol. 53, no. 9, pp. 837-842, 2006.

[19] C. C. Hua, C. N. Long, and X. P. Guan, "New results on stability analysis of neural networks with time-varying delays," *Phys. Lett. A*, vol. 352, pp. 335-340, 2006.

[20] Y.-Y Hou, T.-L. Liao, C.-H. Lien, and J.-J. Yan, "Stability analysis of neural networks with interval time-varying delays," *Chaos*, vol. 17, 033120, 2007.

[21] Y. Wang, L. Xie, and C. E. de Souza, "Robust control of a class of uncertain nonlinear system," *Syst. Contr. Lett.*, vol. 19, pp. 139-149, 1992.

[22] S. Boyd, L. EI Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*. Philadephia, PA: SIAM, 1994.