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Quantum Behaviors and Networks

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***Abstract*— The purpose of this paper is to discuss how Willems' behavioral modeling might be applied to physical systems governed by the laws of quantum physics. A quantum behavior is simply defined in terms of the evolution of physical variables according to quantum mechanics. This evolution is determined by parameters that specify the internal energy of the system, and any interfaces to other systems or fields. A simple framework for modeling open quantum systems and networks of such systems is described; this framework provides tools for determining quantum behaviors. The ideas are illustrated by an example from quantum optics.**

I. INTRODUCTION

The behavioral approach to dynamical systems modeling has been developed by Willems and collaborators (e.g. [9], [11], [12]) to provide general model structures that are intended to be appropriate for applications. Behavioral models describe the range of possibilities that are allowed by the system being considered, and do not depend on notions of state nor inputs and outputs. While these notions can be accommodated and play important roles, behavioral modeling is focused on trajectories of values (usually numerical) of system variables and how they are determined. In particular, the problem of control is seen as finding a control system that can be connected to the plant being controlled so that the behavior of the combined system has desirable properties.

The purpose of this paper is to discuss how behavioral modeling might be applied to physical systems governed by the laws of quantum physics. The unitary dynamical postulate from quantum mechanics determines how physical variables may evolve in time, thus determining the *quantum behavior*. This unitary flow is specified from physical considerations concerning the energy of the system, and is usually expressed in terms of a differential equation, the Schrodinger equation; this may be regarded as the *quantum behavioral equation*. Energy specifications may be regarded as a (non-numerical) *parameterization* of the quantum behavior.

In order to get a feeling for what is involved, consider the quantum optical network illustrated in Figure 1, which shows a pair of optical cavities coupled by an optical medium and an optical interconnect (a light beam). To describe the behavior of such a network, one needs a mathematical model that can represent the physical properties of the components (the cavities), and the mechanisms for interconnection. In quantum mechanics, the variables (represented as operators on a Hilbert space, and called observables) that are used

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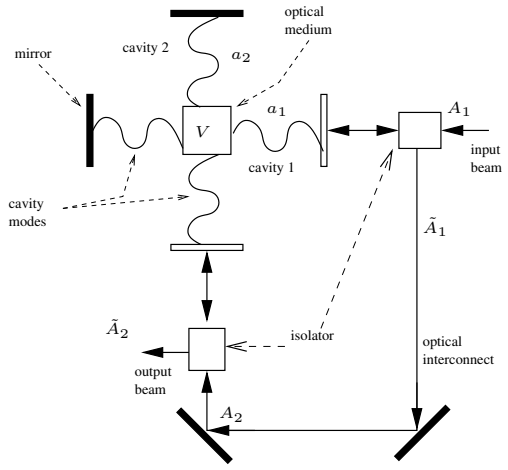

Fig. 1. A pair of optical cavities coupled by an optical medium and an optical interconnect, [13, Fig. 1]. Each cavity consists of a pair of mirrors, one of which is perfectly reflecting (shown solid) while the other is partially transmitting (shown unfilled). The partially transmitting mirror enables the light mode inside the cavity to interact with an external light field. The external field is separated into input and output components by a Faraday isolator. The optical interconnect is formed when light from the output of one cavity is directed into the input of the other, here using additional mirrors.

to describe the cavity include ones that do not commute—this is a fundamental feature of quantum mechanics, and well known consequences include the famous Heisenberg uncertainty principle, [7]. Since the network of Figure 1 includes the use of an external free field channel (light beam) as an interconnect, the modeling framework needs an efficient and tractable quantum mechanical description for such field channels. Quantum noise models [8], [3] can be used to describe the random influence of the optical fields on the cavities. Quantum noise modeling is much like classical white noise modeling, except that the quantum noise includes components that do not commute, and is therefore fundamentally quantum mechanical. A wide range of such open quantum models can be expressed within a quantum noise framework, such as those arising in quantum optics.

The approach we present for quantum network modeling is based on the quantum mechanical models just discussed, and employs a simple parameterization which specifies the internal energy of systems, as well as interfaces to external fields and other systems. The dynamical equations for the quantum system can easily be determined from the parameters. Furthermore, the parameters provide a simple mechanism for describing networks of interconnected systems. This provides simple, algebraic methods for quantum behavioral

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modeling.

Background references. A number of articles and books are available to help readers with the background material on which this paper is based, including [14], [10], [2], [3], [8] and [7].

Notation. In this paper we use matrices $\mathbf{M} = \{m_{ij}\}$ with entries m_{ij} that are operators on an underlying Hilbert space. The asterisk $*$ is used to indicate the Hilbert space adjoint A^* of an operator A, as well as the complex conjugate $z^* = x - iy$ of a complex number $z = x + iy$ (here, $i = \sqrt{-1}$ and x, y are real). The conjugate transpose $\mathbf{M}^\dagger$ of a matrix $\mathbf{M}$ is defined by $\mathbf{M}^\dagger = \{m_{ji}^*\}$. Also defined are the conjugate $\mathbf{M}^* = \{m_{ij}^*\}$ and transpose $\mathbf{M}^T = \{m_{ji}\}$ matrices, so that $\mathbf{M}^\dagger = (\mathbf{M}^T)^* = (\mathbf{M}^*)^T$. In the physics literature, it is common to use the dagger $\dagger$ to indicate the Hilbert space adjoint. The commutator of two operators A, B is defined by $[A, B] = AB - BA$.

II. Preliminaries

In quantum mechanics [7] physical quantities like energy, spin, position, etc are expressed as *observables*; these are usually represented as self-adjoint operators acting on a Hilbert space H. Other physical variables, like annihilation operators—which are not self-adjoint—are also of importance. In physical modeling it is convenient to consider a collection of physical variables for the system, and the appropriate algebraic object is a $*$-algebra. A *$*$-algebra* is a complex vector space equipped with a multiplication and an involution. The space $\mathscr{B}(\mathsf{H})$ of bounded operators forms a $*$-algebra: operators A, B can be multiplied by composition $(AB)\psi = A(B(\psi))$ for $\psi \in \mathsf{H}$, and adjoints A^* define an involution. While $*$-algebras can be considered abstractly, we consider for definiteness $*$-algebras $\mathscr{A}$ that are subalgebras of $\mathscr{B}(\mathsf{H})$ containing the identity I, for some H.

The postulates of quantum mechanics state that in a measurement of a physical quantity represented by an observable A, the possible outcomes a are eigenvalues of A: $a \in \mathrm{spec}(A)$. If

$$A = \sum_{a \in \mathrm{spec}(A)} a P_a$$

is the spectral representation of A, where P_a are orthogonal projections, then the probability of outcome a, $\mathrm{Prob}(a)$, depends on the state of the system as we now describe. A *state* on a $*$-algebra $\mathscr{A}$ is linear functional $\mathbb{P} : \mathscr{A} \to \mathbf{C}$ (the complex numbers) such that $\mathbb{P}(A) \geq 0$ for all non-negative self-adjoint $A \in \mathscr{A}$ and $\mathbb{P}(I) = 1$ (recall $I \in \mathscr{A}$ is the identity). Then given a state $\mathbb{P}$, the *probability* of outcome a is given by $\mathrm{Prob}(a) = \mathbb{P}(P_a)$. The *expected value* of the observable A is $\mathbb{P}(A)$. The simplest example of a state is defined by a state vector $\psi \in \mathsf{H}$ via the Hilbert space inner product $\mathbb{P}(X) = \langle \psi, X\psi \rangle$ for all $X \in \mathscr{A}$.

The postulates of quantum mechanics include unitary dynamics. Let $U(t) \in \mathscr{B}(\mathsf{H})$ be a family of unitary operators. Then, in the so-called Schrodinger picture, states evolve according to $\mathbb{P}_t(X) = \mathbb{P}(U^*(t)XU(t))$, for all $X \in \mathscr{A}$. In the case of state vectors, the state vector at time t is $\psi(t) = U(t)\psi$. The so-called Heisenberg picture is dual to the Schrodinger picture (and therefore equivalent), and describes the evolution of observables, via $X(t) = U^*(t)XU(t)$, for any $X \in \mathscr{A}$. See, e.g., [6] for a discussion of completely positive quantum dynamics which can be regarded as unitary dynamics for a larger system with the additional degrees of freedom averaged out.

III. Quantum Behavioral Modeling

In the behavioral modeling of Willems, a dynamical system is specified by a triple $(\mathbb{T}, \mathbb{W}, \mathfrak{B})$, where $\mathbb{T}$ is the time axis, $\mathbb{W}$ is the *signal space* in which dynamical variables take their values, and $\mathfrak{B} \subset \mathbb{W}^{\mathbb{T}}$ is the *behavior*, the set of possible trajectories for the system, a subset of the *universum* $\mathbb{W}^{\mathbb{T}}$, the set of all trajectories. Typically, but not always, in continuous time behaviors are defined by differential equations. In the case of classical physical systems, the signal space contains all the *numerical values* taken by the physical variables of interest. Behavioral modeling of quantum physical systems will focus on *variables*, not the numerical values they take, because of the fundamental incompatibility concerning observables that do not commute. In what follows we consider systems evolving continuously in time, and we take $\mathbb{T} = [0, \infty)$. We replace the signal space as a set of values by a $*$-algebra $\mathscr{U}$ representing the physical variables of interest, which we call the *physical variable space*.

Given a collection of relevant variables, the main problem of behavioral modeling is to specify how these variables evolve in time. As discussed in section II, in quantum mechanics, dynamical behavior is given by (Heisenberg picture) $X(t) = U^*(t)XU(t)$ for any $X \in \mathscr{U}$, where $\{U(t)\}_{t \in \mathbb{T}}$. The behavioral modeling task is to determine $U(t)$, or a differential equation for it. Representations based on physical parameters can help simplify behavioral modeling.

Before we discuss the parametric description in the following section, we close this section with a list of items that are needed to specify an *open* quantum system, a quantum system that may interact with its environment, and/or other quantum systems, Figure 2. These items are $\mathcal{S} = (\mathscr{U}, \rho, H, \mathcal{I})$, where (i) $\mathscr{U}$ is the $*$-algebra of physical variables of interest; (ii) ρ, is the (initial) state on $\mathscr{U}$; (iii) $H \in \mathscr{U}$ is a self-adjoint operator called the *Hamiltonian*, specifying the self-energy of the system; and (iv) $\mathcal{I}$ is an interface specification. These items, together with the unitary dynamical postulate, determine the behaviour of the open quantum system and its statistics. Quantum measurements can be understood within the framework of quantum conditional expectations and Belavkin's quantum filtering theory, [1], [2].

IV. Open Quantum Systems

The class of systems we consider in this paper are quantum noise models of open quantum systems. These idealized models are based on more complicated first principles models; we refer the reader to the literature for details: [3], [14],

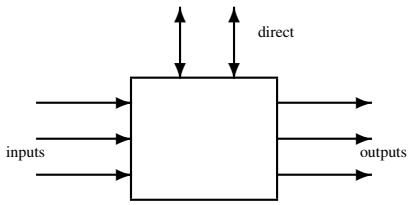

Fig. 2. Diagram of an open quantum system $\mathcal{S}$ showing an interface $\mathcal{I}$ consisting of unidirectional field channel inputs and outputs, and bidirectional direct couplings.

[10], [2], [4]. These models can be expressed in terms of a quantum version of the Ito calculus, [8], [3].

The open system models describe a system, or systems, interacting with boson fields (such as an atom interacting with the electromagnetic field in free space, or phonon vibrations in a material). The fields can be considered as channels that may contain a signal component, and a quantum noise component. The former component may represent modulation of a light beam (e.g. by a classical signal, or by variables from a source), while the latter component describes the volatile nature of the idealized system-field (heat bath) interaction characteristic of white noise models. In addition to bidirectional direct Hamiltonian interactions between systems, field channels provide a mechanism for unidirectional indirect field mediated interactions between systems. For example, in quantum optics laser beams may be used as "quantum wires" interconnecting components, as in Figure 1.

A. Definitions

We now give some more details regarding the open system models. We assume the system is defined on an underlying Hilbert space H. Let $\mathscr{U}$ be a sub-$*$-algebra of $\mathscr{B}(\mathsf{H})$. Boson fields are defined on a Fock space F, a particular type of Hilbert space, with an associated $*$-algebra $\mathscr{F}$ (for full details, see [8], [2]). Open quantum models are defined in terms of operators in the tensor product $\mathscr{U} \otimes \mathscr{F}$. The algebra $\mathscr{U}$ may be regarded as the *initial variable space* for the system, while $\mathscr{U} \otimes \mathscr{F}$ might be called the *full variable space* for the complete system-field model. A collection of n field channels is given by the quantum stochastic processes

$$\mathbf{A} = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}, \quad \mathbf{\Lambda} = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \vdots & \vdots \\ A_{n1} & \dots & A_{nn} \end{pmatrix},$$

which respectively describe annihilation of photons in the field channels, and scattering between channels. We assume that these processes are *canonical*, meaning that we have the following non-vanishing second order Ito products: $dA_j(t)\,dA_k(t)^* = \delta_{jk}dt$, $dA_{jk}(t)\,dA_l(t)^* = \delta_{kl}dA_j(t)^*$, $dA_j(t)\,dA_{kl}(t) = \delta_{jk}dA_l(t)$ and $dA_{jk}(t)\,dA_{lm}(t) = \delta_{kl}dA_{jm}(t)$. The simplest situation is that of a vacuum state ϕ for the field channels, in which case the input processes are purely quantum noise.

Coupling of the system to the field is defined using

$$\mathbf{S} = \begin{pmatrix} S_{11} & \dots & S_{1n} \\ \vdots & \vdots & \vdots \\ S_{n1} & \dots & S_{nn} \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} L_1 \\ \vdots \\ L_n \end{pmatrix},$$

respectively a scattering matrix with operator entries $S_{ij} \in \mathscr{U}$ satisfying $\mathbf{S}^\dagger\mathbf{S} = \mathbf{S}\mathbf{S}^\dagger = \mathbf{I}$, and a vector of coupling operators $L_j \in \mathscr{U}$.

In terms of these parameters, the Schrodinger equation

$$\begin{aligned} dU(t) &= \Big\{\mathrm{tr}[(\mathbf{S}-\mathbf{I})d\mathbf{\Lambda}] + d\mathbf{A}^\dagger\mathbf{L} - \mathbf{L}^\dagger\mathbf{S}d\mathbf{A} \\ &\quad -\frac{1}{2}\mathbf{L}^\dagger\mathbf{L}dt - iHdt\Big\}U(t) \end{aligned} \tag{1}$$

with initial condition $U(0) = I$ determines the unitary motion of the system, in accordance with the fundamental postulate of quantum mechanics. Given an operator X defined on the initial space H, its Heisenberg evolution is defined by $X(t) = \mathrm{j}_t(X) = U(t)^* XU(t)$ and satisfies

$$\begin{aligned} dX(t) &= (\mathcal{L}_{\mathbf{L}(t)}(X(t)) - i[X(t), H(t)])dt \\ &+ d\mathbf{A}^\dagger(t)\mathbf{S}^\dagger(t)[X(t), \mathbf{L}(t)] + [\mathbf{L}^\dagger(t), X(t)]\mathbf{S}(t)d\mathbf{A}(t) \\ &+ \mathrm{tr}[(\mathbf{S}^\dagger(t)X(t)\mathbf{S}(t) - X(t))d\mathbf{\Lambda}(t)]. \end{aligned} \tag{2}$$

In this expression, all operators evolve unitarily (e.g. $\mathbf{L}(t) = \mathrm{j}_t(\mathbf{L})$) (commutators of vectors and matrices of operators are defined component-wise), and tr denotes the trace of a matrix. We also employ the notation

$$\mathcal{L}_{\mathbf{L}}(X) = \frac{1}{2}\mathbf{L}^\dagger[X, \mathbf{L}] + \frac{1}{2}[\mathbf{L}^\dagger, X]\mathbf{L}; \tag{3}$$

this is called the *Lindblad superoperator* in the physics literature. The components of the output fields are defined by $\tilde{\mathbf{A}}(t) = \mathrm{j}_t(\mathbf{A}(t)) \doteq U^*(t)\mathbf{A}(t)U(t)$, $\tilde{\mathbf{\Lambda}}(t) = \mathrm{j}_t(\mathbf{\Lambda}(t)) \doteq U^*(t)\mathbf{\Lambda}(t)U(t)$ and satisfy the quantum stochastic differential equations

$$d\tilde{\mathbf{A}}(t) = \mathbf{S}(t)d\mathbf{A}(t) + \mathbf{L}(t)dt \tag{4}$$

$$d\tilde{\mathbf{\Lambda}}(t) = \mathbf{S}^*(t)d\mathbf{\Lambda}(t)\mathbf{S}^T(t) + \mathbf{S}^*(t)d\mathbf{A}^*(t)\mathbf{L}^T(t) \tag{5}$$

$$+\mathbf{L}(t)d\mathbf{A}(t)\mathbf{S}^T(t) + \mathbf{L}^*(t)\mathbf{L}^T(t)dt, \tag{6}$$

where $\mathbf{L}(t) = \mathrm{j}_t(\mathbf{L})$, etc, as above.

It can be seen that the parameters $\mathbf{G} = (\mathbf{S}, \mathbf{L}, H)$ provide a compact specification of the open system, assuming canonical field inputs, since they determine the behavior of the system, via the flow $\mathrm{j}_t(\cdot)$, as determined by (1). This flow is defined on the full variable space $\mathscr{U} \otimes \mathscr{F}$. We call $\mathrm{j}_t(\cdot)$ the *quantum behavioral flow* of the system. The Schrodinger equation (1) may be regarded as the *quantum behavioral equation* for the system. Note that the parameters $\mathbf{G} = (\mathbf{S}, \mathbf{L}, H)$ are drawn from the variable space $\mathscr{U}$ (they are not numerical).

The *state* of the complete system (including field channels) $\mathbb{P}$ is the tensor product $\mathbb{P} = \rho \otimes \phi$ in the case that the *initial* system state is ρ and the input field channels are in the vacuum state ϕ. The statistics of physical variables in the complete system can be determined using this state and the quantum behavioral equation. The specification or

determination of the initial system state ρ is an important part of the modeling process; but note that the quantum behavioral equation does not depend on ρ.

B. Connections

In our recent paper [4] (see also [5]) we developed an algebraic framework for quantum networks using the parameters $\mathbf{G} = (\mathbf{S}, \mathbf{L}, H)$. In particular, we introduced a series product $\lhd$ to describe series or cascade field-mediated connections, and a concatenation product $\boxplus$ for decomposing or assembling systems. Direct interactions between systems were accommodated using interaction Hamiltonians. In this subsection we develop these ideas further by being more explicit in the manner in which direct connections are accommodated—we introduce a *direct connection* product $\bowtie$ for this purpose. In practice it is much easier and more transparent working with these products than with underlying equations, which can become complex and unwieldy.

To this end, we enlarge the parametrization as follows:

$$\mathbf{G} = (\mathbf{S}, \mathbf{L}, \mathbf{Z}, H). \tag{7}$$

As before, $\mathbf{S}$ is a scattering matrix, $\mathbf{L}$ is vector of field coupling operators, and H is a Hamiltonian. The new item in this parameter list is a vector $\mathbf{Z}$ of variables available for direct connection. All of these operators in $\mathbf{G}$ belong to the $*$-algebra $\mathscr{U}$ for the system. The variables $\mathcal{I} = (\mathbf{S}, \mathbf{L}, \mathbf{Z})$ simply specify the system's *interface*, so that the parameterization $\mathbf{G} = (\mathbf{S}, \mathbf{L}, \mathbf{Z}, H) = (\mathcal{I}, H)$ simply accounts for energy—energy exchanges with external systems, and internal energy; we might call this the *quantum behavioral parameterization*.

Suppose we are given two such systems: $\mathbf{G}_1 = (\mathbf{S}_1, \mathbf{L}_1, \mathbf{Z}_1, H_1)$ and $\mathbf{G}_2 = (\mathbf{S}_2, \mathbf{L}_2, \mathbf{Z}_2, H_2)$, defined in terms of physical variables belonging to $*$-algebras $\mathscr{U}_1$ and $\mathscr{U}_2$, respectively. The products we define below combine these systems to produce new systems defined in terms of parameters drawn from the tensor product of variable spaces $\mathscr{U}_1 \otimes \mathscr{U}_2$ (this replaces the classical behavioral cartesian product of signal spaces $\mathbb{U}_1 \times \mathbb{U}_2$).

The *concatenation* of $\mathbf{G}_1$ and $\mathbf{G}_2$ is the system $\mathbf{G}_1 \boxplus \mathbf{G}_2$ defined by

$$\mathbf{G}_1 \boxplus \mathbf{G}_2 = (\begin{pmatrix} \mathbf{S}_1 & 0 \\ 0 & \mathbf{S}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{pmatrix}, H_1 + H_2). \tag{8}$$

This product simply assembles the components together, without making any connections, Figure 3.

Now suppose $\mathbf{G}_1 = (\mathbf{S}_1, \mathbf{L}_1, \mathbf{Z}_1, H_1)$ and $\mathbf{G}_2 = (\mathbf{S}_2, \mathbf{L}_2, \mathbf{Z}_2, H_2)$ have the same number of field channels (i.e. $\mathbf{L}_1$ and $\mathbf{L}_2$ have the same length. Then the *series product* $\mathbf{G}_2 \lhd \mathbf{G}_1$ is defined by

$$\begin{aligned} \mathbf{G}_2 \lhd \mathbf{G}_1 \quad = \quad & \mathbf{S}_2\mathbf{S}_1, \mathbf{L}_2 + \mathbf{S}_2\mathbf{L}_1, \begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{pmatrix}, \\ & H_1 + H_2 + \frac{1}{2i}(\mathbf{L}_2^\dagger \mathbf{S}_2 \mathbf{L}_1 - \mathbf{L}_1^\dagger \mathbf{S}_2^\dagger \mathbf{L}_2)). \end{aligned} \tag{9}$$

As its name suggests, the series product describes the series or cascade connection using field channels, Figure 4; see [4].

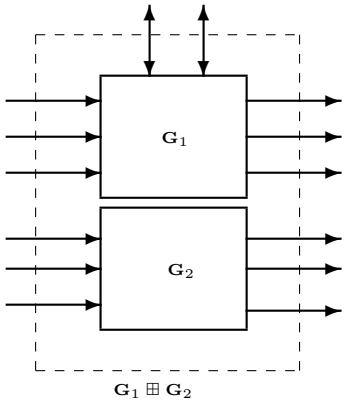

Fig. 3. Concatenation of two systems, $\mathbf{G}_1 \boxplus \mathbf{G}_2$.

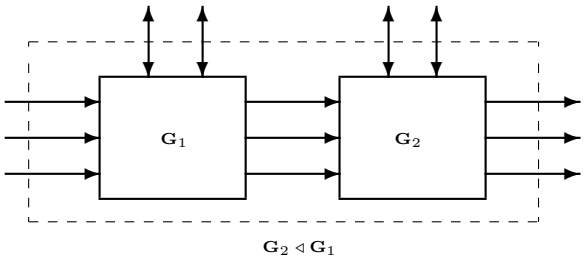

Fig. 4. Series or cascade connection of two systems, $\mathbf{G}_2 \lhd \mathbf{G}_1$.

Finally, if $\mathbf{G}_1 = (\mathbf{S}_1, \mathbf{L}_1, \mathbf{Z}_1, H_1)$ and $\mathbf{G}_2 = (\mathbf{S}_2, \mathbf{L}_2, \mathbf{Z}_2, H_2)$ have the same number of direct connection channels (i.e. $\mathbf{Z}_1$ and $\mathbf{Z}_2$ have the same length. Then the *direct connection product* $\mathbf{G}_2 \bowtie \mathbf{G}_1$ is defined by

$$\begin{aligned} \mathbf{G}_1 \bowtie \mathbf{G}_2 \quad = \quad & (\begin{pmatrix} \mathbf{S}_1 & 0 \\ 0 & \mathbf{S}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \end{pmatrix}, \_, \_, \\ & H_1 + H_2 + \mathbf{Z}_2^\dagger \mathbf{Z}_1 + \mathbf{Z}_1^\dagger \mathbf{Z}_2). \end{aligned} \tag{10}$$

The direct connection product describes interaction between the components in terms of the *interaction Hamiltonian* $\mathbf{Z}_2^\dagger \mathbf{Z}_1 + \mathbf{Z}_1^\dagger \mathbf{Z}_2$, a self-adjoint operator in $\mathscr{U}_1 \otimes \mathscr{U}_2$ quantifying the energy flow between the components, Figure 5.

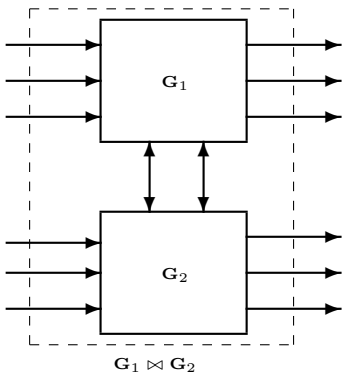

Fig. 5. Direct connection of two systems, $\mathbf{G}_1 \bowtie \mathbf{G}_2$.

Here, the blank _ indicates the absence of an available connection. All products may be extended in a natural way

to describe the absence of a connection using blanks.

We say that a system $\mathbf{G}$ is *reducible* if it can be expressed as

$$\mathbf{G} = (\boxplus_j \mathbf{G}_{fj}) \boxplus (\boxplus_k \mathbf{G}_{dk}) \tag{11}$$

where the subsystems have the form

$$\mathbf{G}_{fj} = (\mathbf{S}_j, \mathbf{L}_j, \_, H_{fj}), \quad \mathbf{G}_{dk} = (\_, \_, \mathbf{Z}_j, H_{dk}). \tag{12}$$

The decomposition (11) identifies any block diagonal structure of the field channels (as reflected in the structure of the scattering matrix $\mathbf{S}$), and separates out the direct interaction terms. It is not unique. This is illustrated in Figure 6.

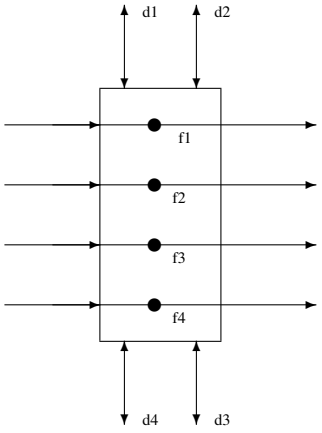

Fig. 6. A reducible system $\mathbf{G} = \mathbf{G}_{f1} \boxplus \mathbf{G}_{f2} \boxplus \mathbf{G}_{f3} \boxplus \mathbf{G}_{f4} \boxplus \mathbf{G}_{d1} \boxplus \mathbf{G}_{d2} \boxplus \mathbf{G}_{d3} \boxplus \mathbf{G}_{d4}$.

The concatenation, series and direct connection products may be used to describe quantum systems. They can be used to represent components, as well as to assemble networks (next section). Note that variables evolve according to the physical laws of both systems and the nature of the connection (cf. [9, sec. 10.8.2])—the behavior of the connected systems is characterized by the parameters specified by the products.

V. Networks

A. Reducible Networks

Let $\{\mathbf{G}_j\}$ be a collection of components, which we may combine together to form an unconnected system $\mathbf{G} = \boxplus_j \mathbf{G}_j$. The components may interact directly via bidirectional exchanges of energy, and this may be specified by *list of direct connections*

$$\mathscr{L}_d = \{\mathbf{G}_{dj} \bowtie \mathbf{G}_{dk}\}. \tag{13}$$

The components may also interact via field interconnects, specified by a *list of series connections*

$$\mathscr{L}_f = \{\mathbf{G}_{fj} \lhd \mathbf{G}_{fk}\}. \tag{14}$$

A *reducible network* $\mathbf{N}$ is the system formed from $\mathbf{G}$ by implementing the connections (13) and (14). An example of a reducible network is shown in Figure 7. Note that a reducible network is itself an open quantum system, and could be considered as a component in a larger network. The parametric representation we use is therefore suitable for hierarchical modeling and network construction.

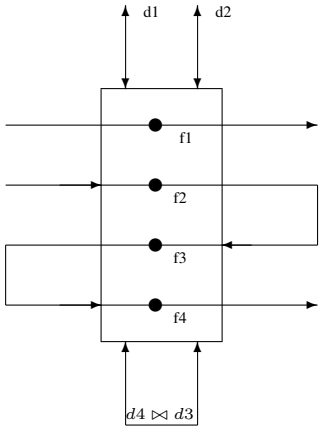

Fig. 7. Given a collection system $\mathbf{G}$ of unconnected components as shown in Figure 6, a reducible network $\mathbf{N} = \mathbf{G}_{f1} \boxplus (\mathbf{G}_{f4} \lhd \mathbf{G}_{f3} \lhd \mathbf{G}_{f2}) \boxplus \mathbf{G}_{d1} \boxplus \mathbf{G}_{d2} \boxplus (\mathbf{G}_{d3} \bowtie \mathbf{G}_{d4})$ is determined by the connection lists $\mathscr{L}_f = \{\mathbf{G}_{f3} \lhd \mathbf{G}_{f2}, \mathbf{G}_{f4} \lhd \mathbf{G}_{f3}\}$ and $\mathscr{L}_d = \{\mathbf{G}_{d3} \bowtie \mathbf{G}_{d4}\}$.

We close this subsection by describing how the classical behavioral interconnection notation $\wedge$, [9, sec. 10.8.2], may be used in our quantum context. Given two systems $\mathbf{G}_1$ and $\mathbf{G}_2$, and specifications for field-mediated and direct connections, $\mathscr{L}_f$ and $\mathscr{L}_d$, we write $\mathbf{G}_1 \wedge \mathbf{G}_2$ for the interconnected system. More generally, for multiple systems $\{\mathbf{G}_j\}$, the interconnected system may be denoted $\wedge_{\mathscr{L}_f, \mathscr{L}_d} \mathbf{G}_j$.

B. Example from Quantum Optics

In this subsection we discuss the quantum behavior for the example from quantum optics, [13, Fig. 1], mentioned in section I, which features a pair of cavities with both field-mediated and direct connections, Figure 1. Each cavity mode is described by an annihilation operator a_1 and a_2, respectively, and is coupled to an external free field A_1 and A_2, respectively. The field-mediated series connection is effected by directing the output of the first channel into the input of the second. The input and output components of a field channel can be separated in the laboratory by a Faraday isolator (not shown in Figure 1). The cavity-field couplings are described by coupling operators $\sqrt{\gamma_1}\, a_1$ and $\sqrt{\gamma_2}\, a_2$, where γ_1 and γ_2 are coupling strength parameters (non-negative real numbers). The direct coupling is defined by an interaction Hamiltonian $V = -ig(a_2 a_1^* - a_2^* a_1)$, where g is a coupling strength parameter. Physically, this coupling could correspond to mode conversion, effected by a polarization rotator, [13, page 4121]. The cavities each have self energies $\Delta_1 a_1^* a_1$ and $\Delta_2 a_2^* a_2$ respectively, due to mismatches (detunings) between the nominal field channel frequency and cavity mode frequencies (the model is expressed in a rotating frame). The $*$-algebras $\mathscr{U}_1$ and $\mathscr{U}_2$ are both copies of the harmonic oscillator algebra (the $*$-algebra generated by an annihilation operator a satisfying the canonical commutation relation $[a, a^*] = 1$).

Before the connections are enabled, the cavity pair could be described as a reducible system $\mathbf{G} = \mathbf{G}_1 \boxplus \mathbf{G}_2$, where

$$\begin{aligned}\mathbf{G}_1 &= (1, \sqrt{\gamma_1}\, a_1, \sqrt{g}\, a_1, \Delta_1 a_1^* a_1) \\ &= (1, \sqrt{\gamma_1}\, a_1, -, \Delta_1 a_1^* a_1) \boxplus (-, -, \sqrt{g}\, a_1, 0) \\ &\doteq \mathbf{G}_{f1} \boxplus \mathbf{G}_{d1},\end{aligned}$$

$$\begin{aligned}\mathbf{G}_2 &= (1, \sqrt{\gamma_2}\, a_2, -i\sqrt{g}\, a_2, \Delta_2 a_2^* a_2) \\ &= (1, \sqrt{\gamma_2}\, a_2, -, \Delta_2 a_2^* a_2) \boxplus (-, -, -i\sqrt{g}\, a_2, 0) \\ &\doteq \mathbf{G}_{f2} \boxplus \mathbf{G}_{d2}.\end{aligned}$$

This expresses the unconnected pair as a concatenation of two cavities, each with field and direct connection channels, as shown in Figure 8.

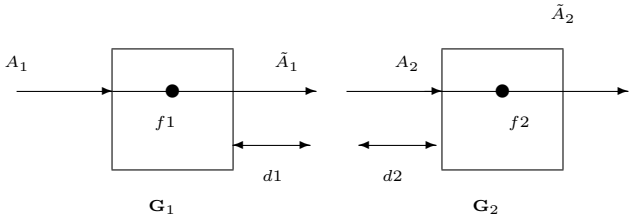

Fig. 8. Network representation of the unconnected cavity pair.

The connections are specified by $\mathbf{G}_{f2} \triangleleft \mathbf{G}_{f1}$ and $\mathbf{G}_{d1} \bowtie \mathbf{G}_{d2}$. After these connections have been made, the connected cavity pair is given by the reducible system

$$\begin{aligned}\mathbf{N} &= \mathbf{G}_1 \wedge \mathbf{G}_2 = (\mathbf{G}_{f2} \triangleleft \mathbf{G}_{f1}) \boxplus (\mathbf{G}_{d1} \bowtie \mathbf{G}_{d2}) \\ &= (1, \sqrt{\gamma_2}\, a_2 + \sqrt{\gamma_1}\, a_1, -, \\ &\quad \Delta_1 a_1^* a_1 + \Delta_2 a_2^* a_2 - ig(a_2 a_1^* - a_2^* a_1)).\end{aligned} \tag{15}$$

This expression simply and transparently describes the network in terms of the interconnections used in forming it, and is illustrated in Figure 9.

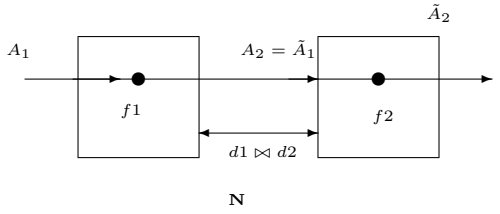

Fig. 9. Network representation of the connected cavity pair.

The connected cavity pair may be considered as an open system with a single field channel and no direct connection channels, as described by the parameters $\mathbf{N}$ given by (15), as shown in Figure 10 (equivalent circuit). This determines the quantum behavior of the connected cavity pair system through the quantum behavioral (Schrodinger) equation (1).

VI. Discussion and Conclusions

In this paper we have discussed some ideas underlying how the "behavioral approach" might be applied to open quantum systems. In the spirit of systems theory, we have

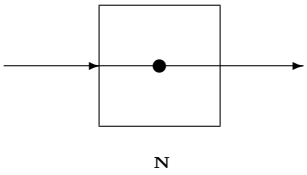

Fig. 10. Equivalent representation of the connected cavity pair.

provided simple algebraic tools for describing open quantum networks using parameters, complete with rules for decomposition and assembly. We remark that classical (i.e. non-quantum) systems, deterministic or stochastic, linear or nonlinear, may be regarded as special cases by considering them as commutative subsystems of open quantum systems (see [4] for examples). It seems that the behavioral ideas we have discussed for open quantum systems and networks, with their focus on the behavior of physical variables, is consistent with much of the behavioral philosophy advocated by Willems and collaborators, e.g. [9], [11], [12]. However, it appears that we may need a *quantum probability space* $(\mathscr{U}, \mathbb{P})$ [2] : the big quantum physical variable space in the sky (cf. [9, page 7])!

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