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Stability analysis for neural networks with time-varying delay

Xun-Lin Zhu and Guang-Hong Yang

Abstract— **This paper studies the problem of stability analysis for neural networks (NNs) with a time-varying delay. The activation functions are assumed to be neither monotonic, nor differentiable, nor bounded. By defining a more general type of Lyapunov functionals, some new less conservative delay-dependent stability criteria are obtained and shown in terms of linear matrix inequalities (LMIs). Since less variables are involved, the computational complexity of the new conditions is reduced. Numerical examples are given to illustrate the effectiveness and the benefits of the proposed method.**

I. INTRODUCTION

Now, neural networks (NNs) are widely studied, because of their immense potentials of application prospective in a variety of areas, such as signal processing, pattern recognition, static image processing, associative memory, and combinatorial optimization. In practice, time delay is frequently encountered in NNs. Due to the finite speed of information processing, the existence of time delays frequently causes oscillation, divergence, or instability in NNs. In recent years, the stability problem of delayed neural networks has become a topic of great theoretic and practical importance [2]-[23]. This issue has gained increasing interest in applications to signal and image processing, artificial intelligence, etc.

The stability criteria for delayed NNs can be classified into two categories, namely, delay-independent [11]-[20] and delay-dependent [6]-[10]. As we know time delay dependent results are looser than the time delay independent ones when the delays are small. So much attention has been paid to the delay-dependent type.

In [9], delay-dependent stability condition was derived by defining a new Lyapunov functional and the obtained condition could include some existing time delay-independent ones. In [10], a less conservative delay-dependent stability criterion for delayed NNs was proposed by using the free weighting matrix method and considering the useful term when estimating the upper bound of the derivative of Lyapunov functional. And the stability result in [10] was improved in [25]. However, these mentioned results are still conservative to some extent, which leave open room for further improvement.

In this paper, the problem of stability analysis for delayed NNs is investigated. Unlike the previous works, the activation functions are assumed to be neither monotonic, nor differentiable, nor bounded, so the considered NNs are more general than the ones in literature. For ensuring larger delay bounds, a new type of Lyapunov functionals is proposed, and some new delay-dependent stability criteria are derived in terms of LMIs. It is shown that the newly obtained results are less conservative and less computationally complex than the existing corresponding ones. Meanwhile, these stability criteria are also more applicable. Finally, numerical examples will be given to show the effectiveness of the main results.

II. PROBLEM FORMULATION

The dynamic behavior of a continuous time-delay neural network can be described as follows:

$$\dot{x}(t) = -Cx(t) + Ag(x(t)) + Bg(x(t-d(t))) + u, \qquad (1)$$

where $x(\cdot) = \begin{bmatrix} x_1(\cdot) & x_2(\cdot) & \cdots & x_n(\cdot) \end{bmatrix}^T \in R^n$ is the neuron state vector, $u = [u_1, \ u_2, \ \cdots, \ u_n]^T \in R^n$ is a constant input vector, and $g(x(\cdot)) = [g_1(x_1(\cdot)) \ \ g_2(x_2(\cdot)) \ \ \cdots \ \ g_n(x_n(\cdot))]^T \in R^n$ denotes the neuron activation function. $C = diag(c_1, \ \cdots, \ c_n)$ with $c_i > 0 \ (i = 1, \ 2, \ \cdots, \ n)$, and $A, \ B$ are the connection weight matrix and the delayed connection weight matrix, respectively. The time delay $d(t)$ is a time-varying differentiable function that satisfies

$$0 \le d(t) \le h, \qquad (2)$$
$$\dot{d}(t) \le \mu, \qquad (3)$$

where h and μ are constants.

In this paper, we assume the activation functions $g_i(\cdot)(i = 1, \ 2, \ \cdots, \ n)$ satisfy the following condition:

$$k_i^- \le \frac{g_i(x) - g_i(y)}{x - y} \le k_i^+ \quad \forall x, \ y \in R, \ x \ne y, \qquad (4)$$

where k_i^-, k_i^+ are some constants. So, such activation functions $g_i(\cdot)(i = 1, \ 2, \ \cdots, \ n)$ are globally Lipschitz continuous of classes $\mathscr{L}$ defined in [26] (see also [27]).

Remark 1. In the literature, the constants k_i^- are all assumed to be zero, which implies that $g_i(\cdot) \ (i = 1, \ 2, \ \cdots, \ n)$ are

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monotonically increasing (see [9], [10], [25]). Correspondingly, k_i^-, k_i^+ are allowed to be positive, negative or zero in this paper.

Assume that $x^* = [x_1^* \ x_2^* \ \cdots \ x_n^*]^T$ is an equilibrium point of system (1), by choosing the coordinate transformation $z(\cdot) = x(\cdot) - x^*$, system (1) is changed into the following error system

$$\dot{z}(t) = -Cz(t) + Af(z(t)) + Bf(z(t-d(t))), \tag{5}$$

where $z(\cdot) = [z_1(\cdot) \quad z_2(\cdot) \quad \cdots \quad z_n(\cdot)]^T$ is the state vector of the transformed system, $f(z) = [f_1(z_1(\cdot)) \quad f_2(z_2(\cdot)) \quad \cdots \quad f_n(z_n(\cdot))]^T$ and $f_i(z_i(\cdot)) = g_i(z_i(\cdot)+x_i^*) - g_i(x_i^*)$ $(i = 1, \ 2, \ \cdots, \ n)$. Then, the functions $f_i(\cdot)$ $i = (1, \ 2, \ \cdots, \ n)$ satisfy the following condition:

$$k_i^- \leq \frac{f_i(z_i)}{z_i} \leq k_i^+, \quad f_i(0) = 0 \quad \forall z_i \neq 0, \tag{6}$$

which implies that

$$(f_i(z_i(t)) - k_i^+ z_i(t)) \cdot (f_i(z_i(t)) - k_i^- z_i(t)) \leq 0 \tag{7}$$

and

$$\begin{aligned}(f_i(z_i(t-d(t))) - k_i^+ z_i(t-d(t)))\\ \cdot (f_i(z_i(t-d(t))) - k_i^- z_i(t-d(t))) \leq 0.\end{aligned} \tag{8}$$

Through this paper, the Jensen integral inequality will be used, so it is listed as the following lemma.

Lemma 1. [1] For any positive definite symmetric constant matrix $M \in R^{n\times n}$, scalars $r_1, \ r_2$ satisfying $r_1 < r_2$, a vector function $\omega : [r_1, \ r_2] \to R^n$ such that the integrations concerned are well defined, then

$$\left(\int_{r_1}^{r_2} \omega(s)ds\right)^T M \int_{r_1}^{r_2} \omega(s)ds \leq (r_2 - r_1)\int_{r_1}^{r_2} \omega^T(s)M\omega(s)ds.$$

III. Main results

For convenience, we denote $K_1 = diag(k_1^+, \ k_2^+, \ \cdots, \ k_n^+)$ and $K_0 = diag(k_1^-, \ k_2^-, \ \cdots, \ k_n^-)$.

From (6), it follows that

$$\int_0^{z_i(t)} (f_i(s) - k_i^- s)ds \geq 0, \tag{9}$$

$$\int_0^{z_i(t)} (k_i^+ s - f_i(s))ds \geq 0. \tag{10}$$

Based on this fact, a Lyapunov-Krasovskii functional can be chosen as

$$V(z_t) = V_1(z_t) + V_2(z_t) + V_3(z_t), \tag{11}$$

where

$$\begin{aligned}V_1(z_t) =& z^T(t)Pz(t) + 2\sum_{i=1}^{n}\Big(\lambda_i \int_0^{z_i(t)} (f_i(s) - k_i^- s)ds\\ &+ \delta_i \int_0^{z_i(t)} (k_i^+ s - f_i(s))ds\Big),\\ V_2(z_t) =& \int_{t-d(t)}^{t} [z^T(s)Q_1 z(s) + f^T(z(s))Q_2 f(z(s))]ds\\ &+ \int_{t-h}^{t} z^T(s)Q_3 z(s)ds,\\ V_3(z_t) =& \int_{-h}^{0}\int_{t+\theta}^{t} \dot{z}^T(s)Z\dot{z}(s)dsd\theta.\end{aligned}$$

Remark 2. Since k_i^- $(i = 1, \ 2, \ \cdots, \ n)$ may be nonzero and the term $2\sum_{i=1}^{n} \delta_i \int_0^{z_i(t)} (k_i^+ s - f_i(s))ds$ is taken into account, it is clear that the Lyapunov-Krasovkii functional in this paper is more general than that the ones in [10] and [25], and the resulting stability results may be more applicable.

A. New stability criteria

First, we give a new stability criterion for the origin of system (5)-(6) as follows.

Theorem 1. For given scalars $h > 0$ and μ, the origin of system (5) with (6) and a time-varying delay satisfying conditions (2) and (3) is globally asymptotically stable if there exist matrices $P = P^T > 0$, $Q_l = Q_l^T \geq 0$ $(l = 1, \ 2, \ 3)$, $Z = Z^T > 0$, $\Lambda = diag(\lambda_1, \ \lambda_2, \ \cdots, \ \lambda_n) \geq 0$, $\Delta = diag(\delta_1, \ \delta_2, \ \cdots, \ \delta_n) \geq 0$, $T_j = diag(t_{1j}, \ t_{2j}, \ \cdots, \ t_{nj}) \geq 0$ $(j = 1, \ 2)$, such that the following LMI holds:

$$\Psi = \begin{bmatrix} \Psi_{11} & h^{-1}Z & \Psi_{13} & \Psi_{14} & 0 & -hC^TZ\\ * & \Psi_{22} & 0 & \Psi_{24} & h^{-1}Z & 0\\ * & * & \Psi_{33} & \Psi_{34} & 0 & hA^TZ\\ * & * & * & \Psi_{44} & 0 & hB^TZ\\ * & * & * & * & \Psi_{55} & 0\\ * & * & * & * & * & -hZ \end{bmatrix} < 0, \tag{12}$$

where

$$\begin{aligned}\Psi_{11} =& -PC - C^TP + Q_1 + Q_3 - (K_1\Delta - K_0\Lambda)C\\ &- C^T(K_1\Delta - K_0\Lambda) - 2K_1T_1K_0 - h^{-1}Z,\\ \Psi_{13} =& PA - C^T(\Lambda - \Delta) + (K_1\Delta - K_0\Lambda)A + (K_0 + K_1)T_1,\\ \Psi_{14} =& PB + (K_1\Delta - K_0\Lambda)B,\\ \Psi_{22} =& -(1-\mu)Q_1 - 2K_1T_2K_0 - 2h^{-1}Z,\\ \Psi_{24} =& (K_1 + K_0)T_2,\\ \Psi_{33} =& Q_2 - 2T_1 + (\Lambda - \Delta)A + A^T(\Lambda - \Delta),\\ \Psi_{34} =& (\Lambda - \Delta)B,\\ \Psi_{44} =& -(1-\mu)Q_2 - 2T_2,\\ \Psi_{55} =& -Q_3 - h^{-1}Z.\end{aligned}$$

Proof: Taking the time derivative of $V_i(z_t)$ $(i = 1, \ 2, \ 3)$ along the trajectory of (5) yields that

$$\begin{aligned}&\dot{V}_1(z_t)\\ &= 2z^T(t)P[-Cz(t) + Af(z(t)) + Bf(z(t-d(t)))]\\ &\quad + 2f^T(z(t))(\Lambda - \Delta)[-Cz(t) + Af(z(t)) + Bf(z(t-d(t)))]\\ &\quad + 2z^T(t)(K_1\Delta - K_0\Lambda)[-Cz(t) + Af(z(t)) + Bf(z(t-d(t)))],\end{aligned} \tag{13}$$

$$\begin{aligned}&\dot{V}_2(z_t)\\ &\leq z^T(t)(Q_1 + Q_3)z(t) + f^T(z(t))Q_2 f(z(t))\\ &\quad - z^T(t-h)Q_3 z(t-h) - (1-\mu)z^T(t-d(t))Q_1 z(t-d(t))\\ &\quad - (1-\mu)f^T(z(t-d(t)))Q_2 f(z(t-d(t))),\end{aligned} \tag{14}$$

$$\begin{aligned}&\dot{V}_3(z_t)\\ &= h\dot{z}^T(t)Z\dot{z}(t) - \int_{t-h}^{t} \dot{z}^T(s)Z\dot{z}(s)ds\\ &= h[-Cz(t) + Af(z(t)) + Bf(z(t-d(t)))]^T Z\\ &\quad \times [-Cz(t) + Af(z(t)) + Bf(z(t-d(t)))]\\ &\quad - \int_{t-d(t)}^{t} \dot{z}^T(s)Z\dot{z}(s)ds - \int_{t-h}^{t-d(t)} \dot{z}^T(s)Z\dot{z}(s)ds.\end{aligned} \tag{15}$$

Next, the upper bound of $\dot{V}_3(z_t)$ can be estimated as follows. By using Lemma 1, it gets that

$$\begin{aligned}
&-\int_{t-d(t)}^{t} \dot{z}^T(s)Z\dot{z}(s)ds \\
&\leq -\frac{1}{d(t)}\Big(\int_{t-d(t)}^{t} \dot{z}(s)ds\Big)^T Z\int_{t-d(t)}^{t} \dot{z}^T(s)ds \\
&\leq -\frac{1}{h}[z(t)-z(t-d(t))]^T Z[z(t)-z(t-d(t))], \qquad (16)\\
&-\int_{t-h}^{t-d(t)} \dot{z}^T(s)Z\dot{z}(s)ds \\
&\leq -\frac{1}{h-d(t)}\Big(\int_{t-h}^{t-d(t)} \dot{z}(s)ds\Big)^T Z\int_{t-h}^{t-d(t)} \dot{z}(s)ds \\
&\leq -\frac{1}{h}[z(t-d(t))-z(t-h)]^T Z[z(t-d(t))-z(t-h)]. \qquad (17)
\end{aligned}$$

On the other hand, for any $T_j = diag(t_{1j},\ t_{2j},\ \cdots,\ t_{nj}) \geq 0\ (j=1,\ 2)$, from (7) and (8), it yields that

$$\begin{aligned}
0 \leq & -2\sum_{i=1}^{n} t_{i1}(f_i(z_i(t))-k_i^+z_i(t))(f_i(z_i(t))-k_i^-z_i(t)) \\
& -2\sum_{i=1}^{n} t_{i2}(f_i(z_i(t-d(t)))-k_i^+z_i(t-d(t))) \\
& \cdot (f_i(z_i(t-d(t)))-k_i^-z_i(t-d(t))) \\
= & -2(f(z(t))-K_1z(t))^T T_1(f(z(t))-K_0z(t)) \\
& -2(f(z(t-d(t)))-K_1z(t-d(t)))^T T_2 \\
& \times (f(z(t-d(t)))-K_0z(t-d(t))). \qquad (18)
\end{aligned}$$

Denoting $\zeta(t) = [z^T(t)\ \ z^T(t-d(t))\ \ f^T(z(t))\ \ f^T(z(t-d(t)))\ \ z^T(t-h)]^T$, and combining (13)-(18), it can be seen that

$$\dot{V}(z_t) \leq \zeta^T(t)\hat{\Psi}\zeta(t), \qquad (19)$$

where

$$\hat{\Psi} = \begin{bmatrix} \Psi_1 & h^{-1}Z & \Psi_{13}-hC^TZA & \Psi_{14}-hC^TZB & 0 \\ * & \Psi_{22} & 0 & \Psi_{24} & h^{-1}Z \\ * & * & \Psi_{33}+hA^TZA & \Psi_{34}+hA^TZB & 0 \\ * & * & * & \Psi_{44}+hB^TZB & 0 \\ * & * & * & * & \Psi_{55} \end{bmatrix},$$

where $\Psi_1 = \Psi_{11}+hC^TZC$. By the Schur complement, it is easy to see that $\dot{V}(z_t) < 0$ if $\Psi < 0$.

Thus, the proof is completed. ■

Remark 3. In Theorem 1, a sufficient condition of global asymptotical stability for the origin of system (5) with (6) and a time-varying delay satisfying conditions (2) and (3) is given in terms of solutions to a set of LMIs. Note that the Lyapunov-Krasovskii functional (11) is more general, and the newly obtained stability criterion is less conservative than that in [10]. Meanwhile, since no any redundant variables are involved, the computational complexity is reduced. The details will be discussed in the sequel.

B. Further development

Next, we estimate the upper bound of $\dot{V}_3(z_t)$ by following the idea of convex combination [24], and an improved stability criterion of Theorem 1 can be developed as follows.

Theorem 2. For given scalars $h > 0$ and μ, the origin of system (5) with (6) and a time-varying delay satisfying conditions (2) and (3) is globally asymptotically stable if there exist matrices $P = P^T > 0$, $Q_l = Q_l^T \geq 0$ $(l = 1,\ 2,\ 3)$, $Z = Z^T > 0$, $\Lambda = diag(\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n) \geq 0$, $\Delta = diag(\delta_1,\ \delta_2,\ \cdots,\ \delta_n) \geq 0$, $T_j = diag(t_{1j},\ t_{2j},\ \cdots,\ t_{nj}) \geq 0$ $(j = 1,\ 2)$, N_1, N_2 and M_1, M_2, such that the following LMIs hold:

$$\Omega_1 = \begin{bmatrix} \Omega & hN & h\mathscr{A}Z \\ * & -hZ & 0 \\ * & * & -hZ \end{bmatrix} < 0, \qquad (20)$$

$$\Omega_2 = \begin{bmatrix} \Omega & hM & h\mathscr{A}Z \\ * & -hZ & 0 \\ * & * & -hZ \end{bmatrix} < 0, \qquad (21)$$

where

$$\Omega = \begin{bmatrix} \Omega_{11} & 0 & \Omega_{13} & \Omega_{14} & 0 \\ * & \Omega_{22} & 0 & \Omega_{24} & 0 \\ * & * & \Omega_{33} & \Omega_{34} & 0 \\ * & * & * & \Omega_{44} & 0 \\ * & * & * & * & \Omega_{55} \end{bmatrix} + \Omega_3 + \Omega_3^T,$$

$$\begin{aligned}
\Omega_{11} &= -PC - C^TP + Q_1 + Q_3 - (K_1\Delta - K_0\Lambda)C \\
&\quad - C^T(K_1\Delta - K_0\Lambda) - 2K_1T_1K_0, \\
\Omega_{13} &= PA - C^T(\Lambda-\Delta) + (K_1\Delta - K_0\Lambda)A + (K_0+K_1)T_1, \\
\Omega_{14} &= PB + (K_1\Delta - K_0\Lambda)B, \\
\Omega_{22} &= -(1-\mu)Q_1 - 2K_1T_2K_0, \\
\Omega_{24} &= (K_1+K_0)T_2, \\
\Omega_{33} &= Q_2 - 2T_1 + (\Lambda-\Delta)A + A^T(\Lambda-\Delta), \\
\Omega_{34} &= (\Lambda-\Delta)B, \\
\Omega_{44} &= -(1-\mu)Q_2 - 2T_2, \\
\Omega_{55} &= -Q_3, \\
\Omega_3 &= \begin{bmatrix} N & -N+M & 0 & 0 & -M \end{bmatrix}, \\
N &= \begin{bmatrix} N_1^T & N_2^T & 0 & 0 & 0 \end{bmatrix}^T, \\
M &= \begin{bmatrix} M_1^T & M_2^T & 0 & 0 & 0 \end{bmatrix}^T, \\
\mathscr{A} &= \begin{bmatrix} -C & 0 & A & B & 0 \end{bmatrix}^T.
\end{aligned}$$

Proof: For any positive definite matrix Z, the following inequality is always true for any vector a and b:

$$-2a^Tb \leq a^TZ^{-1}a + b^TZb,$$

so it is follows that for any $t,\ s \in [0,\ \infty)$

$$-2\zeta^T(t)N\dot{z}(s) \leq \zeta^T(t)NZ^{-1}N^T\zeta(t) + \dot{z}^T(s)Z\dot{z}(s),$$

where $\zeta(t)$ is defined in the proof of Theorem 1.

Thus, integrating on the both sides of the above inequality, it gets that

$$\begin{aligned}
-\int_{t-d(t)}^{t} \dot{z}^T(s)Z\dot{z}(s)ds \leq & d(t)\zeta^T(t)NZ^{-1}N^T\zeta(t) \\
& + 2\zeta^T(t)N[z(t)-z(t-d(t))].
\end{aligned} \qquad (22)$$

Similarly, one can get

$$-\int_{t-h}^{t-d(t)} \dot{z}^T(s)Z\dot{z}(s)ds \leq (h-d(t))\zeta^T(t)MZ^{-1}M^T\zeta(t) + 2\zeta^T(t)M[z(t-d(t))-z(t-h)]. \quad (23)$$

Then, combining (13)-(15), (22)-(23) with (18), it yields that

$$\dot{V}(z_t) \leq \zeta^T(t)\Big(\Omega + h\mathscr{A}Z\mathscr{A}^T + d(t)NZ^{-1}N^T + (h-d(t))MZ^{-1}M^T\Big)\zeta(t). \quad (24)$$

Note that $0 \leq d(t) \leq h$, so

$$\Omega + h\mathscr{A}Z\mathscr{A}^T + d(t)NZ^{-1}N^T + (h-d(t))MZ^{-1}M^T < 0$$

holds if and only if

$$\Omega + h\mathscr{A}Z\mathscr{A}^T + hNZ^{-1}N^T < 0, \quad (25)$$

and

$$\Omega + h\mathscr{A}Z\mathscr{A}^T + hMZ^{-1}M^T < 0. \quad (26)$$

From the Schur complement, inequality (25) is equivalent to $\Omega_1 < 0$, and inequality (26) is equivalent to $\Omega_2 < 0$, respectively. So, $\dot{V}(z_t) < 0$ holds if $\Omega_1 < 0$ and $\Omega_2 < 0$.

This completes the proof. ■

Remark 4. The stability criterion proposed in [10] was improved in [25], and we will prove that Theorem 2 is less conservative and less complex than the one in [25] in the next section.

IV. COMPARISON WITH THE EXISTING RESULTS

In this section, we prove that Theorem 1 is less conservative than that in [10], and Theorem 2 is less conservative than that in [25], respectively. For convenience of comparison, the main results in [10] and [25] are listed as the following lemmas.

Lemma 2. [10] For given scalars $h > 0$ and μ, the origin of system (5) with (6) is globally asymptotically stable if there exist matrices $P = P^T > 0$, $Q_l = Q_l^T \geq 0$ $(l = 1, 2, 3)$, $Z = Z^T > 0$, $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n) \geq 0$, $T_j = diag(t_{1j}, t_{2j}, \cdots, t_{nj}) \geq 0$ $(j = 1, 2)$, N_i, M_i $(i = 1, 2, \cdots, 5)$, such that the following LMI holds:

$$\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 \\ * & \Phi_3 \end{bmatrix} < 0, \quad (27)$$

where

$$\Phi_1 = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & N_5^T - M_1 \\ * & \Phi_{22} & \Phi_{23} & \Phi_{24} & -N_5^T + M_5^T - M_2 \\ * & * & \Phi_{33} & \Lambda B & -M_3 \\ * & * & * & \Phi_{44} & -M_4 \\ * & * & * & * & -Q_3 - M_5 - M_5^T \end{bmatrix},$$

$$\Phi_2 = \begin{bmatrix} hN_1 & hM_1 & -hC^TZ \\ hN_2 & hM_2 & 0 \\ hN_3 & hM_3 & hA^TZ \\ hN_4 & hM_4 & hB^TZ \\ hN_5 & hM_5 & 0 \end{bmatrix},$$

$\Phi_3 = diag(-hZ, -hZ, -hZ)$,
$\Phi_{11} = -PC - C^TP + N_1 + N_1^T + Q_1 + Q_3$,
$\Phi_{12} = N_2^T - N_1 + M_1$,
$\Phi_{13} = PA - C^T\Lambda + K_1T_1 + N_3^T$,
$\Phi_{14} = PB + N_4^T$,
$\Phi_{22} = -(1-\mu)Q_1 - N_2 - N_2^T + M_2 + M_2^T$,
$\Phi_{23} = -N_3^T + M_3^T$,
$\Phi_{24} = K_1T_2 - N_4^T + M_4^T$,
$\Phi_{33} = Q_2 - 2T_1 + \Lambda A + A^T\Lambda$,
$\Phi_{44} = -(1-\mu)Q_2 - 2T_2$.

Lemma 3. [25] For given scalars $h > 0$ and μ, the origin of system (5) with (6) is globally asymptotically stable if there exist matrices $P = P^T > 0$, $Q_l = Q_l^T \geq 0$ $(l = 1, 2, 3)$, $Z = Z^T > 0$, $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n) \geq 0$, $T_j = diag(t_{1j}, t_{2j}, \cdots, t_{nj}) \geq 0$ $(j = 1, 2)$, $X_{11} \geq 0$, $X_{22} \geq 0$, X_{12} and N_i, M_i $(i = 1, 2)$, such that

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & PB & -M_1 & -hC^TZ \\ * & \Theta_{22} & 0 & K_1T_2 & -M_2 & 0 \\ * & * & \Theta_{33} & \Lambda B & 0 & hA^TZ \\ * & * & * & \Theta_{44} & 0 & hB^TZ \\ * & * & * & * & -Q_3 & 0 \\ * & * & * & * & * & -hZ \end{bmatrix} < 0, \quad (28)$$

and

$$\begin{bmatrix} X_{11} & X_{12} & N_1 \\ * & X_{22} & N_2 \\ * & * & Z \end{bmatrix} \geq 0, \quad \begin{bmatrix} X_{11} & X_{12} & M_1 \\ * & X_{22} & M_2 \\ * & * & Z \end{bmatrix} \geq 0, \quad (29)$$

where

$\Theta_{11} = -PC - C^TP + N_1 + N_1^T + Q_1 + Q_3 + hX_{11}$,
$\Theta_{12} = N_2^T - N_1 + M_1 + hX_{12}$,
$\Theta_{13} = PA - C^T\Lambda + K_1T_1$,
$\Theta_{22} = -(1-\mu)Q_1 - N_2 - N_2^T + M_2 + M_2^T + hX_{22}$,
$\Theta_{33} = Q_2 - 2T_1 + \Lambda A + A^T\Lambda$,
$\Theta_{44} = -(1-\mu)Q_2 - 2T_2$.

In [10], it was proved that Lemma 2 is less conservative than the theorem in [9]. For comparing Theorem 1 with Lemma 2, the following lemma is needed.

Lemma 4. Inequality (27) is equivalent to

$$\tilde{\Phi} < 0, \quad (30)$$

where

$$\tilde{\Phi} = \begin{bmatrix} \tilde{\Phi}_{11} & h^{-1}Z & \tilde{\Phi}_{13} & PB & 0 & -hC^TZ \\ * & \tilde{\Phi}_{22} & 0 & K_1T_2 & h^{-1}Z & 0 \\ * & * & \Phi_{33} & \Lambda B & 0 & hA^TZ \\ * & * & * & \Phi_{44} & 0 & hB^TZ \\ * & * & * & * & -Q_3 - h^{-1}Z & 0 \\ * & * & * & * & * & -hZ \end{bmatrix},$$

$\tilde{\Phi}_{11} = -PC - C^TP + Q_1 + Q_3 - h^{-1}Z$,
$\tilde{\Phi}_{13} = PA - C^T\Lambda + K_1T_1$,
$\tilde{\Phi}_{22} = -(1-\mu)Q_1 - 2h^{-1}Z$.

Proof: Denote

$$\bar{\Phi} = \Delta_1\Phi\Delta_1^T, \quad (31)$$

and one can get that

$$\bar{\Phi} = \begin{bmatrix} \tilde{\Phi} & \Delta_2 \\ * & \Delta_3 \end{bmatrix}, \quad (32)$$

where

$$\Delta_1 = \begin{bmatrix} I & 0 & 0 & 0 & 0 & -h^{-1}I & 0 & 0 \\ 0 & I & 0 & 0 & 0 & h^{-1}I & -h^{-1}I & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & h^{-1}I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \end{bmatrix},$$

$$\Delta_2 = \begin{bmatrix} hN_1+Z & hM_1 \\ hN_2-Z & hM_2+Z \\ hN_3 & hM_3 \\ hN_4 & hM_4 \\ hN_5 & hM_5-Z \\ 0 & 0 \end{bmatrix}, \quad \Delta_3 = diag(-hZ,\ -hZ).$$

So, it is obvious that $\tilde{\Phi} < 0$ holds if $\Phi < 0$.

Conversely, if $\tilde{\Phi} < 0$ holds, then $\bar{\Phi} < 0$ is true by letting

$$N_1 = -h^{-1}Z,\ N_2 = h^{-1}Z,\ N_3 = N_4 = N_5 = 0;$$
$$M_2 = -h^{-1}Z,\ M_5 = h^{-1}Z,\ M_1 = M_3 = M_4 = 0.$$

So, $\Phi < 0$ is also true since $\bar{\Phi} < 0$ is equivalent to $\Phi < 0$.

Thus, the proof is completed. ■

Remark 5. Lemma 4 shows that the free weighting matrices $N_i,\ M_i\ (i = 1,\ 2,\ \cdots,\ 5)$ introduced in Lemma 2 are all redundant, and thus the stability condition in [10] can be simplified.

If choosing $k_i^- = 0,\ \delta_i = 0\ (i = 1,\ 2,\ \cdots,\ n)$, then $\Psi < 0$ in Theorem 1 is equivalent to $\tilde{\Phi} < 0$ in Lemma 4, so it yields the following theorem.

Theorem 3. If $\Phi < 0$ in Lemma 2 is feasible, then $\Psi < 0$ in Theorem 1 is also feasible.

Remark 6. From Theorem 3, it is proved that Theorem 1 is less conservative than Lemma 2.

Now, we compare Theorem 2 with Lemma 3.

Theorem 4. If the inequalities in Lemma 3 are feasible, then the inequalities in Theorem 2 are also feasible.

Proof: Note that $\Theta < 0$ in Lemma 3 is equivalent to

$$\tilde{\Theta} + h\tilde{X} + h\mathscr{A}Z\mathscr{A}^T < 0 \qquad (33)$$

where

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_{11} & \tilde{\Theta}_{12} & \Theta_{13} & PB & -M_1 \\ * & \tilde{\Theta}_{22} & 0 & K_1T_2 & -M_2 \\ * & * & \Theta_{33} & \Lambda B & 0 \\ * & * & * & \Theta_{44} & 0 \\ * & * & * & * & -Q_3 \end{bmatrix},$$

$$\tilde{X} = \begin{bmatrix} X_{11} & X_{12} & 0 & 0 & 0 \\ * & X_{22} & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix},$$

$$\tilde{\Theta}_{11} = -PC - C^TP + N_1 + N_1^T + Q_1 + Q_3,$$
$$\tilde{\Theta}_{12} = N_2^T - N_1 + M_1,$$
$$\tilde{\Theta}_{22} = -(1-\mu)Q_1 - N_2 - N_2^T + M_2 + M_2^T,$$

TABLE I

COMPARISON OF THE NUMBERS OF THE VARIABLES INVOLVED

Methods	Number of variables involved
Theorem [9]	$14n^2+6n$
Lemma 2	$12.5n^2+5.5n$
Lemma 3	$8.5n^2+6.5n$
Theorem 1	$2.5n^2+6.5n$
Theorem 2	$6.5n^2+6.5n$

and $\Theta_{13},\ \Theta_{33},\ \Theta_{44},\ X_{11},\ X_{22},\ X_{12},\ P,\ Q_j\ (j = 1,\ 2,\ 3),\ N_i,\ M_i\ (i = 1,\ 2)$ are defined in Lemma 3, $\mathscr{A}$ is defined in Theorem 2. And from (29), it is clear that

$$\tilde{X} \geq NZ^{-1}N^T, \quad \tilde{X} \geq MZ^{-1}M^T, \qquad (34)$$

where $N = [N_1^T\ N_2^T\ 0\ 0\ 0]^T$ and $M = [M_1^T\ M_2^T\ 0\ 0\ 0]^T$.

So, if $\Theta < 0$ is feasible, by setting $\Delta = 0$, then it is easy to see that (25) and (26) hold, which implies that the inequalities in Theorem 2 are feasible.

Thus, the proof is completed. ■

Remark 7. Theorem 4 shows that the stability condition in Theorem 2 is less conservative than that in Lemma 3. Meanwhile, Table 1 provides a comparison of the number of the variables involved in Theorem 1, Theorem 2 and in some existing results.

V. NUMERICAL EXAMPLES

In this section, two examples are given to demonstrate the benefits of the proposed method.

Example 1. Consider the delayed NN (1) with a time-varying delay and

$$C = diag(1.2769,\ 0.6231,\ 0.9230,\ 0.4480)$$

$$A = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix}$$

$$k_1^+ = 0.1137 \quad k_2^+ = 0.1279 \quad k_3^+ = 0.7994 \quad k_4^+ = 0.2368.$$

For the case of $k_i^- = 0\ (i = 1,\ 2,\ 3,\ 4)$, it can be checked that Theorem 1 in [21] and Theorem 1 in [7] are not satisfied. The corresponding upper bounds of h for various μ derived by Theorem 1 and those in [7], [9], [10] and [25] are listed in Table 2, in which $'-'$ means that the results are not applicable to the corresponding cases.

For the case of $k_1^- = -0.1,\ k_2^- = 0.1,\ k_3^- = 0$ and $k_4^- = 0.2$, the upper bounds of h for various μ derived by Theorem 1 and Theorem 2 are listed in Table 3.

Example 2. [25] Consider the delayed NN (1) with

$$C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}$$

$$k_1^+ = 0.4 \quad k_2^+ = 0.8.$$

TABLE II

CALCULATED UPPER BOUNDS OF h FOR EXAMPLE 1

μ	0.1	0.5	0.9	≥ 1
[8] and [9]	3.2775	2.1502	1.3164	1.2598
Theorem 1 [10]	3.2793	2.2245	1.5847	1.5444
Theorem 1 [25]	3.3039	2.5376	2.0853	2.0389
Theorem 1	3.4018	2.2874	1.6234	1.5698
Theorem 2	3.4183	2.5943	2.1306	2.0770

TABLE III

CALCULATED UPPER BOUNDS OF h FOR EXAMPLE 1

μ	0	0.1	0.5	0.9	≥ 1
Theorem 1	3.9224	3.6235	2.5196	1.7808	1.7243
Theorem 2	3.9224	3.6574	2.8467	2.3366	2.2841

For the case of $k_1^- = k_2^- = 0$, the corresponding upper bounds of h for various μ derived by Theorem 1 and methods in [8], [10] and [9] are listed in Table 4.

For the case of $k_1^- = -0.2$, $k_2^- = 0.1$, the upper bounds of h for various μ derived by Theorem 1 are listed in Table 5.

VI. CONCLUSION

In this paper, the delay-dependent stability problem of NNs with a time-varying delay has been investigated. By defining an appropriate Lyapunov functional, new delay-dependent stability criteria are derived in terms of LMIs. The newly obtained results are less conservative, less computationally complex and are also more applicable than the existing ones. Numerical examples are given to illustrate the effectiveness of the presented criteria and their improvement over the existing results.

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TABLE IV

CALCULATED UPPER BOUNDS OF h FOR EXAMPLE 2

μ	0.8	0.9	≥ 1
[8] and [9]	1.2281	0.8636	0.8298
Theorem 1 [10]	1.6831	1.1493	1.0880
Theorem 1 [25]	2.3534	1.6050	1.5103
Theorem 1	1.6849	1.1494	1.0880
Theorem 2	2.3571	1.6050	1.5103

TABLE V

CALCULATED UPPER BOUNDS OF h FOR EXAMPLE 2

μ	0.8	0.9	≥ 1
Theorem 1	1.6417	1.1172	1.0716
Theorem 2	2.3092	1.5646	1.4897

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