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Adaptive regulation of uncertain nonlinear systems with dead-zone

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Abstract— **This paper deals with the adaptive output control of a class of uncertain nonlinear systems with an unknown non-symmetric dead-zone nonlinearity. The nonlinear systems considered are dominated by a triangular system without zero dynamics satisfying polynomial growth in the unmeasurable states. An adaptive control scheme is developed without constructing the dead-zone inverse. The proposed adaptive scheme requires only the information of bounds of the slopes and the breakpoint of dead-zone nonlinearity. The novelty of this paper is that a universal-type adaptive output feedback controller is numerically constructed by using a sum of squares (SOS) optimization algorithm, which can globally regulate all the states of the uncertain systems without knowing the growth rate. An example is presented to show the effectiveness of this methodology.**

I. Introduction

As it is reported recently in many papers, the dead-zone input nonlinearity is a non-differentiable function that characterizes certain non-sensitivity for small control inputs [13]. This kind of input characteristic is ubiquitous in a wide range of mechanical and electrical components such as values, DC servo motors, and other devices. The presence of dead-zone in feedback control systems may cause severe deterioration of the system performance. For example, in most practical motion systems, the dead-zone parameter are poorly known and imperfect knowledge of the non-sensitivity zone causes a serious problem in high precision control and, therefore, poses a fundamental issue on how to cross this zone by adaptation. To cope with this inherent problem, adaptive control techniques may be applied to design controllers. The study of adaptive control for systems subject to dead-zone actuators was initiated in [7],[9] and [10], and the extensions may referred to [12] and [8]. Fussy-logic and neural network approaches were further explored to give different looks in [16], [15], and [22]. Robust stabilization of unknown sandwich systems with known uncertainties bounds was discussed in [20] and [21].

The representation of the non-symmetric dead-zone in this work is similar to [13], and an new adaptive control strategy is proposed without constructing the dead-zone inverse also. The adaptive regulation is considered by *output feedback* for a class of uncertain systems subject to a non-symmetric dead-zone input. The system to be controlled has an uncertainty in unmeasurable states, which has a *polynomial growth* with a *unknown* rate (see Subsection 2.3 for details).

The main contribution of this paper is to *numerically* construct a output feedback control scheme which is *time-invariant* in nature. In this paper, there are some new features in the proposed control scheme for the practical control systems:

i) Instead of using a time-varying strategy to handle the unknown parameter θ (see [19]), we shall design a *universal-type* adaptive output feedback controller which globally regulates all the states of the uncertain system while keeping boundedness of all the signals in the adaption loop.

ii) A dynamic gain is used in the proposed universal-like output feedback controller which is motivated by the theory of *universal control* [1], [14] and the non-separation principle-based output feedback control scheme [3], [4], [5]. The observer gain is updated by an error signal between the system output and its estimate.

iii) In the existing literature, most of the global adaptive control results via output feedback are only applicable to a class of uncertain nonlinear systems in the parametric output form , for instance, (Krstić, Kanellakopoulos, and Kokotović,1995[19]), i.e., $\phi_i(t,x,u) = c \cdot b(y)$ in (6), where $b(y)$ is a smooth function of y—the output of the system. These strategies can not, however, be employed to control nonlinear systems with unknown parameters beyond the output feedback form, such as the uncertain system considered in this paper satisfying Assumption 3 in Subsection 2.3, in which the unknown parameter appears not only in the front of the system output but also in the front of the unmeasurable states.

iv) In this work, we attempt to solve the global regulation problem *numerically* with the polynomial optimization—Sum of squares (SOS) method. SOS formulations can offer a numerically tractable means of attacking the problems that lack an analytical solution. These relaxation schemes have recently been applied to various non-convex problems in control such as Lyapunov stability of nonlinear dynamic systems, robust stability analysis and robust controller synthesis [23]. Consequently, reducing a control design problem to an SOS programming can be considered as a practical solution to this problem. In this paper, it turns out that the SOS technique can be successfully applied to the problem of global stabilization via output-feedback for a family of uncertain nonlinear systems under Assumption 3. (see Subsection 2.3) with the dead-zone nonlinearity. We also consider the adaptive control problem for time-delay systems in another paper.

The rest of this paper is organized as follows. In Section 2, we present some preliminary results concerning the sum of squares decomposition and its application to solving state dependent linear polynomial inequalities. And the intuitive concept and a piece-wise description of dead-zone is introduced also. Then in Section 3, the problem of universal adaptive output control feedback for nonlinear systems with dead-zone is settled under Assumption 1,2,3 and a systemic numerical construction is developed by using the sum of squares approach introduced in Section 2. A numerical example is presented to illustrate the proposed method in Section 4, and finally the paper is ended by some conclusions in Section 5.

II. Preliminaries and problem statement

A. SOS relaxation in polynomial optimization

Definition 2.1: A monomial m_α in n variables is a function as $m_\alpha(x) = x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for $\alpha \in Z_+^n$. The degree of a monomial is defined, $degm_\alpha := \sum_{i=1}^n \alpha_i$.

Definition 2.2: A polynomial f in n variables is a finite linear combination of monomials,

$$f := \sum_\alpha c_\alpha m_\alpha = \sum_\alpha x_\alpha$$

with $c_\alpha \in R$. Define R_n to the set of all polynomials in n variables. The degree of f is defined as $degf := max(degm_\alpha)$.

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For a multivariate polynomial, it is not easy to determine whether it is positive semidefinite (PSD) or not. However, if it can be decomposed as a sum of squares (SOS) of several polynomials, that is

$$f(x) = \sum_{i=1}^{m} f_i^2(x) \tag{1}$$

then it clearly implies $f(x) > 0$ for any $x \in R$. It provides a sufficient condition for nonnegativity of a multivariate polynomial.

The SOS condition (1) is equivalent to the existence of a positive definite matrix Q, such that

$$f(x) = Z(x)^T QZ(x) \tag{2}$$

where $Z(x)$ is some properly chosen vector of monomials. It can be reformed as a feasibility problem of convex optimization to determine whether a multivariate polynomials can be decomposed as an SOS form or not.

Proposition 2.3. Let $F(x)$ be an $N \times N$ symmetric polynomial matrix of degree $2d$ in $x \in R^n$, furthermore, let $Z(x)$ be a column vector whose entries are all monomials in x with degree no greater than d, and consider the following conditions.

(1) $F(x) \geq 0$ *for all* $x \in R^n$.
(2) *There exists a real vector* $v \in R^N$ *such that* $v^T F(x) v$ *is a sum of squares.*
(3) *There exists a positive semidefinite matrix* Q *such that* $v^T F(x) v = (v \otimes Z(x))^T Q (v \otimes Z(x))$,

where $\otimes$ denotes the Kronecker product.
Then (1)$\Longleftarrow$(2) and (2) $\Longleftrightarrow$ (3).

Proof: see [23] ■

B. Dead-zone model and its intuitive properties

The non-symmetric dead-zone input nonlinearities $D(u)$ is described as follows:

$$D(u(t)) \triangleq \begin{cases} m_r(u-b_r) & \text{if } u(t) \geq b_r, \\ 0 & \text{if } -b_r < u(t) < b_r, \\ m_l(u+b_l) & \text{if } u(t) \leq -b_l. \end{cases} \tag{3}$$

The non-symmetric dead-zone input is shown in Figure 1. The parameters m_r and m_l stand for the right and the left slope of the dead-zone characteristic. b_r and b_l represent the breakpoints of the input nonlinearity. In this section, the following assumptions are considered.

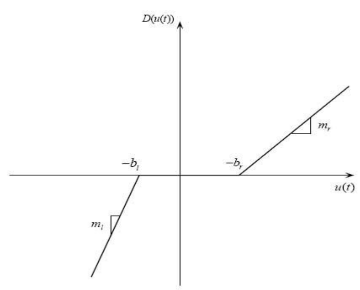

Fig. 1. Non-symmetric dead-zone nonlinearity

Assumption 1. The coefficients m_r, m_l , b_l and b_r are strictly positive and unknown.

Assumption 2. The maximum and the minimum values of the characteristic slopes and the breakpoints are known: $\max\{m_l, m_r\} = \bar{m}$, $\min\{m_l, m_r\} = \underline{m}$; $\max\{b_l, b_r\} = \bar{b}$, $\min\{b_l, b_r\} = \underline{b}$.

Assumption 1 and 2 are not restrictive conditions, since a priori knowledge of the upper bounds and the lower bounds of the slopes and breakpoints seems to be a natural assumption in engineering practice. According to the above notation, the dead-zone (3) can be redefined as a slowly time-varying input-dependent function of the following form:

$$D(u) = m(t)u + d(t), \tag{4}$$

where

$$m(t) \triangleq \begin{cases} m_l & \text{if } u \leq 0, \\ m_r & \text{if } u \geq 0 \end{cases}$$

and

$$d(t) \triangleq \begin{cases} -m_l b_r & \text{if } u \geq b_r, \\ -m(t)u & \text{if } -b_l < u < b_r, \\ m_l b_l & \text{if } u \leq -b_r. \end{cases}$$

Remark 2.1 From Assumption 1 and 2, one can conclude that $d(t)$ is bounded, and satisfies

$$\underline{d} = \underline{mb} \leq |d(t)| \leq \bar{m}\bar{b} = \bar{d}.$$

C. Control problem statement

In this paper, the problem of global states regulation is considered by output feedback for a family of single-input single-output (SISO) uncertain nonlinear systems without zero dynamics subject to the non-symmetric dead-zone input nonlinearity (4):

$$\begin{aligned} \dot{x}_1 &= x_2 + \phi_1(t,x,u), \\ \dot{x}_2 &= x_3 + \phi_2(t,x,u), \\ &\vdots \\ \dot{x}_n &= D(u) + \phi_n(t,x,u), \\ y &= x_1, \end{aligned} \tag{5}$$

where $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the system state, input and output, respectively. The functions $\phi_i : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, n$, are C^1 with respect to all the variables. Here ϕ_i are introduced to represent the system uncertainty, so they need not to be precisely known. Throughout this paper, we focus our attention on a sub-family of uncertain nonlinear systems (1) characterized by the following *unknown polynomial growth condition* in states.

Assumption 3. There is an *unknown* constant $c \geq 0$ such that

$$|\phi_i(t,x,u)| \leq \quad c \cdot p_i(x_1, \cdots, x_i), \quad i = 1, \ \cdots, \ n \tag{6}$$

where $p_i(x_1, \cdots, x_i)$ for $i = 1, \cdots, n$ are positive definite polynomials composed of all monomials of $|x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_i^{\alpha_i}|$, where α_i satisfies $\sum_{j=1}^{i} j\alpha_j \leq i, \ \alpha_j \in \overline{Z^-}$ when $1 \leq i \leq n$.

Remark 2.2 For explanation, $|\phi_3(t,x,u)| \leq c \cdot p_3(x_1,x_2,x_3) = c \cdot (|x_1^3| + |x_1^2| + |x_1 x_2| + |x_1| + |x_2| + |x_3|)$. So this condition can be seen as an extension of the *linear growth condition* in [11] under which $p_i(x_1, \cdots, x_i) = |x_1| + \cdots + |x_i|, i = 1, \cdots, n.$

Remark 2.3 It can be seen that the unknown parameters appear not only in the front of the system output y but also in the front of the unmeasurable states $(x_2, \cdots, x_n)$. This makes the conventional observer design method difficult, because the constructed observer contains a copy of the original system with unknown parameters and hence the conventional method is not implementable.

To be precise, the main contribution of this paper is that under Assumption 1,2,3, a systemic numerical construction by sum of squares approach is developed to design a C^1 universal output controller of the form (see Theorem 1.):

$$\dot{\hat{x}} = f(\hat{x}, L, y), \quad \hat{x} \in \mathbb{R}^n,$$
$$\dot{L} = h(\hat{x}, L, y), \quad u = g(\hat{x}, L, y), \tag{7}$$

such that for every $(x(0), \hat{x}(0)) \in \mathbb{R}^n \times \mathbb{R}^n$ and a fixed $L(0) > 0$, the closed-loop system (5)-(7) has a unique solution which exists on $[0, +\infty]$ and is bounded. Moreover,

$$\lim_{t\to+\infty}(x(t), \hat{x}(t)) = (0,0), \quad \lim_{t\to+\infty} L(t) = \bar{L} \in R_+.$$

In this paper, we develop an output feedback control scheme which is *time − invariant in nature*. Instead of using a time-varying strategy to handle the unknown parameter c, a universal-type adaptive output feedback controller which globally regulates all the states of the uncertain system (5) is designed. In [11], the authors firstly refer such an adaptive control strategy as *universal adaptive regulation by output feedback*.

III. Universal adaptive output feedback control and its SOS optimization algorithm

In this section, we will prove that without knowing the bound of the growth rate c in (6), it is still possible to globally regulate the whole family of uncertain system (5) subject to unknown dead-zone nonlinearity (4) by a universal-type output feedback controller, and such controller can be numerically constructed by sum of squares approach. Formally, the main result of this paper is summarized as below.
Denote

$$D = diag\{1,2,\cdots,n\}, \varepsilon = (\varepsilon_1,\cdots,\varepsilon_n)^T, z = (z_1,\cdots,z_n)^T,$$
$$\Phi(\cdot) = \Big[\frac{1}{L}\phi_1(t,x,u), \frac{1}{L^2}\phi_2(t,x,u), \cdots, \frac{1}{L^n}\phi_n(t,x,u)\Big]^T,$$
$$P(\cdot) = \Big[\frac{1}{L}p_1(x_1), \frac{1}{L^2}p_2(x_1,x_2), \cdots, \frac{1}{L^n}p_n(x_1,\cdots,x_n)\Big]^T,$$
and
$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \overrightarrow{a} = (a_1,\cdots,a_n)^T,$$
$$\overrightarrow{b} = [0,\cdots,1]^T, m^* = \bar{m} + 1.$$

Theorem 1. *For the family of uncertain system (5) satisfying Assumption 1, if the following inequalities hold*

$$\Psi_1(\varepsilon) \ge \|\varepsilon\|^2, \Psi_2(z) \ge \|z\|^2, V_\varepsilon(\varepsilon) \ge \lambda_\varepsilon\|\varepsilon\|^2, V_z(z) \ge \lambda_z\|z\|^2,$$
$$\lambda_\varepsilon > 0, \lambda_z > 0, \tag{8}$$
$$\frac{\partial V_\varepsilon}{\partial \varepsilon}D\varepsilon \ge 0, \frac{\partial V_\varepsilon}{\partial \varepsilon}A\varepsilon + |\frac{\partial V_\varepsilon}{\partial \varepsilon}\overrightarrow{a}\varepsilon_1)| \le -(\bar{d}+2)\Psi_1(\varepsilon), \tag{9}$$
$$|\frac{\partial V_\varepsilon}{\partial \varepsilon}P(\varepsilon,z)| \le \Psi_1(\varepsilon) + \Psi_2(z),$$
$$\frac{\partial V_\varepsilon}{\partial \varepsilon}\overrightarrow{b}\big((m(t)-1)U(z) + \frac{d(t)}{L^{n+1}}\big) \le \bar{d}\Psi_1(\varepsilon) + m^*\Psi_2(z),$$
$$\frac{\partial V_z}{\partial z}Dz \ge 0, \frac{\partial V_z}{\partial z}(Az + \overrightarrow{b}U(z)) \le -(m^*+2)\Psi_2(z),$$
$$\frac{\partial V_z}{\partial z}\overrightarrow{a}\varepsilon_1 \le \varepsilon_1^2 + \Psi_2(z), \tag{10}$$

then the problem of universal adaptive regulation for the uncertain nonlinear system (5) is solvable by a dynamic output compensator of the form (7), in particular, by

$$\dot{\hat{x}}_1 = \hat{x}_2 + La_1(y - \hat{x}_1),$$
$$\dot{\hat{x}}_2 = \hat{x}_3 + L^2a_2(y - \hat{x}_1),$$
$$\vdots$$
$$\dot{\hat{x}}_n = u + L^na_n(y - \hat{x}_1), \tag{11}$$
$$\dot{L} = \frac{1}{L^2}(y - \hat{x}_1)^2 \quad with \quad L(0) = 1, \tag{12}$$
$$u = U(L, \hat{x})$$
$$= L^nQ_1(\hat{x}_1) + L^{n-1}Q_2(\hat{x}_1,\hat{x}_2) + \cdots + LQ_n(\hat{x}_1,\cdots,\hat{x}_n) \tag{13}$$

where $V_\varepsilon(\varepsilon)$, $V_z(z)$, $\Psi_1(\varepsilon)$, $\Psi_2(z)$ and are polynomial functions in the corresponding variables. $Q_i(\hat{x}_1,\cdots,\hat{x}_i)$ are polynomials made up of all monomials of $\hat{x}_1^{\alpha_1}\hat{x}_2^{\alpha_2}\cdots\hat{x}_i^{\alpha_i}$, where α_j satisfies $\sum_{j=1}^i j\alpha_j = i$, $i = 1,\cdots,n$. $U(z) = Q_1(z_1) + \cdots + Q_n(z_1,\cdots,z_n)$, λ_ε, λ_z and $a_1,\cdots,a_n$ are scalers.

Proof: Let $e_i = x_i - \hat{x}_i$ be the estimation error. Then, the error dynamics is given by

$$\dot{e}_1 = e_2 - La_1e_1 + \phi_1(t,x,u),$$
$$\dot{e}_2 = e_3 - L^2a_2e_1 + \phi_2(t,x,u),$$
$$\vdots$$
$$\dot{e}_n = \big(D(u) - u\big) - L^na_ne_1 + \phi_n(t,x,u). \tag{14}$$

With the help of the following rescaleing transformation ($i = 1,\cdots,n$)

$$\varepsilon_i = \frac{e_i}{L^i} \quad \text{and} \quad z_i = \frac{\hat{x}_i}{L^i}, \tag{15}$$

the closed-loop system can be represented in the following compact form:

$$\dot{\varepsilon} = LA\varepsilon + L\overrightarrow{b}\big((m(t)-1)U(z) + d(t)\big) - L\overrightarrow{a}\varepsilon_1 + \Phi(t,x,u,L) - \frac{\dot{L}}{L}D\varepsilon,$$
$$\dot{z} = LAz + L\overrightarrow{b}U(z) + L\overrightarrow{a}\varepsilon_1 - \frac{\dot{L}}{L}Dz, \tag{16}$$

Observe that by construction, $\dot{L} \ge 0$ and hence $L \ge L(0) = 1$. As a result, the vector fields of the closed-loop system (16)-(12) are C^1 with its variables. Thus, the differential equations (16) satisfy a local Lipschitz condition in a neighborhood of the corresponding solution $(\varepsilon(0), z(0), L(0)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$, and the corresponding solution $(\varepsilon(0), z(0), L(0))$ of the system (16)-(12) exists and is unique on $[0, T_f)$ for some $T_f \in (0, +\infty]$. Without loss of generality, we assume from now on that $(0, T_f]$ is the maximally extended interval of the solution of (16)-(12). This fact will be used in the rest of the proof.

Choose the polynomial Lyapunov function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$V(\varepsilon, z) = V_\varepsilon(\varepsilon) + V_z(z) \tag{17}$$

for the nonlinear system (5). Then, the derivative of V along the solution of (16) is given by

$$\dot{V}_\varepsilon = \frac{\partial V_\varepsilon}{\partial \varepsilon}\Big(LA\varepsilon - L\overrightarrow{a}\varepsilon_1 + L\overrightarrow{b}\big((m(t)-1)U(z) + \frac{d(t)}{L^{n+1}}\big) + \Phi - \frac{\dot{L}}{L}D\varepsilon\Big)$$
$$\dot{V}_z = \frac{\partial V_z}{\partial z}(LAz + L\overrightarrow{b}U(z) + L\overrightarrow{a}\varepsilon_1 - \frac{\dot{L}}{L}Dz)$$
$$\dot{V} = L\frac{\partial V_\varepsilon}{\partial \varepsilon}(A\varepsilon - \overrightarrow{a}\varepsilon_1) + \frac{\partial V_\varepsilon}{\partial \varepsilon}\Phi + L\frac{\partial V_z}{\partial z}(Az + \overrightarrow{a}\varepsilon_1 + \overrightarrow{b}U(z))$$
$$-\frac{\dot{L}}{L}(\frac{\partial V_\varepsilon}{\partial \varepsilon}D\varepsilon + \frac{\partial V_z}{\partial z}Dz) + L\frac{\partial V_\varepsilon}{\partial \varepsilon}\overrightarrow{b}\big((m(t)-1)U(z) + \frac{d(t)}{L^{n+1}}\big)$$

In addition, observe that

$$\Big|\frac{\phi_i(\cdot)}{L^i}\Big| \le \frac{c}{L^i} p_i(x_1,\cdots,x_i) = \frac{c}{L^i} p_i(e_1+\hat{x}_1,\cdots,e_i+\hat{x}_i) \\ \le c p_i(\varepsilon, z)$$

From (8), (9), (10), we can see that

$$\begin{aligned}\dot{V} &\le -(\bar{d}+2)L\Psi_1(\varepsilon) + c(\Psi_1(\varepsilon)+\Psi_2(z)) \\ &\quad + L\big(\bar{d}\Psi_1(\varepsilon)+m^*\Psi_2(z)\big) \\ &\quad - L(m^*+2)\Psi_2(z) + L(\Psi_1(\varepsilon)+\Psi_2(z)) \\ &\le -(L-c)(\Psi_1(\varepsilon)+\Psi_2(z)) \qquad (18)\end{aligned}$$

With the help of (18), it can be proved that Theorem 1 holds. That is, starting from any initial condition $(\varepsilon(0),z(0)) \in \mathbb{R}^n \times \mathbb{R}^n$ and $L(0)=1$, the closed-loop system (16) with (12) has the following two properties:

(i) The corresponding solution$(\varepsilon(t),z(t),L(t))$ of (16)-(12) exists on$[0,+\infty)$, and is unique;

(ii) $\lim_{t\to+\infty}(z(t),\varepsilon(t)) = 0$, $\lim_{t\to+\infty} L(t) = \bar{L} \in R_+$.

Recall that the solution $(\varepsilon(t),z(t),L(t))$ of (16)-(12) exists and is unique on the maximally extended interval $[0,+T_f)$. Thus, conclusion (i) and (ii) follow immediately if one can prove that $\varepsilon(t), z(t)$ and $L(t)$ are bounded on $[0,+\infty)$. This can be done by a contradiction argument.

Suppose

$$\lim_{t\to T_f} sup\|(L(t),\varepsilon(t),z(t))^T\| = +\infty$$

We first claim that $L(t)$ cannot escape at $t=T_f$. To prove this claim, suppose that $\lim_{t\to T_f} L(t) = +\infty$. Since $\dot{L} = \varepsilon_1^2 \ge 0$, $L(t)$ is a monotone non-decreasing function. Thus, there exists a finite time $t^* \in (0,T_f)$, such that

$$L(t) \ge c, \quad \text{when} \quad t^* \le t \le T_f.$$

From (18), it follows that

$$\dot{V}(\eta(t)) \le -[\Psi_1(\varepsilon)+\Psi_2(z)] = -\Psi(\eta(t)), \quad \forall t \in [t^*, T_f), \\ \text{where} \quad \eta(t) = (\varepsilon(t), z(t))^T.$$

As a consequence,

$$\int_{t^*}^{T_f} \varepsilon_1^2 dt \le \int_{t^*}^{T_f} \Psi(\eta(t))dt \le V(\eta(t^*)) = \text{constant}. \qquad (19)$$

Using (19), one has

$$\begin{aligned}+\infty = L(T_f) - L(t^*) &= \int_{t^*}^{T_f} \dot{L}(t)dt \qquad (20) \\ &= \int_{t^*}^{T_f} \varepsilon_1^2(t)dt \le V(\eta(t^*)) = \text{constant}\end{aligned}$$

which leads to a contradiction. Thus, the dynamic gain L is well defined and bounded on $[0,T_f)$. From $\dot{L} = \varepsilon_1^2$, it is concluded that $t \mapsto \int_0^t \varepsilon_1^2 dt$ is bounded on $[0,T_f)$.

Next, we claim that z is well defined and bounded on the interval $[0,T_f)$. To see why, consider the Lyapunov function $V_2(z)$ for the z-dynamic system of (16). Clearly, a direct computation gives

$$\begin{aligned}\dot{V}_z(z) &\le L\frac{\partial V_z}{\partial z}(Az + \overrightarrow{b}U(z)) + L\frac{\partial V_z}{\partial z}\overrightarrow{a}\varepsilon_1 \\ &\le -(m^*+1)L\Psi_2(z) + L\varepsilon_1^2 \\ &\le -(m^*+1)\Psi_2(z) + L\dot{L}.\end{aligned}$$

Thus, from (8) in turn, leads to $\forall t \in [0,T_f)$,

$$\begin{aligned}\lambda_z\|z(t)\|^2 - V_z(z(0)) &\le V_z(z(t)) - V_z(z(0)) \\ &\le \frac{1}{2}[L^2(t)-1] - (m^*+1)\int_0^t \Psi_2(z(t))dt\end{aligned}$$

from which it follows that $\forall t \in [0,T_f)$,

$$\begin{aligned}&\|z(t)\|^2 \le V_z(z(0)) + \frac{1}{2\lambda_\varepsilon}[L^2(t)-1], \\ &\int_0^t \|z(t)\|^2 dt \le \int_0^t \Psi_2(z(t))dt \\ &\qquad \le \frac{1}{m^*+1}\Big(\frac{1}{2}[L^2(t)-1] + V_z(z(0))\Big).\end{aligned}$$

Since L is bounded on $[0,T_f)$, the inequalities above imply the boundedness of z , $t \mapsto \int_0^t \Psi_2(z(t))dt$ on $[0,T_f)$ and $t \mapsto \int_0^t \|z(t)\|^2 dt$ on $[0,T_f)$.

Finally, we prove that ε is bounded on $[0,T_f)$. To this end, we introduce the change of coordinates

$$\xi_i = \frac{e_i}{L^{*i}}, \qquad i = 1,\cdots,n, \qquad (21)$$

where L^* is a constant satisfying $L^* = max\{L(T_f), c+1\}$.

Then, the error dynamics (14) is transformed into

$$\begin{aligned}\dot{\xi}_1 &= L^*\xi_2 - La_1\xi_1 + \frac{\phi_1(\cdot)}{L^*}, \\ \dot{\xi}_2 &= L^*\xi_3 - L(\frac{L}{L^*})a_2\xi_1 + \frac{\phi_2(\cdot)}{L^{*2}}, \\ &\vdots \\ \dot{\xi}_n &= L^*\Big(\big(m(t)-1\big)(\frac{L}{L^*})^{n+1}U(z) + \frac{d(t)}{L^{*n+1}}\Big) - L(\frac{L}{L^*})^{n-1} a_n\xi_1 \\ &\qquad + \frac{\phi_n(\cdot)}{L^{*n}},\end{aligned}$$

which can be written in the following compact form:

$$\begin{aligned}\dot{\xi} = L^*A\xi + L^*\overrightarrow{b}\Big(\big(m(t)-1\big)(\frac{L}{L^*})^{n+1}U(z) + \frac{d(t)}{L^{*n+1}}\Big) \\ - L\Gamma\overrightarrow{a}\xi_1 + \Phi^*(\cdot) \qquad (22)\end{aligned}$$

where $\Gamma = diag\{1, L/L^*, \cdots, (L/L^*)^{n-1}\}$ and $\Phi^*(\cdot) = \Big[\frac{\phi_1(t,x,u)}{L^*}, \frac{\phi_2(t,x,u)}{(L^*)^2}, \cdots, \frac{\phi_n(t,x,u)}{(L^*)^n}\Big]^T$. Now, consider the Lyapunov function $V_\xi(\xi) = V_\varepsilon(\xi)$ for system (22). A straightforward calculation shows that along the trajectories of (22),

$$\begin{aligned}\dot{V}_\xi(\xi) &= \frac{\partial V_\xi}{\partial \xi}[L^*A\xi + L^*\overrightarrow{b}\Big(\big(m(t)-1\big)U(z) + \frac{d(t)}{L^{*n+1}}\Big) \\ &\qquad - L\Gamma\overrightarrow{a}\xi_1 + \Phi^*] \\ &= L^*\frac{\partial V_\xi}{\partial \xi}\big[A\xi + \overrightarrow{b}\Big(\big(m(t)-1\big)(\frac{L}{L^*})^{n+1}U(z) + \frac{d(t)}{L^{*n+1}}\Big) \\ &\qquad - (\frac{L}{L^*})\Gamma\overrightarrow{a}\xi_1\big] + \frac{\partial V_\xi}{\partial \xi}\Phi^*. \qquad (23)\end{aligned}$$

With the help of (9), and considering the convexity of $\frac{L}{L^*}$ and $\frac{L}{L^*}\Gamma$ when $\frac{L}{L^*} \in [0,1]$ in (23), we can get

$$\begin{aligned}\dot{V}_\xi &\le -(\bar{d}+2)L^*\Psi_1(\xi) + c(\Psi_1(\xi)+\Psi_2(z)) \\ &\qquad + L^*(\bar{d}\Psi_1(\xi) + m^*\Psi_2(z)) \\ &\le -(L^*-c)\Psi_1(\xi) + (m^*L^*+c)\Psi_2(z) \\ &\le -\Psi_1(\xi) + (m^*L^*+c)\Psi_2(z) \qquad (24)\end{aligned}$$

From (24),(8), it follows that $\forall t \in [t, T_f)$,

$$\lambda_\varepsilon \|\xi(t)\| \le V_\varepsilon(\xi(0)) + c\int_0^t \Psi_2(z(t))dt \tag{25}$$

$$\begin{aligned}\int_0^t \|\xi(t)\|^2 dt &\le \int_0^t \Psi_1(\xi(t))dt \\ &\le \Big(V_\xi(\xi(0)) + (m^*L^*+c)\int_0^t \Psi_2(z(t))dt\Big)\end{aligned} \tag{26}$$

Since $t \mapsto \int_0^t \Psi_2(z(t))dt$ is bounded on $[0,T_f)$, it is concluded immediately from (25),(26) that $t \mapsto \int_0^t \Psi_1(\xi(t))dt$ and $t \mapsto \int_0^t \|\xi(t)\|^2 dt$ is bounded on $[0,T_f)$. This, in view of (21) and (15), results in the boundedness of $t \mapsto \int_0^t \|\varepsilon(t)\|^2 dt$ and ε on $[0,T_f)$.

In summary, we have shown that L, ε and z are well defined and all bounded on the maximally extended interval $[0,T_f)$. The conclusion is certainly contradictory to the assumption that $\lim_{t\to T_f} sup\|(L(t),\varepsilon(t),z(t))\| = +\infty$. In other words, $T_f = +\infty$ as T_f is maximal. Consequently, all the states of the closed-loop system are well defined and bounded on $[0,+\infty)$. This, together with (28), implies that $\int_0^{+\infty}\|z\|^2 dt < +\infty$. Similarly, it follows from the relation $\dot L = \varepsilon_1^2$ that $\int_0^{+\infty}\|\varepsilon_1\|^2 dt < +\infty$. Finally, with the help of (26), (21), (15), it is not difficult to conclude that $\int_0^{+\infty}\|\varepsilon\|^2 dt$ is bounded as well.

On the other hand, using the boundedness of (L,ε,z) on $[0,+\infty)$, it is straightforward to deduce that $\dot\varepsilon \in L_\infty$ and $\dot z \in L_\infty$. This, together with the properties of $\varepsilon \in L_2$ and $z \in L_2$ yields (by the Barbalat's Lemma)

$$\lim_{t\to+\infty} z(t) = 0 \text{ and } \lim_{t\to+\infty}\varepsilon(t) = 0. \tag{27}$$

■

In the following lemma, an SOS relaxation of the problem of universal adaptive regulation for the uncertain nonlinear system (5) with dead-zone input (4) under Assumption 1,2,3 is formulated.

Lemma 1. *Suppose that the following SOS programme is solvable, then the solution also satisfies the inequalities (8),(9) and (10) in Theorem 1.*

$$\begin{aligned}&\Psi_1(\varepsilon) - \|\varepsilon\|^2 \textit{ and } \Psi_2(z) - \|z\|^2 \textit{ are SOS polynomials,}\\ &V_\varepsilon(\varepsilon) - \lambda_\varepsilon\|\varepsilon\|^2 \textit{ and } V_z(z) - \lambda_z\|z\|^2 \textit{ are SOS polynomials,}\\ &\lambda_\varepsilon, \lambda_z > 0\end{aligned} \tag{28}$$

$$\frac{\partial V_\varepsilon}{\partial \varepsilon} D\varepsilon \textit{ and } \frac{\partial V_z}{\partial z} Dz \textit{ are SOS polynomials,} \tag{29}$$

$$\begin{bmatrix} I_3 & \mathbf{0} & -\frac{\partial V_\varepsilon}{\partial \varepsilon}^T \\ \mathbf{0} & I_3 & -\vec{a}\,\varepsilon_1 \\ -\frac{\partial V_\varepsilon}{\partial \varepsilon} & -\vec{a}^T\varepsilon_1 & -2\frac{\partial V_\varepsilon}{\partial \varepsilon}A\varepsilon - 2(\bar d+2)\Psi_1(\varepsilon)\end{bmatrix} \textit{ is an SOS matrix,} \tag{30}$$

$$\begin{bmatrix} I_3 & \mathbf{0} & -\frac{\partial V_z}{\partial z}^T \\ \mathbf{0} & I_3 & -\vec{b}\,U(z) \\ -\frac{\partial V_z}{\partial z} & -\vec{b}^T U(z) & -2\frac{\partial V_z}{\partial z}Az - 2(m^*+2)\Psi_2(z)\end{bmatrix} \textit{ is an SOS matrix,} \tag{31}$$

$$\begin{bmatrix} I_3 & \mathbf{0} & -\frac{\partial V_\varepsilon}{\partial \varepsilon}^T \\ \mathbf{0} & I_3 & -m^*\vec{b}\,U(z) \\ -\frac{\partial V_\varepsilon}{\partial \varepsilon} & -m^*\vec{b}^T U(z) & \bar d\Psi_1 + m^*\Psi_2 - \underline{d}\frac{\partial V_\varepsilon}{\partial \varepsilon}\vec{b}\end{bmatrix} \textit{ is an SOS matrix,} \tag{32}$$

$$\begin{bmatrix} I_3 & \mathbf{0} & -\frac{\partial V_\varepsilon}{\partial \varepsilon}^T \\ \mathbf{0} & I_3 & -m^*\vec{b}\,U(z) \\ -\frac{\partial V_\varepsilon}{\partial \varepsilon} & -m^*\vec{b}^T U(z) & \bar d\Psi_1 + m^*\Psi_2 - \bar d\frac{\partial V_\varepsilon}{\partial \varepsilon}\vec{b}\end{bmatrix} \textit{ is an SOS matrix,} \tag{33}$$

$$\begin{bmatrix} I_3 & -\frac{\partial V_\varepsilon}{\partial \varepsilon}^T \\ -\frac{\partial V_\varepsilon}{\partial \varepsilon} & 2\Psi_1 + 2\Psi_2 - \overline{P}(\varepsilon,z)^T\overline{P}(\varepsilon,z)\end{bmatrix} \textit{ is an SOS matrix,} \tag{34}$$

$$\begin{bmatrix} I_3 & \mathbf{0} & -\frac{\partial V_z}{\partial z}^T \\ \mathbf{0} & I_3 & -\vec{a}\,\varepsilon_1 \\ -\frac{\partial V_z}{\partial z} & -\vec{a}^T\varepsilon_1 & 2\varepsilon_1^2 + 2\Psi_2(z)\end{bmatrix} \textit{ is an SOS matrix,} \tag{35}$$

where the entry $\overline{p_i}(\varepsilon,z)$ of $\overline{P}(\varepsilon,z)$ is a positive definite polynomial which is lager than the corresponding entry $p_i(\varepsilon,z)$ of $P(\varepsilon,z)$.

Proof: By the definition of the SOS decomposition of polynomial, (8) is obviously implied by (28). By the Schur's complement, (30),(31),(34) and (35) imply that

$$2\frac{\partial V_\varepsilon}{\partial \varepsilon}A\varepsilon + 2(\bar d+2)\Psi_1(\varepsilon) + [\frac{\partial V_\varepsilon}{\partial \varepsilon}\quad \vec{a}^T\varepsilon_1][\frac{\partial V_\varepsilon}{\partial \varepsilon}\quad \vec{a}^T\varepsilon_1]^T \le 0$$

$$\begin{aligned}&2\frac{\partial V_z}{\partial z}Az + 2(m^*+2)\Psi_2(z)\\ &\qquad + [\frac{\partial V_z}{\partial z}\quad \vec{b}^T U(z)][\frac{\partial V_z}{\partial z}\quad \vec{b}^T U(z)]^T \le 0\end{aligned}$$

$$\begin{aligned}&-2\bar d\Psi_1 - 2m^*\Psi_2(z) + 2\underline{d}\frac{\partial V_\varepsilon}{\partial \varepsilon}\vec{b}\\ &\qquad + [\frac{\partial V_z}{\partial z}\quad m^*\vec{b}^T U(z)][\frac{\partial V_z}{\partial z}\quad m^*\vec{b}^T U(z)]^T \le 0\end{aligned}$$

$$\begin{aligned}&-2\bar d\Psi_1 - 2m^*\Psi_2(z) + 2\bar d\frac{\partial V_\varepsilon}{\partial \varepsilon}\vec{b}\\ &\qquad + [\frac{\partial V_z}{\partial z}\quad m^*\vec{b}^T U(z)][\frac{\partial V_z}{\partial z}\quad m^*\vec{b}^T U(z)]^T \le 0\end{aligned}$$

$$\begin{aligned}&-2\Psi_1(\varepsilon) - 2\Psi_2(z) + P^T(\varepsilon,z)P(\varepsilon,z) + \frac{\partial V_\varepsilon}{\partial \varepsilon}\frac{\partial V_\varepsilon}{\partial \varepsilon}^T\\ &\le -2\Psi_1(\varepsilon) - 2\Psi_2(z) + \overline{P}^T(\varepsilon,z)\overline{P}(\varepsilon,z) + \frac{\partial V_\varepsilon}{\partial \varepsilon}\frac{\partial V_\varepsilon}{\partial \varepsilon}^T \le 0\end{aligned}$$

$$-2\varepsilon_1^2 - 2\Psi_2(z) + [\frac{\partial V_z}{\partial z}\quad \vec{a}^T\varepsilon_1][\frac{\partial V_z}{\partial z}\quad \vec{a}^T\varepsilon_1]^T \le 0.$$

Considering $\pm 2M^TN \le M^TM + N^TN,\ M,N \in \mathbb{R}^n$, then (8), (9) and (10) hold. ■

The formulated linear polynomial matrix inequalities (LPMI) in Lemma 1. can be tested algorithmically using the SOS decomposition, as explained in Section II. Here we use the scalarization approach to formulate an SOS programme (LMI conditions taking explicitly into account the matrix structure can be used alternatively to reduce the complexity [2], [6]), and the resulting SOS programme can be solved algorithmically with the aid of SOSTOOLS or GloptiPoly.

IV. Example and Simulation

In the section, we present an example to illustrate the application of Theorem 1 and the corresponding SOS algorithm. A mass-spring system with a softening spring, linear viscous damping, and an external force F with unknown dead-zone nonlinearity $\mathrm{D}(\cdot)$ can be represented by the Duffing's equation, shown in Figure 1. (see, for instance, Khalil (2002, p. 8)),

$$m\ddot y + \mu\dot y + ky - a^2y^3 = D(F)$$

where m is the quality of the mass, k is the spring constant, μ is viscous friction coefficient, they are all unknown system parameters. The restoring force of the *softening spring* is modeled in

$$g(y) = k(1-a^2y^2)y, \quad |ay| < 1 \tag{36}$$

where, beyond a certain displacement, a large displacement produces a small force increment.

With $x_1 = y$ and $x_2 = \dot{y}$, the state model is built in

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -\frac{k}{m}x_1 - \frac{\mu}{m}x_2 - \frac{k}{m}(1 - a^2x_1^2)x_1 + D(F) \quad (37)$$

Our control goal is to adaptively regulate the stats by output feedback, i.e., using only the output signal x_1 that represents the displacement of the mass when the parameters m, k, μ are unknown. A simple analysis indicates that the mass-spring system (37) does

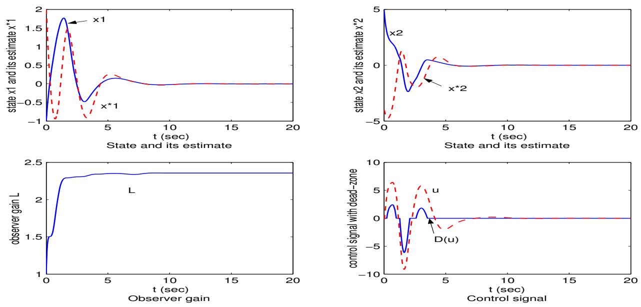

Fig. 2. The transient-response of the closed-loop system (37)-(38)

satisfy the *unknown polynomial growth condition* in Assumption 3. As a matter of fact, from (36) the following estimation

$$|-\frac{k}{m}x_1 - \frac{\mu}{m}x_2 - \frac{k}{m}(1-a^2x_1^2)x_1| \le \frac{2k}{m}|x_1| + \frac{\mu}{m}|x_2| + \frac{ak}{m}|x_1^2|$$

can be easily obtained. Hence, Assumption 3 holds with $c = \max\{\frac{2k}{m}, \frac{\mu}{m}, \frac{ak}{m}\}$, where c is an unknown constant. By Theorem 1, with the help of the software SOSTOOLS or GloptiPoly, we can construct a universal output feedback controller of form (11)-(15)-(13)-(12) such that all the states of the nonlinearly parameterized system (37) are globally regulated.

Following the aforementioned design procedure by SOS algorithm in Lemma 1, a universal output feedback controller is found as follows:

$$\dot{\hat{x}}_1 = \hat{x}_2 + L(y - \hat{x}_1),$$
$$\dot{\hat{x}}_2 = u + L^2(y - \hat{x}_1),$$
$$\dot{L} = \left(\frac{y - \hat{x}_1}{L}\right)^2$$
$$u = -0.787L\hat{x}_2 - 0.897L^2\hat{x}_1 - 0.323L\hat{x}_1^2 \quad (38)$$

which does the job.

A numerical simulation is given in Figure 2., illustrating the effectiveness of the universal output feedback controller (38). The simulation is carried out with the system parameters $\frac{k}{m} = 0.5, \frac{\mu}{m} = 1, a = 0.1$, $m_l = 0.8, m_r = 1.2, b_l = 0.2, b_r = 0.5$, and the universal controller used is composed of the high-gain observer with $a = (1,1)^T$. The initial condition is $(x_1(0), x_2(0)) = (-1,5)^T$ and $(\hat{x}_1(0), \hat{x}_2(0)) = (-2,-4)^T$.

V. Conclusion

By integrating the idea of universal adaptive output feedback control and sum of squares (SOS) method, we have presented systematic methodology for a universal-type output feedback controller that achieves global state regulation. This universal output feedback control law would simultaneously regulate a whole family of nonlinear systems with unknown dead-zone nonlinearity, as long as the uncertainty of the system is dominated by a polynomially growing triangular system with unknown growth rate. It was demonstrated, by means of example and simulation, that the proposed output feedback controller can be numerically constructed and efficiently implemented.

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