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Stochastic Lyapunov Function Design using Quantization of Markov Process

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Abstract— **We propose a new procedure for designing approximate stochastic Lyapunov functions (SLFs) for nonlinear Ito stochastic systems. We approximate nonlinear stochastic Lyapunov equations (SLEs) by linear Schrödinger-like equations with Kushner's scheme of difference approximation and Alcaraz et al.'s "quantization of Markov processes." We construct time-invariant functions concerned with the solutions of Schrödinger-like equations and then obtain sufficient conditions for converting the time-invariant functions into approximate SLFs.**

I. Introduction

Lyapunov's method is useful in various cases for designing nonlinear deterministic dynamical systems. The existence of Lyapunov functions ensures that the origins of the systems are stable. We can assess the robustness and obtain better controllers for closed-loop systems with ad-hoc controllers by using Lyapunov functions (see [10], [18], [20]).

Many researchers have studied the problems related to the design of Lyapunov functions, e.g., Krasovskii [10], [18], Schultz [10], Zubov [18], and Vannelli and Vidyasagar [21]. However, these problems have yet not been resolved completely.

Lyapunov's method is also useful for designing nonlinear stochastic dynamical systems. The existence of SLFs guarantees that the origins of the systems are *stable in probability*. The basic properties of the SLFs have been studied by Bucy [2], Has'minskiĭ [9], Kushner [11], Mao [13], and so on. The problems related to the design of SLFs are also difficult to solve.

This paper proposes a new procedure for designing approximate stochastic Lyapunov functions for nonlinear Ito stochastic systems. We approximate nonlinear SLEs by using linear Schrödinger-like equations with Kushner's scheme of difference approximation (see [8], [12]) and Alcaraz et al.'s quantization of Markov processes (see [1], [17]). Further, we construct time-invariant functions concerned with the solutions of the Schrödinger-like equations and then obtain sufficient conditions for converting these functions into approximate SLFs.

Section II introduces notations, definitions, Lyapunov theory for the stochastic systems, Kushner's scheme of difference approximation, and Alcaraz et al.'s quantization of Markov processes. In Section IV, we propose a method of designing approximate SLFs by using previous works. Section VI concludes this paper.

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II. Preliminary Discussion

In this paper, $\mathbb{R}^n$ denotes an n-dimensional Euclidian space and $\mathbb{C}^n$, an n-dimensional unitary space. Let $\mathbb{R}_{>0} = (0,\infty)$, $\mathbb{R}_{\geq 0} = [0,\infty)$, $\mathbb{N}_{>0} = \{1,2,3,\ldots\}$, and $\mathbb{N}_{\geq 0} = \{0,1,2,\ldots\}$. We define A_{ij} as elements of the $n\times n$ matrix A and $\mathrm{tr}(A)$ as the trace of A. Thus,

$$\sum_{j\neq i} A_{ij} := \sum_{j=1}^{n} A_{ij} - A_{ii}, \quad \forall i \in \mathbb{N}_{>0}. \tag{1}$$

We define $\mathrm{E}(X|X_0)$ as the expectation of some event X given some other event X_0.

Let us consider the following nonlinear Ito stochastic system:

$$dx(t) = f(x(t))dt + \sigma(x(t))dw(t), \tag{2}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $f:\mathbb{R}^n\to\mathbb{R}^n$ is a continuous mapping; $\sigma(x(t))\in\mathbb{R}^n\times\mathbb{R}^d$, the diffusion coeffients of $x(t)$; and $w(t)\in\mathbb{R}^d$, the standard Wiener process. Moreover, let us assume that the origin is an isolated equilibrium point. Let

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T, \tag{3}$$

$$a(x) = \sigma(x)\cdot\sigma(x)^T = \begin{bmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{bmatrix}. \tag{4}$$

A. Definitions

We define $\mathscr{L}$ as an infinitesimal operator of (2) that satisfies

$$\mathscr{L}v(x(t)) = \lim_{h\to 0}\frac{\mathrm{E}\left[v(x(t+h))|x(t)=x_t\right] - v(x(t))}{h}, \tag{5}$$

where $x_t\in\mathbb{R}^n$, $h\in\mathbb{R}_{>0}$, and $v:\mathbb{R}^n\to\mathbb{R}$.

Definition 1 (SLF) *We assume that* $W:\mathbb{R}^n\to\mathbb{R}$ *is continuous and non-negative in subspace* Q_m *satisfying* $0\in Q_m = \{x|W(x)<m,\ m\in\mathbb{R}_{>0}\}\subset\mathbb{R}^n$. *If*

$$\mathscr{L}W(x)\leq 0,\ x\in Q_m, \tag{6}$$

the function $W(x)$ *is said to be an SLF of* (2). ■

Definition 2 (SLE) *Let* $L,V:\mathbb{R}^n\times\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$. *Then, the equation*

$$\frac{\partial V}{\partial t}(x,t) = -L(x,t) - \mathscr{L}V(x,t) \tag{7}$$

is said to be an SLE. ■

We will now define *an approximate stochastic Lyapunov equation (A-SLE)* and a master equation in order to use Kushner's difference approximation. Let

$$f_i^+(x) = \max(f_i(x),0), \quad f_i^-(x) = \max(-f_i(x),0), \tag{8}$$
$$a_{ij}^+(x) = \max(a_{ij}(x),0), \quad a_{ij}^-(x) = \max(-a_{ij}(x),0). \tag{9}$$

We define a discrete state space as

$$M_d := \left\{\delta \sum_{i=1}^{n} \gamma_i e_i,\ \forall \gamma_i \in \mathbb{Z}\right\} = \{x_d, x_d', x_d'', \ldots\}, \tag{10}$$

where δ is a spatial step and $e_1, e_2, \ldots, e_n$ are the orthogonal bases of the state vector. Let us consider the first-order difference quotients

$$\mathscr{D}_i^+ v(x_d,t) := \frac{v(x_d+\delta_i e_i,t) - v(x_d,t)}{\delta_i}, \tag{11}$$
$$\mathscr{D}_i^- v(x_d,t) := \frac{v(x_d,t) - v(x_d-\delta_i e_i,t)}{\delta_i}, \tag{12}$$

and the second-order difference quotients

$$\mathscr{D}_{ii} v(x_d,t) := \frac{1}{\delta_i}(\mathscr{D}_i^+ v(x_d,t) - \mathscr{D}_i^- v(x_d,t)), \tag{13}$$
$$\begin{aligned}\mathscr{D}_{ij}^+ v(x_d,t) := \frac{1}{2\delta_j}\{&-\mathscr{D}_i^+ v(x_d,t) + \mathscr{D}_i^- v(x_d,t)\\ &+ \mathscr{D}_i^+ v(x_d+\delta_j e_j,t) - \mathscr{D}_i^- v(x_d-\delta_j e_j,t)\},\end{aligned} \tag{14}$$
$$\begin{aligned}\mathscr{D}_{ij}^- v(x_d,t) := \frac{1}{2\delta_j}\{&\mathscr{D}_i^+ v(x_d,t) - \mathscr{D}_i^- v(x_d,t)\\ &- \mathscr{D}_i^+ v(x_d-\delta_j e_j,t) + \mathscr{D}_i^- v(x_d+\delta_j e_j,t)\},\end{aligned} \tag{15}$$

where $v : M_d \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $i \neq j$. Let

$$\begin{aligned}\mathscr{L}_d(\cdot) := &\sum_{i=1}^{n}\left\{f_i^+(x_d)\mathscr{D}_i^+(\cdot) - f_i^-(x_d)\mathscr{D}_i^-(\cdot)\right\}\\ &+ \frac{1}{2}\sum_{i=1}^{n}\Big[a_{ii}(x_d)\mathscr{D}_{ii}(\cdot)\\ &+ \sum_{i\neq j}\left\{a_{ij}^+(x_d)\mathscr{D}_{ij}^+(\cdot) - a_{ij}^-(x_d)\mathscr{D}_{ij}^-(\cdot)\right\}\Big].\end{aligned} \tag{16}$$

Then, $\mathscr{L}_d$ is said to be a difference operator of (2).

Definition 3 (A-SLE) *Let $L_d, V_d : M_d \times \mathbb{R}_{\geq 0} \to \mathbb{R}$. Then, the equation*

$$\frac{\partial V_d}{\partial t}(x_d,t) = -L_d(x_d,t) - \mathscr{L}_d V_d(x_d,t) \tag{17}$$

is said to be an A-SLE. ■

Let $p^1(x_d'|x_d)$ and $p^2(x_d'|x_d)$ be the transition probability rates from x_d to x_d'. If $V_d : M_d \times \mathbb{R}_{\geq 0} \to \mathbb{R}$,

$$\frac{\partial V_d}{\partial t}(x_d,t) = \sum_{x_d' \in M_d}\left\{p^1(x_d|x_d')V_d(x_d',t) - p^2(x_d'|x_d)V_d(x_d,t)\right\} \tag{18}$$

is said to be the master equation.

We define some representations appearing in quantum mechanics [7] in order to explain the quantization of Markov processes.

Consider a complex separable Hilbert space $\mathbb{H}$, whose orthogonal bases are elements of M_d. The space $\mathbb{H}^*$ denotes a dual space of $\mathbb{H}$. A vector in $\mathbb{H}$ is written as $|\cdots\rangle$, and a vector in $\mathbb{H}^*$ is written as $\langle\cdots|$. The vectors $|\cdots\rangle$ and $\langle\cdots|$ are said to be a ket vector and a bra vector, respectively. The inner product of a bra vector and a ket vector is expressed as $\langle\cdots|\cdots\rangle$.

For a function $V_d : M_d \times [0,\infty) \to \mathbb{R}$, we define

$$|\bar{V}(t)\rangle := \sum_{x_d \in M_d} V_d(x_d,t)|x_d\rangle. \tag{19}$$

Then,

$$V_d(x_d,t) = \langle x_d|\bar{V}(t)\rangle \tag{20}$$

is satisfied. An operator $\mathscr{H}$ is called a Hamiltonian operator if

$$\frac{\partial}{\partial t}|\bar{V}(t)\rangle = -\mathscr{H}|\bar{V}(t)\rangle. \tag{21}$$

Equation (21) is called a Schrödinger-like equation.

Moreover, we define *a discretized stochastic Lyapunov function (D-SLF)* in order to introduce our result.

Definition 4 (D-SLF) *We assume that $W_d : M_d \to \mathbb{R}$ is non-negative in subspace Q_m^d satisfying $0 \in Q_m^d = \{x|W_d(x_d) < m,\ m \in \mathbb{R}_{>0}\} \subset M_d$. If*

$$\mathscr{L}_d W_d(x_d) \leq 0,\ x_d \in Q_m^d, \tag{22}$$

the function $W_d(x_d)$ is said to be a D-SLF of (2). ■

B. Stochastic Lyapunov Function Approach

Stochastic Lyapunov theory is mostly analogous with deterministic Lyapunov theory but for some differences (see [9], [11], [13], [15]). For example, we use the infinitesimal operator $\mathscr{L}$ and not the derivative d/dt because the Wiener process is not differentiable. Besides, some different concepts have been proposed for the stochastic Lyapunov theory: stability in probability, stability with probability one, moment stability, and so on. We are mainly concerned with stability in probability. In addition, an isolated equilibrium point of a system is stable in probability if the point is stable with probability one.

Proposition 1 *The infinitesimal operator $\mathscr{L}$ is expressed as*

$$\mathscr{L}(\cdot) = \left(\frac{\partial(\cdot)}{\partial x}\right)^T \cdot f(x) + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial(\cdot)}{\partial x}\right)^T \cdot a(x)\right\}. \tag{23}$$

♦

The proof of Proposition 1 is shown by Øksendal [15] in Theorem 7.3.3.

Theorem 1 (Kushner [11], Chapter 2, Theorem 2) *Assume that $W(x)$ exists as an SLF of* (2). *Let $x_0 \in Q_m$ be an initial state and*

$$P_m = Q_m \cap \{x | \mathscr{L}W(x) = 0\}. \quad (24)$$

Then, $x(t)$ converges to P_m with a probability no less than $1 - W(x_0)/m$. ♦

In Theorem 1, the origin of (2) is asymptotically stable in probability if $P_m = \{0\}$, and the origin is asymptotically stable with probability one if $P_m = \{0\}$ and $Q_m = \mathbb{R}^n$.

C. Kushner's Difference Approximation [12]

Kushner introduces a method of difference approximation for solving Hamilton-Jacobi-Bellman equations. The solutions of the equations yield the optimal control laws of stochastic systems with inputs [12]. When Kushner's method is applied to the no-input system (2), the method acts as a scheme of approximating SLEs.

SLE (7) is replaced by A-SLE (17) if the first and second partial derivatives of $V(x,t)$ in x are approximated by the following:

$$\frac{\partial V}{\partial x_i}(x,t) \approx \begin{cases} \mathscr{D}_i^+ V_d(x_d,t) & \text{if} \quad f_i(x) \geq 0 \\ \mathscr{D}_i^- V_d(x_d,t) & \text{if} \quad f_i(x) < 0 \end{cases}, \quad (25)$$

$$\frac{\partial^2 V}{\partial x_i^2}(x,t) \approx \mathscr{D}_{ii} V_d(x,t), \quad (26)$$

$$\frac{\partial^2 V}{\partial x_i \partial x_j}(x,t) \approx \begin{cases} \mathscr{D}_{ij}^+ V_d(x_d,t) & \text{if} \quad a_{ij}(x) \geq 0 \\ \mathscr{D}_{ij}^- V_d(x_d,t) & \text{if} \quad a_{ij}(x) < 0 \end{cases}, \quad (27)$$

where $i \neq j$.

Kushner approximates $\partial V / \partial t$ by using the backward difference and defines the one-step transition probabilities. We keep the time continuous and define $p^\flat(x_d'|x_d)$ as the transition probability rates from x_d to x_d'. Thus,

$$p^\flat(x_d + \delta e_i | x_d) = \frac{1}{\delta} f_i^+(x_d) + \frac{1}{2\delta^2} \Big\{ a_{ii}(x_d) - \sum_{j \neq i} |a_{ij}(x_d)| \Big\}, \quad (28)$$

$$p^\flat(x_d - \delta e_i | x_d) = \frac{1}{\delta} f_i^-(x_d) + \frac{1}{2\delta^2} \Big\{ a_{ii}(x_d) - \sum_{j \neq i} |a_{ij}(x_d)| \Big\}, \quad (29)$$

$$p^\flat(x_d + \delta e_i + \delta e_j | x_d) = p^\flat(x_d - \delta e_i - \delta e_j | x_d) = \frac{a_{ij}^+(x_d)}{2\delta^2}, \quad i \neq j, \quad (30)$$

$$p^\flat(x_d - \delta e_i + \delta e_j | x_d) = p^\flat(x_d + \delta e_i - \delta e_j | x_d) = \frac{a_{ij}^-(x_d)}{2\delta^2}, \quad i \neq j, \quad (31)$$

$$p^\flat(x_d | x_d) = -\sum_{i=1}^n \Big\{ \frac{|f_i(x_d)|}{\delta} + \frac{a_{ii}(x_d)}{\delta^2} - \sum_{j \neq i} \frac{|a_{ij}(x_d)|}{2\delta^2} \Big\}, \quad (32)$$

$$p^\flat(x_d' | x_d) = 0, \quad (33)$$

where $x_d' \neq x_d, x_d + \delta e_i, x_d - \delta e_i, x_d + \delta e_i + \delta e_j, x_d + \delta e_i - \delta e_j, x_d - \delta e_i + \delta e_j, x_d - \delta e_i - \delta e_j$.

Proposition 2 *A-SLE* (17) *is equivalent to*

$$\frac{\partial V_d}{\partial t}(x_d,t) = -L_d(x_d,t) - \sum_{x_d' \in M_d} p^\flat(x_d | x_d') V_d(x_d', t). \quad (34)$$

♦

Kushner's method is presented in Section IX-3 of Fleming and Soner [8] and Section 9-1 of Kushner [12].

D. Alcaraz et al.'s Quantization of Markov Processes [1]

Alcaraz et al. introduces the quantization of Markov processes, which is a procedure for quantizing classical dynamical systems (see [1], [17]). In this subsection, we immediately extend the results.

We present specific examples of bra-ket notations. Let $\psi_1, \psi_2 : M_d \to \mathbb{C}$ and $\mathscr{O}$ be a linear operator. Then, we obtain

$$\langle \psi_1 | \mathscr{O} | \psi_2 \rangle = \sum_{x_d' \in M_d} \psi_1^*(x_d') (\mathscr{O}\psi_2)(x_d'), \quad (35)$$

$$\langle \psi_1 | \psi_2 \rangle = \sum_{x_d' \in M_d} \psi_1^*(x_d') \psi_2(x_d'), \quad (36)$$

$$\langle x_d | \psi_1 \rangle = \psi_1(x_d), \quad (37)$$

$$\langle x_d | x_d' \rangle = \delta(x_d | x_d'), \quad (38)$$

where ψ_1^* is the complex conjugate of ψ_1.

Proposition 3 *If a Hamiltonian operator $\mathscr{H}$ satisfies*

$$\langle x_d | \mathscr{H} = \sum_{x_d' \in M_d} \left\{ -p^1(x_d | x_d') \langle x_d' | + p^2(x_d' | x_d) \langle x_d | \right\}, \quad (39)$$

the master equation (18) *is equivalent to the Schrödinger-like equation* (21). ♦

Hamiltonian operator $\mathscr{H}$ can be represented as a matrix. The elements of the matrix are obtained as $\langle x_d | \mathscr{H} | x_d' \rangle$.

Proposition 4 *If a Hamiltonian operator satisfies* (39),

$$\langle x_d | \mathscr{H} | x_d' \rangle = \begin{cases} -p^1(x_d' | x_d) & \text{if } x_d' \neq x_d \\ -p^1(x_d | x_d) + \sum_{x_d'' \in M_d} p^2(x_d'' | x_d) & \text{if } x_d' = x_d. \end{cases} \quad (40)$$

♦

If $p^1 = p^2$, Propositions 3 and 4 are consistent with Alcaraz et al.'s results given in Section 3 of Alcaraz et al. [1] and Section 2 of Rajewsky et al. [17]. The transformation from the master equation (18) to the Schrödinger-like equation (21) is called the quantization of Markov processes.

Remark 1 *The Hamiltonian operator of the Schrödinger-like equation* (21) *generally has complex eigenvalues and eigenfunctions because $\mathscr{H}$ is non-Hermite [1], [17]. This is different from the cases in quantum mechanics [7], [19].* ■

III. Problem Formulation

The purpose of this paper is to propose a method of designing SLFs for system (2).

In Subsection IV-A, we show that the A-SLE is equivalent to a Schrödinger-like equation by using the quantization of Markov processes. Moreover, we describe the general solution of the A-SLE by using the eigenvalues and eigenfunctions of the Hamiltonian operator.

In Subsection IV-B, we obtain sufficient conditions for a time-invariant function to become a D-SLF. Note that the D-SLF $W_d(x_d)$ is an approximation of the SLF $W(x)$ because the difference operator $\mathscr{L}_d$ is an approximation of the infinitesimal operator $\mathscr{L}$.

In Subsection IV-C, we propose a method of designing D-SLFs.

We consider the SLE (7) with

$$L(x,t) = -q(x)V(x,t), \tag{41}$$

where $q:\mathbb{R}^n \to \mathbb{R}_{>0}$. System (2) does not have any restrictions if (41) is considered because the following proposition holds:

Proposition 5 *If there exists an SLF of system* (2) *in* Q_m*, there exists an SLF of* (2) *in* Q_m *satisfying* (7) *and* (41). ♦

Proof: Let t_0 be the initial time and x_0, the initial state. If $W(x(t))$ is an SLF of system (2) in Q_m,

$$W'(x(t)) := e^{-\int_{t_0}^{t}\tilde{q}(x(s))ds}W(x_0),\ \forall x_0 \in Q_m \tag{42}$$

is a non-negative function defined in Q_m, where $\tilde{q}(x(s))$ is non-negative and submartingale. When $x(t) = x_t \in \mathbb{R}$,

$$\begin{aligned}\mathscr{L}W'(x(t)) &= -\left\{\mathscr{L}\int_{t_0}^{t}\tilde{q}(x(s))ds\right\}W'(x(t))\\ &= -\lim_{h\to 0}\frac{\mathrm{E}\left[\int_{t_0}^{t+h}\tilde{q}(x(s))ds|x(t)=x_t\right]-\int_{t_0}^{t}\tilde{q}(x(s))ds}{h}W'(x(t))\\ &\leq -\tilde{q}(x(t))W'(x(t)).\end{aligned} \tag{43}$$

For all x_t, (43) holds. Therefore, $W'(x(t))$ is an SLF satisfying (7) and (41). ■

IV. Stochastic Lyapunov Function Design

A. Solutions of A-SLE

We define

$$p^\sharp(x_d'|x_d) := p^\flat(x_d|x_d'),\quad x_d' \neq x_d \tag{44}$$

$$p^\sharp(x_d|x_d) := q(x_d) - \sum_{x_d'\neq x_d} p^\sharp(x_d'|x_d). \tag{45}$$

Thus, we obtain the following lemma.

Lemma 1 *If Hamiltonian operator* $\mathscr{H}$ *satisfies*

$$\langle x_d|\mathscr{H} = \sum_{x_d'\in M_d}\{p^\flat(x_d|x_d')\langle x_d'| + p^\sharp(x_d'|x_d)\langle x_d|\}, \tag{46}$$

A-SLE (17) *with* (41) *is equivalent to the Schrödinger-like equation* (21). ♦

Proof: By using (44)–(45), we obtain

$$q(x_d) = \sum_{x_d'\in M_d} w^\sharp(x_d'|x_d). \tag{47}$$

By using (34), (41), and (47), we obtain

$$\frac{\partial V}{\partial t}(x_d,t) = -\sum_{x_d'\in M_d}\{p^\flat(x_d|x_d')V(x_d',t) + p^\sharp(x_d'|x_d)V(x_d,t)\}. \tag{48}$$

Equation (48) is the master equation (18) with $p^1 = -p^\flat$ and $p^2 = p^\sharp$. Hence, (48) is transformed into the Schrödinger-like equation (21) with (46) by using Proposition 3. ■

Let us show that the general solution of A-SLE (17) with (41) can be represented by using the eigenvalues and eigenfunctions of a Hamiltonian operator satisfying (46). Let

$$E_j := E_{jR} + \mathbf{i}E_{jI},\ E_j^* := E_{jR} - \mathbf{i}E_{jI} \tag{49}$$

be the eigenvalues of $\mathscr{H}$ with (46) and

$$\begin{aligned}\phi_j(x_d) &:= \phi_{jR}(x_d) + \mathbf{i}\phi_{jI}(x_d)\\ \phi_j^*(x_d) &:= \phi_{jR}(x_d) - \mathbf{i}\phi_{jI}(x_d)\end{aligned} \tag{50}$$

be the eigenfunctions corresponding to E_j and E_j^*, where $j \in \mathbb{N}_0$, $E_{jR}, E_{jI} \in \mathbb{R}$, $\phi_{jR}, \phi_{jI} : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{i}$ is the imaginary unit, and $|E_0| \leq |E_1| \leq \cdots$.

Theorem 2 *The general solution of A-SLE* (17) *with* (41) *is represented as*

$$V^\natural(x_d,t) = \sum_{j\in\mathbb{N}_0}\{C_j e^{-E_j t}\phi_j(x_d) + C_j^* e^{-E_j^* t}\phi_j^*(x_d)\}, \tag{51}$$

where C_j *is an arbitrary complex constant and* C_j^**, the complex conjugate of* C_j. ♦

Proof: By using the linearity of the Schrödinger-like equation (21), the general solution of (21) with (46) is represented as

$$V^\natural(x_d,t) = \sum_{j\in\mathbb{N}_0}\{C_j e^{-E_j t}\phi_j(x_d) + D_j e^{-E_j^* t}\phi_j^*(x_d)\}, \tag{52}$$

where C_j and D_j are arbitrary complex constants. By using Lemma 1 and (52), (51) is obtained as the general solution of the approximate Lyapunov equation (17) with (41). ■

B. Sufficient Conditions for D-SLFs

Consider the following time-invariant function:

$$W^\natural(x_d) := V^\natural(x_d,0) = \sum_{j\in\mathbb{N}_0}\left(C_j\phi_j(x_d) + C_j^*\phi_j^*(x_d)\right). \tag{53}$$

We can obtain sufficient conditions for $W^\natural(x_d)$ to be a D-SLF.

We define

$$W_j^\flat(x_d) := C_j\phi_j(x_d) + C_j^*\phi_j(x_d) \tag{54}$$

$$W_j^\sharp(x_d) := -\mathbf{i}(C_j\phi_j(x_d) - C_j^*\phi_j(x_d)). \tag{55}$$

Assumption 1 *Let* $0 \in M_d^\natural \subset M_d$*. There exists a combination* $(C_0, C_1, \ldots)$ *that satisfies all of the following conditions:*

1) $W^\natural(x_d)$ *is non-negative in* $M_d^\natural$.
2) *For all* $x_d \in M_d^\natural$, *there exists a real constant K satisfying*

$$\sum_{j\in\mathbb{N}_{\geq 0}} \left(|W_j^\sharp(x_d)| + |W_j^\flat(x_d)|\right) \leq K W^\natural(x_d). \tag{56}$$

■

We then obtain the following theorem:

Theorem 3 *Suppose that Assumption 1 holds. Function* $W^\natural(x_d)$ *is a D-SLF of* (2) *if*

$$K \max_{j\in N^\natural}(|E_{jR}|,|E_{jI}|) \leq q(x_d),\ \forall x_d \in M_d^\natural, \tag{57}$$

where $N^\natural := \{j | C_j \neq 0\}$. ♦

Proof: By using

$$C_j := k_j(\cos\mu_j + \mathbf{i}\sin\mu_j),\ k_j,\mu_j \in \mathbb{R}, \tag{58}$$

the general solution of the approximate Lyapunov equation (51) is represented as

$$V^\natural(x_d,t) = \sum_{j\in\mathbb{N}_{\geq 0}} e^{-E_{jR}t} k_j\{\phi_{jR}(x_d)\cos(\mu_j - E_{jI}t) - \phi_{jI}(x_d)\sin(\mu_j - E_{jI}t)\}. \tag{59}$$

By using (59), we obtain

$$\frac{\partial V^\natural}{\partial t}(x_d,t) = -2\sum_{j\in\mathbb{N}_{\geq 0}} e^{-E_{jR}t}k_j \times\{(E_{jR}\phi_{jR}(x_d) - E_{jI}\phi_{jI}(x_d))\cos(\mu_j - E_{jI}t) - (E_{jI}\phi_{jR}(x_d) + E_{jR}\phi_{jI}(x_d))\sin(\mu_j - E_{jI}t)\} \tag{60}$$

and

$$\mathscr{L}_d V^\natural(x_d,t) = 2\sum_{j\in\mathbb{N}_{\geq 0}} e^{-E_{jR}t}k_j \times\{\mathscr{L}_d\phi_{jR}(x_d,t)\cos(\mu_j - E_{jI}t) - \mathscr{L}_d\phi_{jI}(x_d,t)\sin(\mu_j - E_{jI}t)\}. \tag{61}$$

By substituting (59)–(61) into A-SLE (17) with (41), we obtain

$$\mathscr{L}_d\phi_{jR}(x_d,t) = E_{jR}\phi_{jR}(x_d) - E_{jI}\phi_{jI}(x_d) - q(x_d)\phi_{jR}(x_d), \tag{62}$$

$$\mathscr{L}_d\phi_{jI}(x_d,t) = E_{jI}\phi_{jR}(x_d) + E_{jR}\phi_{jI}(x_d) - q(x_d)\phi_{jI}(x_d). \tag{63}$$

By using (58), (62), and (63), we derive

$$\mathscr{L}_d W^\natural(x_d) = 2\sum_{j\in\mathbb{N}_{\geq 0}} k_j(\cos\mu_j\mathscr{L}_d\phi_{jR}(x_d) - \sin\mu_j\mathscr{L}_d\phi_{jI}(x_d)) = -q(x)W(x_d) + \sum_{j\in\mathbb{N}_{\geq 0}}\{E_{jR}W^\flat(x_d) - E_{jI}W^\sharp(x_d)\}. \tag{64}$$

By using Assumption 1-2) and $N^\natural$, we obtain

$$\begin{aligned}\sum_{j\in\mathbb{N}_{\geq 0}}&\{E_{jR}W^\flat(x_d) - E_{jI}W^\sharp(x_d)\}\\ &\leq \sum_{j\in N^\natural}\{|E_{jR}||W^\flat(x_d)| + |E_{jI}||W^\sharp(x_d)|\}\\ &\leq K\max_{j\in N^\natural}(|E_{jR}|,|E_{jI}|)\sum_{j\in N^\natural}(|W^\flat(x_d)| + |W^\sharp(x_d)|)\\ &\leq K\max_{j\in N^\natural}(|E_{jR}|,|E_{jI}|).\end{aligned} \tag{65}$$

By using (64) and (65), we obtain

$$\mathscr{L}_d W^\natural(x_d) \leq \left\{-q(x_d) + K\max_{j\in N^\natural}(|E_{jR}|,|E_{jI}|)\right\}W^\natural(x_d). \tag{66}$$

By using Assumption 1-1), (57), and (66), we obtain

$$\mathscr{L}_d W^\natural(x_d) \leq 0, \quad x_d \in M_d^\natural. \tag{67}$$

Therefore, $W^\natural(x_d)$ is a D-SLF of (2). ■

In addition, a simple case can be described as follows:

Corollary 1 *Let* $\alpha \in \mathbb{N}_{\geq 0}$ *and* $C_\alpha, E_\alpha, \phi_\alpha \in \mathbb{R}$. *Consider a combination* $(C_0, C_1, \ldots) = (0,\ldots,0,C_\alpha/2,0,\ldots,0)$. *The function* $W_\alpha^\natural(x_d) := C_\alpha\phi_\alpha(x_d)$ *is a D-SLF of* (2) *if*

$$E_\alpha < q(x_d), \quad \forall x_d \in M_d^\natural \tag{68}$$

and $W_\alpha^\natural$ *is non-negative in* $M_d^\natural$. ♦

Proof: Let

$$V_\alpha^\natural(x_d,t) = \exp(-E_\alpha t)W_\alpha^\natural(x_d). \tag{69}$$

Because $V_\alpha^\natural(x_d,t)$ is a solution of A-SLE (17) with (41), we obtain

$$\frac{\partial V_\alpha^\natural}{\partial t}(x_d,t) + \mathscr{L}_d V_\alpha^\natural(x_d,t) = -q(x_d)V_\alpha^\natural(x_d). \tag{70}$$

Hence, we obtain

$$\mathscr{L}_d W_\alpha^\natural(x_d,t) = (E_\alpha - q(x_d))V_\alpha^\natural(x_d). \tag{71}$$

Therefore, $W_\alpha^\natural(x_d)$ is a D-SLF of (2). ■

C. D-SLF Design Procedure

A new procedure for designing D-SLFs is proposed. The procedure involves the following four steps:

[Step 1] Determine positive function $q(x)$.
[Step 2] Discretize state space $\mathbb{R}^n$ into M_d.
[Step 3] Solve the eigenvalue problem for Hamiltonian operator $\mathscr{H}$.
[Step 4] Choose a combination $(C_0, C_1, \ldots)$ satisfying Assumption 1 and (57).

By using Steps 1–4, we can obtain $W^\natural(x_d)$ as a D-SLF of (2).

Previous work with finite-difference approximations, e.g., Kushner [12], assumed that the boundary satisfied the Neumann condition $\partial V/\partial x = 0$. Hence, if the boundary condition does not hold, the iteratively calculated values may diverge.

Our new method can be used in domains where the system is asymptotically stable in probability. The restrictions of an SLF do not impose any restrictions on the system because Proposition 5 holds. Moreover, there are no divergence problems because our method does not involve any iterative calculations.

Hamiltonian operator $\mathscr{H}$ can be represented as a matrix because $\mathscr{H}$ is a linear operator. The matrix is sparse if $\mathscr{H}$ satisfies (46). This is derived from the transition probability rates (28)–(33) and Proposition 4 with $p^1 = -p^\flat$ and $p^2 = p^\sharp$. Previous work, e.g., Press et al. [16], proposed various algorithms for solving the eigenequations of sparse matrices. In our previous work [14], we discussed the matrix representation of a Hamiltonian operator for deterministic dynamical systems.

Certain classes of systems, e.g., controlled nonholonomic systems, require that their SLFs be nonsmooth. Our method lends itself to such systems because viscosity solutions, which are weak solutions of partial differential equations [6], are characterized as solutions of the Lyapunov equations in Kushner's method [8].

Camilli et al. extend the scheme called Zubov's method to perturbed systems [3], stochastic systems [5], and control systems [4]. The method also allows viscosity solutions. Hence, our method can be widely applied to control dynamical systems if Camilli et al.'s and our methods are combined.

V. Numerical Example

We can illustrate our scheme by using a simple example. Consider a two-dimensional system

$$\begin{aligned} dx_1 &= a_1 x_2 dt \\ dx_2 &= -(a_2 \sin x_1 + a_3 x_2)dt + a_4 x_2 dw, \end{aligned} \tag{72}$$

where $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, and $a_4 = 1/5$. We can calculate a D-SLF $W^\natural(x_d)$ of the system (72) with a bounded region $M_d = \{(x_1, x_2) | -1 \le x_1 \le 1,\ -1 \le x_2 \le 1\}$Clattice points $l = 21 \times 21$Cand a non-negative function $q(x) = 1$. Figure 1 shows the D-SLF $W^\natural(x_d)$, and Figure 2 shows $\mathscr{L}_d W^\natural(x_d)$ obtained by using multilinear approximation. The region labeled Ω denotes the largest connected level set of $W^\natural(x_d)$, and $\partial\Omega$ denotes the boundary of Ω.

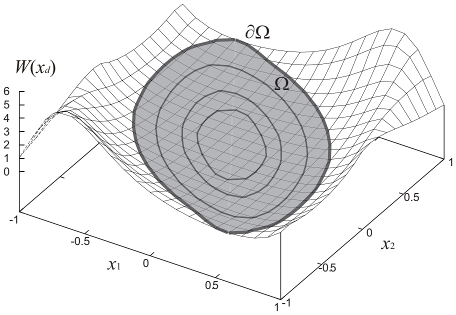

Fig. 1. Discretized Stochastic Lyapunov function $W^\natural(x_d)$

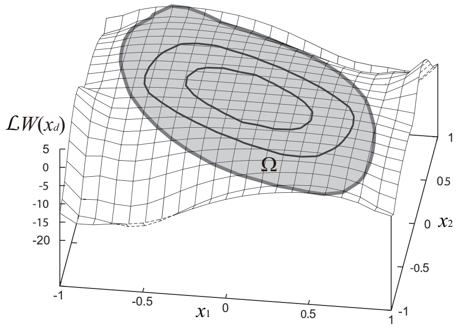

Fig. 2. Approximation of $\mathscr{L}W(x_d)$

VI. Conclusions

We have proposed a design procedure for designing D-SLFs. First, we have approximated SLEs by using Kushner's scheme of difference approximation. Second, we have used the quantization of Markov processes to replace A-SLEs with Schrödinger-like equations. Third, we have obtained sufficient conditions for time-invariant functions to become D-SLFs. Finally we have provided a method of constructing D-SLFs.

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