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# On the Observability Properties of Homogeneous and Heterogeneous Networked Dynamic Systems

Daniel Zelazo and Mehran Mesbahi†

*Abstract*— **This work provides a framework for the observability analysis of linear networked dynamic systems (NDS). A distinction is made between NDS that have homogeneous agent dynamics and NDS that have heterogeneous dynamics. In each case, conditions for the observability of such a system are presented; we will also quantify the relative degree of observability of these systems. Moreover, an *index of homogeneity* and an *index of heterogeneity* are introduced as the means of quantitatively measuring how homogeneous a particular NDS is.**

## I. INTRODUCTION

There has a been a recent research surge in the controls community focusing on the study of networked dynamic systems (NDS). A NDS is a collection of dynamic systems that are coupled together through some kind of network. The network may represent a communication topology through which each dynamic agent can exchange information, or a sensing topology to coordinate high-level objectives such as formations. Examples of such systems include multiple space, air, and land vehicles [1], [2], [3], [4]. What makes this class of problems interesting is the role that the network plays in the dynamics of the entire system.

When studying linear and time-invariant systems, all the essential properties of the system can be derived from the quadruple $(A, B, C, D)$. In a NDS, however, there is an additional parameter in the underlying connection topology. Although this connection topology can easily be embedded into the system matrices, it is more enlightening to consider how changes in that topology explicitly affects certain systems theoretic properties. Therefore, it is becoming increasingly important to consider the quintuple $(A, B, C, D, \mathcal{G})$, where $\mathcal{G}$ denotes the underlying connection topology graph, when performing analysis of linear NDS. Examples include relating controllability to graph symmetry [5], and observability properties in consensus seeking filters [6]. In this paper, we consider NDS where the network couples the agents at their outputs. Such systems are prevalent in formation flying applications where high level objectives are obtained via relative sensing. An example includes work on formation flying of spacecraft [7].

The contribution of this paper is twofold. We first introduce two distinct classes of NDS consisting of homogeneous and heterogeneous agent dynamics. Although the homogeneous case can be considered as a subset of the heterogeneous case, it turns out to be enlightening, whether through algebraic simplicity or development of intuition, to consider both cases separately. The second contribution is a rigorous observability analysis of NDS based on the relative sensing problem. Specifically, this paper highlights how both the qualitative and quantitative notions of observability change as the dynamics of each agent become more homogeneous. In this direction, an *index of homogeneity* and an *index of heterogeneity* are introduced to quantitatively capture how the dynamics of each agent in the ensemble and the underlying connection topology affect the overall observability properties of the overall system.

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†The authors are with the Department of Aeronautics and Astronautics, University of Washington, Box 352400, Seattle, WA 98195-2400. Emails: *dlz@aa.washington.edu* and *mesbahi@aa.washington.edu*

## II. PRELIMINARIES AND NOTATIONS

We provide some notations and preliminaries that will be used throughout the paper.

### A. Graphs and their Algebraic Representation

An undirected (simple) graph $\mathcal{G}$ is specified by a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$ whose elements characterize the incidence relation between distinct pairs of $\mathcal{V}$. Two vertices $i$ and $j$ are called *adjacent* (or neighbors) when $\{i, j\} \in \mathcal{E}$; we denote this by writing $i \sim j$. An *orientation* of an undirected graph $\mathcal{G}$ is the assignment of directions to its edges, i.e., an edge $e_k$ is an ordered pair $(i, j)$ such that $i$ and $j$ are, respectively, the initial and the terminal nodes of $e_k$.

In this work we make extensive use of the $|\mathcal{V}| \times |\mathcal{E}|$ incidence matrix, $E(\mathcal{G})$, for a graph with arbitrary orientation. The columns of $E(\mathcal{G})$ are indexed by the edges, and the $i^{th}$ row entry takes the value $+1$ if it is the initial node of the corresponding edge, $-1$ it it is the terminal node, and 0 otherwise.

From the definition of the incidence matrix it follows that the null space of its transpose, $\mathcal{N}(E(\mathcal{G})^T)$, contains $\mathbf{span}\,\{\mathbf{1}\}$, where $\mathbf{1}$ is the vector of all ones. The rank of the incidence matrix depends only on $|\mathcal{V}|$ and the number of its connected components [8].

The (graph) Laplacian of $\mathcal{G}$, $L(\mathcal{G}) := E(\mathcal{G})E(\mathcal{G})^T$, is a rank deficient positive semi-definite matrix. The spectrum of the graph Laplacian $\{\lambda_i(L(\mathcal{G}))\}_{1\leq i\leq n}$ can be ordered as $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{|\mathcal{V}|}$. Two graphs, $\mathcal{G}_1$ and $\mathcal{G}_2$, are said to be *cospectral* with respect to the graph Laplacian if the spectrum of $L(\mathcal{G}_1)$ is identical to the spectrum of $L(\mathcal{G}_2)$.

### B. Homogeneous and Heterogeneous NDS

Henceforth, we will be referring to two classes of NDS; those with 1) homogeneous dynamics, and 2) heterogeneous dynamics. For both cases, we will work with linear and time-invariant systems for each agent, driven by a generalized disturbance

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i w_i(t) \\ y_i(t) &= C_i x_i(t). \end{aligned} \quad (1)$$

Each agent is indexed by the sub-script $i$, and the sub-script is dropped for the homogeneous case.

In the homogeneous case, it is assumed that each dynamic agent in the NDS is described by the same set of linear state-space dynamics (e.g., $(A_i, B_i, C_i) = (A_j, B_j, C_j),\ \forall i, j)$. In the heterogeneous case, each agent is assumed to have different linear dynamics.

For both cases, we will assume a stable, strictly proper system ($D_i = 0$) with a minimal realization. Finally, we also assume that each agent has compatible outputs (e.g. system outputs correspond to the same physical quantity) and dimensions.

The observability grammian of a dynamic system is an important operator that will be used throughout this paper. The observability grammian for an individual agent based on the dynamics in (1) is defined as

$$Y_o^{(i)} = \int_0^\infty e^{A_i^T t} C_i^T C_i e^{A_i t} dt\,. \quad (2)$$

As each agent is assumed to be minimal, the grammian is a positive-definite matrix and can be expressed in terms of its singular value decomposition, $Y_o^{(i)} = U_i \Sigma_i U_i^T$. We denote, respectively, the largest and smallest singular values of $Y_o^{(i)}$ as $\overline{\sigma}(Y_o^{(i)})$ and $\underline{\sigma}(Y_o^{(i)})$.

The grammian can be calculated by solving a system of linear equations, described by the Lyapunov equation

$$A_i^T Y_o^{(i)} + Y_o^{(i)} A_i + C_i^T C_i = 0. \quad (3)$$

The grammian can be used as a quantitative way to compare the relative observability of different modes in the system, as $\|y(t)\|_2 = \|Y_o^{1/2} x(0)\|_2$.

In both cases, we will represent the parallel interconnection of all the agents with the following state-space description:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(\mathbf{t}) + \mathbf{B}\mathbf{w}(\mathbf{t}) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(\mathbf{t}), \end{aligned} \quad (4)$$

with $\mathbf{x}(t)$, $\mathbf{w}(t)$, and $\mathbf{y}(t)$ denoting respectively, the concatenated state vector, generalized disturbance vector, and output vector of all the agents in the NDS. The dimensions of each agents state, control, and output vectors are $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$, and $y_i(t) \in \mathbb{R}^r$ respectively. In the heterogeneous case, the dimension of the state and the control need not be the same, but for notational convenience we only examine the above case. The matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are the block diagonal aggregation of each agents' state-space matrices.

The model we examine in this paper is motivated by the relative sensing problem. The sensed output of the NDS is a vector $\mathbf{y}_\mathcal{G}(t) \in \mathbb{R}^{r|\mathcal{E}|}$ containing the relative information of each agent and its neighbors . For example, the output sensed across an edge $e = (i, i')$ would be of the form $y_i(t) - y_{i'}(t)$. This can be compactly written as

$$\mathbf{y}_\mathcal{G}(t) = (E(\mathcal{G})^T \otimes I_r)\mathbf{y}(t); \quad (5)$$

here, "$\otimes$" denotes the Kronecker product, and $I_r$ is the $r \times r$ identity matrix. An important result on the singular value decomposition of Kronecker products will prove useful in subsequent discussions.

*Theorem 2.1 ([9]):* Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ each have a singular value decomposition of $A = U_A \Sigma_A V_A^T$ and $B = U_B \Sigma_B V_B^T$. The singular value decomposition of the Kronecker product of $A$ and $B$ is

$$A \otimes B = (U_A \otimes U_B)(\Sigma_A \otimes \Sigma_B)(V_A^T \otimes V_B^T). \quad (6)$$

In subsequent sections, we will refer to homogeneous and heterogeneous NDS by $\Sigma_{hom}(\mathcal{G})$ and $\Sigma_{het}(\mathcal{G})$ respectively. We also refer to $\Sigma_{hom}(\mathcal{G})$ and $\Sigma_{het}(\mathcal{G})$ in an operator context. Using the above notations, we have the following compact descriptions for homogeneous and heterogeneous NDS:

$$\Sigma_{hom}(\mathcal{G}) \begin{cases} \dot{\mathbf{x}}(t) &= (I_N \otimes A)\mathbf{x}(t) + (I_N \otimes B)\mathbf{w}(t) \\ \mathbf{y}_\mathcal{G}(t) &= (E(\mathcal{G})^T \otimes C)\mathbf{x}(t) \end{cases} \quad (7)$$

$$\Sigma_{het}(\mathcal{G}) \begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{w}(t) \\ \mathbf{y}_\mathcal{G}(t) &= (E(\mathcal{G})^T \otimes I_r)\mathbf{C}\mathbf{x}(t) \end{cases} \quad (8)$$

## III. Observability Properties of NDS

Examination of observability properties of a system is an important tool in the analysis of dynamical systems. It can be used to characterize, for example, the $\mathcal{H}_2$ norm of a system, as well as provide answers to the existence of a state-estimator. In this section we present observability analysis for the homogeneous and heterogeneous linear NDS separately. The results of both are valuable, noting that each agent can be individually compensated to achieve homogeneous or heterogeneous dynamics. Furthermore, it will be shown that the homogeneous system is a specialization of the heterogeneous case.

We also include in this discussion expressions for the observability grammian for both the homogeneous and heterogeneous cases. As mentioned in §II-B, the observability grammian of a dynamic system can give additional insight about the observability properties of the system. In the context of NDS, the grammian leads to explicit characterization of how the underlying topology affects the observability properties.

### A. Homogeneous System

For the homogeneous case, we have the following result on its observability properties.

*Proposition 3.1:* The homogeneous networked dynamic system (7) is unobservable.

*Proof:* Using the PHB test for observability of a linear system, it suffices to show that we can construct a nonzero vector $q$ such that

$$(I_N \otimes A)q = \lambda q \text{ and } (E(\mathcal{G})^T \otimes C)q = 0,$$

where $\lambda$ is an eigenvalue of $(I_N \otimes A)$.

Let $\tilde{q}$ be an eigenvector of $A$ such that $A\tilde{q} = \lambda\tilde{q}$. To construct an eigenvector for $I_N \otimes A$, we only need concatenate $N$ versions of $\tilde{q}$ into one vector. Thus, $q = \mathbf{1}\otimes\tilde{q}$.

Exploiting properties of the Kronecker product reveals that $q$ is in the null space of $E(\mathcal{G})^T \otimes C$, completing the proof.

$$(E(\mathcal{G})^T \otimes C)(\mathbf{1} \otimes \tilde{q}) = (E(\mathcal{G})^T\mathbf{1}) \otimes (C\tilde{q}) = 0.$$

■

It is also beneficial to discuss how the observable and unobservable subspaces of the NDS relate to the structure of the network. One way to examine these subspaces is to find a transformation matrix $S$ that separates the system into its observable and unobservable components. There are many ways to construct such a transformation, and we will do so by exploiting properties of the incidence matrix associated with the underlying graph.

First, we define a partition of the network into a tree and its cycles. Denote $E_\tau$ as the incidence matrix corresponding to any spanning tree subgraph of $\mathcal{G}$. The remaining edges necessarily complete the cycles in $\mathcal{G}$, and $E_c$ denotes the incidence matrix for those edges. Therefore, with an appropriate permutation of the columns of $E(\mathcal{G})$, we can always write $E(\mathcal{G}) = \left[\begin{array}{cc} E_\tau & E_c \end{array}\right]$.

One important property of $E_\tau$ is that its columns are linearly independent. We can construct the transformation matrix as $S = (\left[\begin{array}{cc} E_\tau & \mathbf{1} \end{array}\right]) \otimes I_n$.

Now, we define the new state $\mathbf{z}(t)$ such that $S\mathbf{z}(t) = \mathbf{x}(t)$. The transformed state-space is thus

$$\begin{aligned} \dot{\mathbf{z}}(t) &= S^{-1}(I_N \otimes A)S\mathbf{z}(t) + S^{-1}(I_N \otimes B)\mathbf{w}(t) \\ \mathbf{y}_{\mathcal{G}}(t) &= (E^T \otimes C)S\mathbf{z}(t). \end{aligned} \quad (9)$$

Using properties of the Kronecker product, we note the following simplifications:

$$S^{-1}(I_N \otimes A)S = I_N \otimes A \quad (10)$$

$$(E(\mathcal{G})^T \otimes C)S = \left[\begin{array}{cc} \left(\left[\begin{array}{c} E_\tau^T E_\tau \\ E_c^T E_\tau \end{array}\right] \otimes C\right) & 0 \end{array}\right]. \quad (11)$$

This transformation clearly shows that the unobservable subspace is spanned by the $\mathbf{1}$ vector. Physically, this corresponds to a rigid-body type motion of the NDS. That is, we are not able to observe the inertial position of the formation. For the estimation problem, if the objective is to estimate the relative states between each agent, then we can accept the unobservable subspace. However, if we require an estimate of the inertial states then we must affectively anchor one of the nodes to reconstruct the states of all the other nodes. We should also note that based on our earlier assumption that each agent is minimal and stable, we are at least guaranteed that the unobservable mode is stable.

An expression for the observability grammian of the entire NDS in (7) is

$$\begin{aligned} \mathbf{Y_o} &= \int_0^\infty e^{(I_N\otimes A)^T t}(E^T \otimes C)^T(E^T \otimes C)e^{(I_N\otimes A)t}dt \\ &= L(\mathcal{G}) \otimes Y_o, \end{aligned} \quad (12)$$

where $Y_o$ represents the observability grammian of a single agent in the network (described in (2)).

The form of (12) explicitly shows how the network structure affects the observability grammian. In fact, (12) can be used as an alternative proof to Proposition 3.1 by invoking Theorem 2.1. Since zero is an eigenvalue of $L(\mathcal{G})$, it must also be an eigenvalue of $\mathbf{Y}_o$ with multiplicity $|\mathcal{V}|$, resulting in a positive-semidefinite grammian. This is equivalent to the system being unobservable.

### B. Heterogeneous System

We give conditions for when (8) is observable or unobservable.

*Proposition 3.2:* The heterogeneous networked dynamic system (8) is observable if there is *no* eigenvalue of $\mathbf{A}$ that is an eigenvalue for each $A_i$.

*Proof:* We must show that we can *not* construct a nonzero vector $q$ that satisfies $\mathbf{A}q = \lambda q$ and $(E^T \otimes I_r)\mathbf{C}q = 0$. Assume that no agents share the same eigenvalues. The proof is similar if a subset of agents do share an eigenvalue. In this case, an eigenvector of $\mathbf{A}$ must have the form

$$q = \left[\begin{array}{ccccccc} 0_n^T & 0_n^T & \cdots & \tilde{q}_i^T & 0_n^T & \cdots & 0_n^T \end{array}\right]^T,$$

where $\tilde{q}_i$ is the eigenvector for $A_i$.

We now check to see if $q \in \mathcal{N}\left\{(E(\mathcal{G})^T \otimes I_r)\mathbf{C}\right\}$.

$$\begin{aligned} &(E(\mathcal{G})^T \otimes I_r)\mathbf{C}\mathbf{q} = \\ &\qquad (E(\mathcal{G})^T \otimes I_r)\left[\begin{array}{ccccc} 0_n^T & \cdots & (C_i\tilde{q}_i)^T & 0_n^T & \cdots \end{array}\right]^T. \end{aligned}$$

By assumption, $C_i\tilde{q}_i \neq 0$, and (8) is observable. ■

Proposition 3.2 only provides a sufficient condition for observability. In order for a heterogeneous system to be unobservable, not only does each agent need to share a common eigenvalue, but the outputs of each agent associated with a certain direction must be indistinguishable. This is characterized in the following proposition.

*Proposition 3.3:* The heterogeneous networked dynamic system (8) is unobservable if the following conditions are met:

1) There exists an eigenvalue, $\lambda^*$, of $\mathbf{A}$ that is common to each $A_i$
2) $C_iq_i = C_jq_j \;\forall\, i,j$ with $A_iq_i = \lambda^*q_i \;\forall\, i$.

*Proof:* By assumption, there exists a $\lambda^*$ that is an eigenvalue for each $A_i$. We can construct an eigenvector for $\mathbf{A}$ as $q = [\,q_1^T, \cdots, q_{|\mathcal{V}|}^T\,]^T$, with $A_iq_i = \lambda^*q_i$. By condition 2, we have that $\mathbf{C}q = \mathbf{1} \otimes r$, where $r = C_iq_i \neq 0$ for all $i$.

Using properties of the Kronecker product we have

$$(E(\mathcal{G})^T \otimes I_r)\mathbf{C}q = \quad (E(\mathcal{G})^T\mathbf{1} \otimes r) \quad = 0.$$

This shows the system is unobservable. ■

It is clear that Proposition 3.3 is a generalization of the homogeneous case.

The advantages of an observable heterogeneous system is the ability to reconstruct the inertial states of each agent using an observer (given that the conditions of Proposition 3.3 are not met). However, a heterogeneous system introduces another degree of complexity. For the homogeneous case, the assignment of an agent to a certain position in the network topology does not change the observability properties. In the heterogeneous case, the assignment of an agent to a certain position can have a dramatic affect on the system observability.

As in the homogeneous case, we can derive an expression for the observability grammian of the heterogeneous NDS. Using the definition directly, we have

$$\mathbf{Y_o} \;=\; \int_0^\infty e^{\mathbf{A}^T t}\mathbf{C}^T(L(\mathcal{G})\otimes I_r)\mathbf{C}e^{\mathbf{A}t}dt.$$

The above form, however, is not as satisfying as the form derived for the homogeneous case. The reason is that the graph structure does not decouple cleanly from the expression. However, with some manipulation, the expression can be derived to highlight the role of the network in a more transparent way.

We begin by first noting that

$$\mathbf{C}e^{\mathbf{A}t} \;=\; \sum_{i=1}^{|\mathcal{V}|}(e_i e_i^T\otimes C_i e^{A_i t}),$$

where $e_i \in \mathbb{R}^{|\mathcal{V}|}$ is the $i$-th unit coordinate basis vector for $\mathbb{R}^{|\mathcal{V}|}$.

It can also be verified that

$$L(\mathcal{G}) \;=\; \sum_{i=1}^{|\mathcal{V}|}\sum_{j=1}^{|\mathcal{V}|} e_i e_i^T L(\mathcal{G}) e_j e_j^T.$$

Using these results, the expression for the observability grammian can be further simplified to

$$\mathbf{Y}_o = \sum_{i=1}^{|\mathcal{V}|}\sum_{j=1}^{|\mathcal{V}|}\int_0^\infty e_i e_i^T L(\mathcal{G}) e_j e_j^T \otimes \left(e^{A_i^T t}C_i^T C_j e^{A_j t}\right)dt.$$

We now can introduce a notational simplification by defining the observability operator and its adjoint:

$$\Psi_i(x) = C_i e^{A_i t}x \quad \text{and} \quad \Psi_i^*(y(t)) = \int_0^\infty e^{A_i^T t}C_i^T y(t)dt.$$

Each agent is assumed to be stable and minimal, so we have that $\Psi_i : \mathbb{R}^n \mapsto L_2^m[0,\infty)$ and the adjoint $\Psi_i^* : L_2^m[0,\infty) \mapsto \mathbb{R}^n$. We also note that the composition of $\Psi_i^*$ with its adjoint, as in $\Psi_i^*\Psi_i$, is precisely equal to the observability grammian of agent i, $Y_o^{(i)}$. More generally, $Y_{ij} = \Psi_i^*\Psi_j$ can be calculated by solving the Sylvester equation

$$A_i^T Y_{ij} + Y_{ij}A_j + C_i^T C_j = 0. \tag{13}$$

All the results above can be used to derive the following expression for the observability grammian of a heterogeneous NDS:

$$\mathbf{Y}_o \;=\; (L(\mathcal{G})\otimes J_n)\circ(\mathbf{\Psi}^*\mathbf{\Psi}), \tag{14}$$

where $J_n$ is the $n\times n$ matrix of all ones, $\mathbf{\Psi} = \begin{bmatrix}\Psi_1 & \cdots & \Psi_{|\mathcal{V}|}\end{bmatrix}$, and $A\circ B$ denotes the Hadamard product of $A$ and $B$.

The form of (14) is appealing in how it separates the role of the network from each agent. A precise characterization of the eigenvalues of (14) is non-trivial, but we can construct bounds on those values, as presented in [10]. In particular, since both terms in the Hadamard product are positive-semidefinite matrices, we can apply Schur's Theorem to obtain the following bound:

$$\underline{d}\,\underline{\sigma}(\mathbf{\Psi}^*\mathbf{\Psi}) \le \underline{\sigma}(\mathbf{Y}_o) \le \overline{\sigma}(\mathbf{Y}_o) \le \overline{d}\,\overline{\sigma}(\mathbf{\Psi}^*\mathbf{\Psi}), \tag{15}$$

where $\underline{d} = \min_i\,[L(\mathcal{G})\otimes J_n]_{ii}$ and $\overline{d} = \max_i\,[L(\mathcal{G})\otimes J_n]_{ii}$. These correspond, respectively, to the minimum and maximum degree vertices of the underlying graph.

The grammian expression (14) can be represented alternatively as a *node weighted* Laplacian. Consider scalar weights $w_i$ on each node collected together in a diagonal matrix $W = \mathbf{diag}\{w_1,\ldots,w_{|\mathcal{V}|}\}$. The *node weighted* Laplacian can be defined as

$$\hat{L}(\mathcal{G}) \;=\; WL(\mathcal{G})W \;=\; L(\mathcal{G})\circ ww^T. \tag{16}$$

This can be generalized to $n\times n$-block matrix weights, and (16) can be equivalently written as

$$\hat{L}(\mathcal{G}) = \mathbf{W}(L(\mathcal{G})\otimes I_n)\mathbf{W}^T. \tag{17}$$

Using (17) leads to a new interpretation of the expression in (14). Each node in the graph is weighted by the observability operator of the agent assigned to that node.

$$\mathbf{Y}_o \;=\; \mathbf{diag}\{\mathbf{\Psi}^*\}(L(\mathcal{G})\otimes I_n)\mathbf{diag}\{\mathbf{\Psi}\}. \tag{18}$$

### *C. Necessary and Sufficient Conditions for Observability of NDS*

The results of the previous sections provide necessary and sufficient conditions for the observability of a NDS. These conditions do not depend on the homogeneity of the NDS, as the general conditions capture both scenarios. We combine the results into the following theorem.

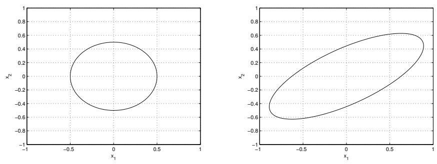
*Theorem 3.1:* Consider a NDS composed of homogeneous or heterogeneous dynamics that are individually observable. The NDS is unobservable if and only if the following conditions are met:

1) There exists an eigenvalue of $\mathbf{A}$, $\lambda^*$, that is common to each $A_i$
2) $C_i q_i = C_j q_j\ \forall\, i,j$ with $A_i q_i = \lambda^* q_i\ \forall\, i$.

*Proof:* The proof follows immediately from the proofs of Propositions 3.1, 3.2, and 3.3. ■

## IV. CHARACTERIZING OBSERVABILITY IN A NDS: INDEX OF HOMOGENEITY AND HETEROGENEITY

The previous section only provides a "yes" or "no" answer to the question of observability in a NDS. As discussed in §II-B, the singular values of the observability grammian can be used to give a quantitative comparison of the relative observability between different modes of the system. In the context of a single agent, the symmetry of the observability ellipsoid could be considered as a description of the homogeneity of that agents' initial condition to output map. As an example, the ellipsoid in Figure 1(a) is symmetric, which corresponds to the output energy being independent of the direction of the initial condition of the system. On the other hand, the ellipsoid in Figure 1(b) shows the output energy is strongly dependent on the direction of the initial condition.



(a) Symmetric Ellipsoid (b) Stretched Ellipsoid

Fig. 1. Visualization of observability grammian ellipsoids for a "symmetric" system and "stretched" system

The shape of the ellipsoid, of course, corresponds to the relative magnitude of the singular values of the observability grammian.

This notion can be extended for NDS to answer the following questions:

1) How does the structure of the underlying network topology affect the relative observability of the NDS?
2) How does the placement of agents in the network affect the relative observability of the NDS?

More fundamentally, these questions suggest that certain topologies in a homogeneous system might be "more homogeneous" then others. Similarly, placing heterogeneous agents in different locations of a NDS might result in a "more heterogeneous" NDS. This would correspond to a symmetry, or lack thereof, of the observability ellipsoid of the NDS.

This section aims to develop an *index of homogeneity* for homogeneous NDS, and an *index of heterogeneity* for heterogeneous NDS that can be used to answer these questions. It is natural that these measures should some how relate to the observability grammian of the NDS. As in §III, we separate the discussion into the homogeneous and heterogeneous settings.

### A. NDS Index of Homogeneity

In the homogeneous case, as indicated by (12), we recognize that the network topology has a direct affect on the observability grammian. Furthermore, the statement of Theorem 2.1 shows that the eigenvalues of $\mathbf{Y_o}$ are the eigenvalues of $Y_o$ scaled by the eigenvalues of the graph Laplacian, $L(\mathcal{G})$. The index of homogeneity should capture the affect of the network on the overall observability properties. Using the symmetry analogy developed earlier, a more homogeneous NDS should correspond to a more symmetric observability grammian.

The index of homogeneity will be denoted as $\rho(\Sigma_{hom}(\mathcal{G}))$. One choice for this index is

$$\rho(\Sigma_{hom}(\mathcal{G})) = \left(\frac{\lambda_2(\mathcal{G})}{\lambda_{|\mathcal{V}|}(\mathcal{G})}\right)\frac{\underline{\sigma}(Y_o)}{\overline{\sigma}(Y_o)}, \qquad (19)$$

where $\lambda_2(\mathcal{G})$ and $\lambda_{|\mathcal{V}|}(\mathcal{G})$ denote, respectively, the second smallest and largest eigenvalue of the graph Laplacian.

Using this index for characterizing the relative observability properties of the homogeneous NDS leads to some interesting observations. First, note that whenever the graph is disconnected, $\rho(\Sigma_{hom}(\mathcal{G})) = 0$. This corresponds to the intuitive result that a disconnected graph should somehow be "less homogeneous" than a connected one. In terms of this specific index, the homogeneity of the NDS is lower bounded by 0, and is indistinguishable from any disconnected graph on $|\mathcal{V}|$ nodes.

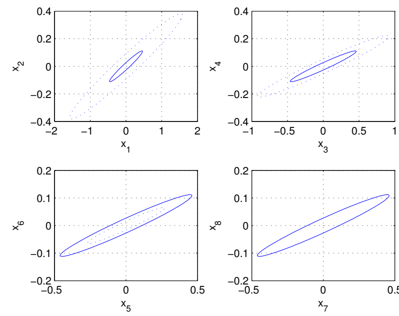


Fig. 2. Visualization of Observability Grammian Ellipsoids

This index is also upper-bounded by $\underline{\sigma}(Y_o)/\overline{\sigma}(Y_o)$. This upper-bound is achieved whenever the underlying graph is complete. The complete graph is the only graph where $\lambda_2(\mathcal{G}) = \lambda_{|\mathcal{V}|}(\mathcal{G})$.

Finally, we note the set of graphs that are cospectral with respect to the graph Laplacian will all result in the same index of homogeneity. This property could prove to be useful if reconfiguration of the connection topology is required.

The motivation for choosing such a function has a more intuitive explanation relating to the symmetry arguments of the observability grammian.

The term containing the ratio of the smallest and largest singular values of $Y_o$ corresponds loosely to a measure of the eccentricity of the grammian ellipsoid. The closer this ratio is to the value 1, the more symmetric the ellipsoid is. Conversely, as this ratio approaches 0, the ellipsoid becomes more elongated (along one plane). As we have assumed a minimal realization for the system dynamics, we are guaranteed that this ratio will always be strictly positive.

Next, consider the observability grammian of the parallel configuration of homogeneous NDS, corresponding to the system in (4). The grammian can be written as $\tilde{Y}_o = I_N \otimes Y_o$.

In the $N$ agent case, the ellipsoid of agent $i$ is oriented orthogonally to the ellipsoid of agent $j$. This is illustrated by the solid lines in Figure 2. In this example, we look at the grammian for a 4-agent homogeneous system with 2 states. The grammian for each agent is the same, and its 2-d projection is plotted for each pair of state variables.

When the parallel NDS is coupled by a network, say a path graph, the ellipsoid becomes scaled and rotated. This is visualized by the dotted lines in Figure 2. We immediately notice that one ellipsoid is scaled by the 0 eigenvalue of graph Laplacian. Using the statement of Theorem 2.1, we see that $\underline{\sigma}(\mathbf{Y}_o) = \lambda_2\,\underline{\sigma}(Y_o)$ and $\overline{\sigma}(\mathbf{Y}_o) = \lambda_N\,\overline{\sigma}(Y_o)$ are respectively, the minimum and maximum non-zero singular values of $\mathbf{Y}_o$. We thus have the following relationship:

$$0 < \lambda_2\,\underline{\sigma}(Y_o) \le \lambda_N\,\overline{\sigma}(Y_o). \qquad (20)$$

In the homogeneous NDS, $Y_o$ represents a fixed property of the system, determined by the agent dynamics. Thus, in terms of the symmetry argument, a more homogeneous NDS should preserve as closely as possible the shape of the grammian. Scaling the eigenvalues of $\mathbf{Y}_o$ by $\lambda_{|\mathcal{V}|}(\mathcal{G})\overline{\sigma}(Y_o)$ is

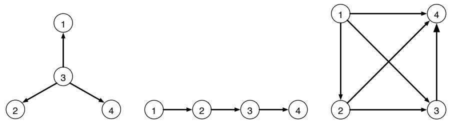


Fig. 3. Three topologies on 4 nodes

effectively normalizing the observability grammian singular values to 1.

### B. NDS Index of Heterogeneity

In the heterogeneous case, we wish not only to characterize how the topology affects the observability properties, but also how the placement of agents within that topology affects the observability of the NDS as well. Contrary to the homogeneous case, the interplay between the graph Laplacian eigenvalues and the eigenvalues of the NDS grammian is less straightforward. A nice property of the index of homogeneity is that it can be computed by studying-independently- the spectral properties of the graph and the observability properties of the homogeneous agents. Finding an analogous approach for the index of heterogeneity reduces to understanding the spectral properties of (14) or (18), which requires further examination.

An index of heterogeneity can be developed using the numerical evaluation of the grammian. The index of heterogeneity will be denoted as $\rho(\Sigma_{het}(\mathcal{G}))$. One choice for this index is

$$\rho(\Sigma_{het}(\mathcal{G})) = \left(\min_{\sigma_i(\mathbf{Y}_o)\neq 0} \sigma_i(\mathbf{Y}_o)\right)^{-1} \overline{\sigma}(\mathbf{Y}_o), \qquad (21)$$

where $\mathbf{Y}_o$ is given in (18).

Although not as transparent as the index of homogeneity, some useful observations can be made about this choice of index. It can be seen that the index is upper-bounded by 1, which corresponds to an upper-bound on the homogeneity of the NDS. It is interesting to note that this upper-bound can be achieved by a homogeneous NDS with a complete graph topology, and with the agent grammian ellipsoid being completely symmetric.

In fact, if all the agents in the NDS are homogeneous, then the index of heterogeneity reduces to (19). It might be natural to assume that the observed properties of (19) also apply to the heterogeneous case. Unfortunately, this is not the case, and is best illustrated with a simple example.

We consider a heterogeneous NDS with 4 agents and three different topologies. The topologies used are a star graph, a path graph, and the complete graph, which are shown in Figure 3.

Note that there are only four unique node assignments for the star graph, twelve unique assignments for the path graph, and one for the complete graph. For each permutation of the agent's position, the index of heterogeneity was calculated and plotted in Figure 4. As indicated in the above discussion, larger values of $\rho\left(\Sigma_{het}(\mathcal{G})\right)$ correspond to the NDS being "more homogeneous". The important point to notice in the figure is that the topology alone is not sufficient to determine which systems are more homogeneous. Furthermore, it can be seen that the complete graph does not correspond to the most homogeneous system.

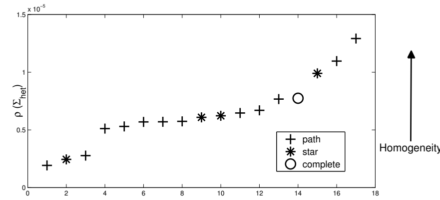


Fig. 4. Index of Heterogeneity

## V. Conclusions

In this paper we presented an observability analysis for certain classes of NDS. A distinction was made between NDS with homogeneous agent dynamics and NDS with heterogeneous agent dynamics. One important distinction between homogeneous and heterogeneous NDS relates to the observability of the system. In the homogeneous case, the NDS is always unobservable whereas in the heterogeneous case the NDS can be either observable or unobservable depending on the structure of the individual agent dynamics.

By studying the observability grammian, further quantitative results on the relative degree of observability were developed. In the homogeneous case, the relative degree of observability can be attributed to the underlying network topology. Specifically, the spectrum of the graph Laplacian provides a sufficient characterization of the relative degree of observability. In the heterogeneous case, an expression for the observability grammian was developed that involves a Hadamard product, or alternatively as a node weighted graph Laplacian. In both representations, the analysis of the spectrum is non-trivial and is the subject of ongoing research.

One of the contribution of the present paper is the introduction of an *index of homogeneity* and an *index of heterogeneity* for the two distinct classes of NDS. This index provides a quantitative means for comparing the homogeneity of different NDS which could be used as a performance metric for the synthesis of controllers and estimators for networked dynamic systems.

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