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On Design of Reduced-Order H_∞ Filters for Discrete-Time Systems from Incomplete Measurements

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***Abstract*—In networked control systems, when data are transmitted over wireless channels, measurements of the transmitted data are often incomplete due to transmission errors and/or packet losses. The problem of designing reduced-order H_∞ filters for discrete-time systems from incomplete measurements is investigated in this paper. A Bernoulli distributed white sequence is adopted as a model for the normal operating condition of packet delivery and transmission failure. The reduced-order filter to be designed from incomplete measurements is required to to ensure the mean-square stability of the filtering error system and to guarantee a prescribed H_∞ filtering performance level. It is shown that such a desired reduced-order H_∞ filter can be constructed under a sufficient condition expressed in terms of two linear matrix inequalities (LMIs) subject to a rank constraint.**

I. INTRODUCTION

The problem of H_∞ filtering for dynamic systems is concerned with designing an estimator which guarantees that the L_2-induced gain from the noise signals to the estimation error is less than a prescribed level. During the past decade, design of H_∞ filters has been an active area of research (see, e.g., [4], [6], [22], [25]).

It is noticed that in the existing literature most approaches to H_∞ filter design assume that the measurements of the system output are complete without any loss. How to use lossy measurements in estimating signals presents a practically important issue that requires much attention. The lossy measurements maybe arise in information transmissions across limited bandwidth wireless channels [2], [14]. In memoryless communication channel, the lossy measurement is commonly modeled as a stochastic Bernoulli process while in fading communication channel it is modeled as a finite-state Markov chain [3], [9]. Estimation with lossy measurement for dynamic systems has been a hot research topic recently [2], [5], [14]. Some signal estimation results for dynamic systems with lossy measurements have been reported in the literature [2], [14], [17], [21]. A jump estimation technique was presented to cope with lossy information in [2]. The authors in [14] investigated the Kalman filtering problem with intermittent observations. A variance-constrained filtering approach was proposed for systems with lossy measurements in [17].

On the other hand, in many real-world problems, there is need to utilize lower order filters, which has inspired the research on reduced-order H_∞ filtering. Based on the projection lemma, the authors in [19] proposed an LMI (linear matrix inequality) with rank constraint approach to the reduced-order H_∞ filter design for a class of stochastic systems. The reduced-order H_∞ filtering problem for linear systems with Markovian jump parameters was studied in [16]. More recently, an LMIs with equality constraint approach to the fixed-order H_∞ filtering problem of uncertain systems with Markovian jump parameters was presented in [18].

The purpose of this paper is to investigate the reduced-order H_∞ filtering problem for dynamic systems with lossy measurements. The works in [14], [19], [24] provide the impetus to carry out the present investigation. Specifically, we are interested in designing reduced-order filters by using lossy measurement such that the filtering error system is exponentially mean-square stable and a prescribed H_∞ filtering performance level is achieved.

Notations: Throughout this paper, Z^+ denotes the set of positive integers; $\mathbb{R}^n$ denotes the n dimensional Euclidean space; $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices. A real symmetric matrix $P > 0 (\geq 0)$ denotes P being a positive definite (or positive semi-definite) matrix, and $A > (\geq) B$ means $A - B > (\geq) 0$. I denotes an identity matrix of appropriate dimension. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations. The superscript 'τ' represents the transpose. $*$ is used as an ellipsis for terms that are induced by symmetry. For a $x \in \mathbb{R}^n$,

$$||x||^2 := x^\tau x.$$

Any matrix whose columns form the basis of the right null space of M is denoted by $\mathcal{N}(M)$ or $\mathcal{N}_M$. The notation $l_2[0,\infty)$ represents the space of square summable infinite vector sequences with the usual norm $\|\cdot\|_2$. A sequence

$$v = \{v_k\} \in l_2[0,\infty)$$

if

$$\|v\|_2 = \sqrt{\sum_{k=1}^{\infty} v_k^\tau v_k} < \infty.$$

$Prob\{.\}$ stands for the occurrence probability of an event; $\mathbb{E}\{.\}$ denotes the expectation operator with respect to some probability measure.

II. Problem Formulation

Consider the discrete-time dynamic system Σ:

$$x_{k+1} = Ax_k + A_\omega \omega_k \quad (1)$$

$$z_k = Lx_k + L_\omega \omega_k \quad (2)$$

where $x_k \in \mathbb{R}^n$ is the state; $\omega_k \in \mathbb{R}^{\bar{p}}$ is the deterministic disturbance signal in $l_2[0,\infty)$; $z_k \in \mathbb{R}^q$ is the signal to be estimated; and A, A_ω, L and L_ω are known constant matrices with compatible dimensions. The measurement is modeled by

$$y_k = Cx_k \quad (3)$$

$$y_{ck} = (1-\theta_k)y_k + \theta_k y_{k-1} \quad (4)$$

where $y_k \in \mathbb{R}^p$ is the output, $y_{ck} \in \mathbb{R}^p$ is the measured output, $C \in \mathbb{R}^{p\times n}$ is a known matrix, and the stochastic variable θ_k is a Bernoulli distributed white sequence taking value on 0 and 1 with

$$Prob\{\theta_k = 1\} = \mathbb{E}\{\theta_k\} = \rho \quad (5)$$

$$Prob\{\theta_k = 0\} = \mathbb{E}\{1-\theta_k\} = 1-\rho \quad (6)$$

where $\rho \in [0,1]$ and is a known constant.

Remark 1: The system measurement modeled in (3) and (4) was first introduced in [13] and has been used to characterize the effect of communication data loss in information transmissions across limited bandwidth communication channels over a wide area, such as navigating a vehicle based on the estimations from a sensor web of its current position and velocity [14]. The output y_k produced at a time k is sent to the observer through a communication channel. If no packet-loss occurs, the measurement output y_{ck} takes value y_k; otherwise, the measurement output y_{ck} takes value y_{k-1}. When the probability of event packet-loss occurring is assumed as ρ, the measurement output y_{ck} in (4) thus takes the value y_k with probability $1-\rho$, and the value y_{k-1} with probability ρ.

We consider the following filter for the estimation of z_k:

$$\left.\begin{array}{rcl}\hat{x}_{k+1} & = & A_f\hat{x}_k + B_f y_{ck} \\ \hat{z}_k & = & C_f\hat{x}_k + D_f y_{ck}\end{array}\right\} \quad (7)$$

where $\hat{x}_k \in \mathbb{R}^{\hat{n}}$, $0 < \hat{n} \le n$, and $\hat{z}_k \in \mathbb{R}^q$. A_f, B_f, C_f and D_f are to be determined.

Remark 2: The filter in the form of *(7)* reduces to a full order one when $\hat{n} = n$. There are many results for the designs of the full order filters (see, e.g., [12], [25]).

Combining (1)–(4) and (7) together, the filtering error dynamics can be represented as $\tilde{\Sigma}$:

$$\left.\begin{array}{rcl}\bar{x}_{k+1} & = & \mathcal{A}(\theta_k)\bar{x}_k + \mathcal{A}_1(\theta_k)H\bar{x}_{k-1} + \mathcal{A}_\omega\omega_k \\ \bar{z}_k & = & \mathcal{L}(\theta_k)\bar{x}_k + \mathcal{L}_1(\theta_k)H\bar{x}_{k-1} + \mathcal{L}_\omega\omega_k\end{array}\right\} \quad (8)$$

where

$$\left.\begin{array}{l}\bar{x}_k = \left[\begin{array}{cc} x_k^\tau & \hat{x}_k^\tau \end{array}\right]^\tau \\ \bar{z}_k = z_k - \hat{z}_k \\ H = \left[\begin{array}{cc} I, & 0 \end{array}\right]\end{array}\right\} \quad (9)$$

$$\left.\begin{array}{l}\mathcal{A}(\theta_k) = \left[\begin{array}{cc} A & 0 \\ (1-\theta_k)B_fC & A_f \end{array}\right] \\ \mathcal{A}_1(\theta_k) = \left[\begin{array}{c} 0 \\ \theta_k B_f C \end{array}\right] \\ \mathcal{A}_\omega = \left[\begin{array}{c} A_\omega \\ 0 \end{array}\right] \\ \mathcal{L}(\theta_k) = \left[\begin{array}{cc} L-(1-\theta_k)D_fC, & -C_f \end{array}\right] \\ \mathcal{L}_1(\theta_k) = -\theta_k D_f C \\ \mathcal{L}_\omega = L_\omega \end{array}\right\} \quad (10)$$

Let

$$\mathcal{F} = \left[\begin{array}{cc} A_f & B_f \\ C_f & D_f \end{array}\right] \quad (11)$$

It can be checked via (10) that

$$\begin{aligned}&\left[\begin{array}{ccc} \mathcal{A}(\rho) & \mathcal{A}_1(\rho) & \mathcal{A}_\omega \\ \mathcal{L}(\rho) & \mathcal{L}_1(\rho) & \mathcal{L}_\omega \end{array}\right] \\ = & \left[\begin{array}{cc|c|c} A & 0 & 0 & A_\omega \\ 0 & 0 & 0 & 0 \\ \hline L & 0 & 0 & L_\omega \end{array}\right] \\ & + \left[\begin{array}{cc} 0 & 0 \\ I & 0 \\ \hline 0 & -I \end{array}\right] \mathcal{F} \left[\begin{array}{cc|c|c} 0 & I & 0 & 0 \\ (1-\rho)C & 0 & \rho C & 0 \end{array}\right] \quad (12)\end{aligned}$$

where the ρ-dependent matrices are defined as in (10) with θ_k replaced by ρ.

Throughout the paper, we make the following assumptions for system (1)-(4).

Assumption 1: The matrix A is Schur stable *(*i.e., all eigenvalues of A are located within the unit circle in the complex plane*)*.

Assumption 2: $x_{-1} = 0$.

Remark 3: Assumption 1 is a common assumption in dealing with the filtering problem. Assumption 2 implies from (3) that

$$y_{-1} = 0,$$

which gives the initial condition for the lossy measurement model (4).

It is noted that the filtering error dynamics (8) is a system with stochastic parameters since some of the parametric matrices in (10) are associated with the stochastic variable θ_k. For the problem formulation, we adopt the notion of stochastic stability in the mean-square sense from [15].

Definition 1: The filtering error dynamics $\tilde{\Sigma}$ is said to be exponentially mean-square stable if with

$$\omega_k \equiv 0,$$

there exist constants $\alpha > 0$ and $\tau \in (0, 1)$ such that

$$\mathbb{E}\{\|\bar{x}_k\|^2\} \le \alpha\tau^k\mathbb{E}\{\|\bar{x}_0\|^2\},$$
$$\text{for all } \bar{x}_0 \in \mathbb{R}^{n+\hat{n}},\ k \in Z^+.$$

The H_∞-type filtering problem addressed in this paper is to design a filter in the form of (7) such that the filtering error system $\tilde{\Sigma}$ is exponentially mean-square stable and under the zero initial condition, the filtering error $\bar{z}_k$ satisfies

$$\sum_{k=0}^{\infty}\mathbb{E}\{\|\bar{z}_k\|^2\} \le \gamma^2\sum_{k=0}^{\infty}\|\omega_k\|^2 \quad (13)$$

for a given scalar γ and all nonzero ω_k. In such a case, the filtering error system is called to be exponentially mean-square stable with H_∞ filtering performance γ.

To begin with, we establish a condition of mean-square stability and H_∞ performance for the filtering error dynamics $\tilde{\Sigma}$, which will be fundamental in the derivation of our H_∞ filter design methodology.

Lemma 1: Consider the filtering error dynamics $\tilde{\Sigma}$. Given a scalar $\gamma > 0$, the filtering error system $\tilde{\Sigma}$ is exponentially mean-square stable and has a guaranteed γ level of disturbance attenuation, if there exist matrices P and Q such that

$$\begin{bmatrix} -P & 0 & P\mathcal{A}(\rho) & P\mathcal{A}_1(\rho) & P\mathcal{A}_\omega \\ * & -I & \mathcal{L}(\rho) & \mathcal{L}_1(\rho) & \mathcal{L}_\omega \\ * & * & H^\tau QH - P & 0 & 0 \\ * & * & * & -Q & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix} < 0 \quad (14)$$

where $*$ denotes the corresponding transposed block matrix due to symmetry.

III. Reduced-order H_∞ Filter Design

Based on Lemma 1 we will give a sufficient condition for the existence of the H_∞ filter in the form of (7) and present a method to construct the filter. We first give the following lemma (i.e., Projection Lemma) which will be used in the derivation of the main result in this section.

Lemma 2: [8] Given a real symmetric Ψ and two real matrices U and V, the following linear matrix inequality problem

$$\Psi + U^\tau X^\tau V + V^\tau XU < 0$$

is solvable with respect to X if and only if

$$\mathcal{N}_U^\tau \Psi \mathcal{N}_U < 0$$

$$\mathcal{N}_V^\tau \Psi \mathcal{N}_V < 0$$

where $\mathcal{N}_U$ and $\mathcal{N}_V$ denote matrices whose columns form bases of right null spaces of U and V, respectively.

The following theorem provides us with a solution to the reduced-order H_∞ filtering for dynamic systems with lossy measurements in terms of two linear matrix inequalities and a rank constraint condition.

Theorem 1: Consider an H_∞ filter (7), of order $\hat{n}$, with the H_∞ filtering performance level γ, for system (1)-(2) with lossy measurements (3)-(4). Suppose that $0 < \hat{n} \leq n$. There exist a filter matrix $\mathcal{F}$, and matrices P and Q satisfying (14), if and only if there exist matrices $X > 0,\ Y > 0$ and $Q > 0$ such that

$$\begin{bmatrix} -Y & YA_\omega & 0 & YA \\ * & -\gamma^2 I & 0 & 0 \\ * & * & -Q & 0 \\ * & * & * & Q-Y \end{bmatrix} < 0 \quad (15)$$

$$\begin{bmatrix} -X & 0 & XA_\omega & XA\mathcal{N}_2 \\ * & -I & L_\omega & L\mathcal{N}_2 \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -\mathcal{N}_1^\tau Q\mathcal{N}_1 + \mathcal{N}_2^\tau (Q-X)\mathcal{N}_2 \end{bmatrix} < 0 \quad (16)$$

$$X - Y \geq 0 \quad (17)$$

where

$$\begin{bmatrix} \mathcal{N}_1 \\ \mathcal{N}_2 \end{bmatrix} := \mathcal{N}_{[\ \rho C \quad (1-\rho)C\ ]} \quad (18)$$

and

$$\mathrm{rank}(X - Y) \leq \hat{n} \quad (19)$$

In this case, if matrices $X,\ Y$, and Q are solutions to linear matrix inequalities (15)-(17) with rank constraint (19), then there always exist matrices $X_{22} \in \mathbb{R}^{\hat{n}\times\hat{n}}$ with $X_{22} > 0$ and $X_{12} \in \mathbb{R}^{n\times\hat{n}}$ satisfying

$$X_{12}X_{22}^{-1}X_{12}^\tau = X - Y \quad (20)$$

The parametric matrix $\mathcal{F}$ defined in (11), of the filter in the form of (7) with order $\hat{n}$, can be obtained via solving the linear matrix inequality:

$$\Psi + U^\tau \mathcal{F}^\tau V + V^\tau \mathcal{F} U < 0 \quad (21)$$

where

$$\Psi = \left[\begin{array}{cc|c|cc|c|c} -X & -X_{12} & 0 & XA_\omega & 0 & 0 & XA \\ * & -X_{22} & 0 & X_{12}^\tau A_\omega & 0 & 0 & X_{12}^\tau A \\ \hline * & * & -I & L_\omega & 0 & 0 & L \\ * & * & * & -\gamma^2 I & 0 & 0 & 0 \\ \hline * & * & * & * & -X_{22} & 0 & X_{12}^\tau \\ \hline * & * & * & * & * & -Q & 0 \\ \hline * & * & * & * & * & * & Q-X \end{array}\right] \tag{22}$$

$$U = \left[\begin{array}{cc|c|cc|c|c} 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho C & (1-\rho)C \end{array}\right] \tag{23}$$

$$V = \left[\begin{array}{cc|c|cc|c|c} X_{12}^\tau & X_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & 0 \end{array}\right] \tag{24}$$

Remark 4: Theorem 1 provides not only a sufficient condition for the solvability of the reduced-order H_∞ filtering problem for the discrete-time systems with lossy measurement, but also an equivalent condition to Lemma 1. This equivalence implies that the filter design result derived from Lemma 1 is more general. It should be pointed out that the rank-constrained linear matrix inequalities (15)-(19) are non-convex due to the rank constraint (19). Many techniques have been presented to solve rank-constrained linear matrix inequalities (see [7], [11] and the references therein).

IV. Conclusions

This paper has examined the problem of designing reduced-order H_∞ filters for a class of discrete-time systems with lossy measurements. The main contribution has been the development of a reduced-order H_∞ filter design approach by using the projection lemma. The solvability of the reduced-order H_∞ filtering problem with lossy measurements has been linked to the feasibility of two linear matrix inequalities with a rank constraint, which significantly facilitates finding out the desired solution.

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