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A Hopf-Algebraic Formula for Compositions of Noncommuting Flows

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Abstract— **The Chen-Fliess series is known to be an exponential Lie series. Previously explicit formulas for the iterated integral coefficients were known only for its factorization into a directed infinite product of exponentials. This factorization uses Hall sets and the Zinbiel product.**

We use the underlying Hopf algebra structure to derive explicit formulas for the corresponding coefficients in the logarithm of the series. This allows one to express the series as a single exponential.

This work is closely related to Fer and Magnus expansions, and has interpretations in terms of a continuous Campbell-Baker-Hausdorff formula. The result facilitates work in nonlinear control, numerical integration and various applications that involve compositions of noncommuting flows.

I. Introduction and prior work

Solutions formulas in terms of infinite series have a long history in the context of both uncontrolled and controlled dynamical systems, e.g. [2]. They also underlie most numerical integration schemes for differential equations.

Series expansions are of particular interest for systems that naturally split – due to different scales involved, or due to the presence of distinguished control or perturbation terms. Our interest lies in systems defined by a collection of analytic vector fields f_i on $\mathbb{R}^n$ with time varying locally integrable coefficients u_i

$$\dot{x} = \sum_{i=1}^{m} u_i(t) f_i(x), \qquad x \in \mathbb{R}^n,\ u \in U \subseteq \mathbb{R}^m. \tag{1}$$

In the general control setting u typically takes values in a compact convex subset of $\mathbb{R}^m$, often normalized to a ball or the cube $[-1,1]^m$. Bang-bang optimal controls might take values in the vertex set $\{-1,1\}^m$, whereas in hybrid or switched systems one might require $u \in \{0,1\}^m$ with the additional constraint $\sum u_i = 1$. Commonly one specializes to piecewise constant u, in the extreme case as simple as $(u_1, u_2) = (1, \varepsilon)$ in averaging theory and perturbation methods. Generally the flows of the vector fields f_i do not commute. It is primary interest to construct the solution of (1) from the knowledge of the flow of each f_i. One such well-studied application is the numerical integration of ordinary differential equations with algebraic constraints, especially geometric integration schemes on manifolds.

Volterra series [36] and the Campbell-Baker-Hausdorff series [3] which originated in the late 1800s are distinguished by rapidly increasing complexity of higher order terms. Similarly, the expansions introduced in the 1950s by e.g. Chen [5], Fer [8], Magnus [23], as well as Fliess' interpretation [9] of the latter in terms of control systems, and Agrachev's and Gamkrelidze's exponential representations [1] also suffer from many redundant terms. The quest to simplify these expansion to the minimal number of terms and to obtain explicit formulas for the kernels and iterated integral coefficients has been going hand-in-hand with efforts to uncover the underlying algebraic and combinatorial structures. It is now well-understood that instead of directly manipulating complicated iterated integrals, it is advantageous to encode these by simple algebraic objects such as linear combinations of *words* or labeled trees, and utilize their respective algebraic structures. In this context, the original Chen series and Fliess' reinterpretation of it may be recognized as images of the identity map on the free associative algebra under suitable algebra homomorphisms. For analytic vector fields f_i in (1), the corresponding images of formal power series are interpreted as asymptotic series which will converge uniformly on compact sets, compare e.g. [30], [33]. The focus here is on the combinatorics and algebra, and we will not discuss convergence issues in detail.

While earlier work focused on Lie and shuffle algebra structures, it is now understood that even more basic pre-Lie structures lead to deeper insights and simpler formulas. In particular, the coefficients in the factorization of the identity into an infinite directed product of exponentials indexed by a Hall set take a particularly simple form in terms of a product that generates a Zinbiel algebra – the name coined by Loday [22] in the context of studying Leibniz algebras. In terms of the iterated integral functionals in control, this product maps to what we may consider *the product of nonlinear control theory*, e.g., for absolutely continuous functions $U, V \colon [0,T] \mapsto \mathbb{R}$

$$(U * V)(t) = \int_0^t U(s) \cdot V'(s)\, ds. \tag{2}$$

This fundamental building block of nonlinear control systems is easily verified to satisfy the right Zinbiel identity

$$U_1 * (U_2 * U_3) = (U_1 * U_2) * U_3 + (U_2 * U_1) * U_3. \tag{3}$$

Recent work by Ebrahimi-Fard et.al [7], [6] uses dendriform algebras, which basically combine simultaneous right and left Zinbiel structures together with a compatibility condition, to demonstrate the intimate relationships between Fer and Magnus expansions. However, that work does not (yet) utilize bases to minimize the size of the expansions in a free setting. This article combines Hall bases and the Zinbiel product with a Hopf-algebra approach. Starting from formulas for the *coordinates of the second kind* (in a product

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of exponentials) which are based on Hall sets and Zinbiel products, we derive formulas for the corresponding formulas for *coordinates of the first kind* (in the exponential of a series).

Earlier work by the second author [18] obtained formulas for the first 14 terms in the logarithm of the Chen-Fliess series by a brute force calculation relying on computer algebra. Recently, Murua [28] and Rocha [31] also proposed formulas for these coordinates of the first kind. Further related work by Gray and Li [21], [11] is on the Laplace-Borel transform of Fliess operators, by Ignatovich and Sklyar [14] on power moments, by Monaco, Normand-Cyrot and Califano [25] also consider discrete-time dynamics in their description of the Magnus exponent. In the last few years, much related activity has been centered about uncovering the algebraic structures that underlie geometric (numerical) integration algorithms, compare e.g. the work by Iserles [4], [15] Ebrahimi-Fard [7], [6], Munthe-Kaas [26], and co-authors, and the works cited therein.

In the setting of control theory the directed infinite exponential product expansion of Sussmann [34] has given significant insight into the geometry, and much simplified both analysis of optimality and controllability, and applications such as tracking and path planning. One commonly restricts attention to finitely parameterized families of controls and try to invert the *endpoint map* to solve for parameter values that steer the system as desired. Typical examples include Jacob [16] using piecewise constant and polynomial inputs, Lafferierre and Sussmann [20], and Murray and Sastry [27] using sinusoidal inputs. Working with a single exponential (of a Lie series), rather than an infinite product of exponentials is expected to simplify such applications.

The organization of this article is as follows: After this introduction surveying the history and relevance of the problem, the next sections quickly review some algebraic and combinatorial background, formally state the main result and provide some illustrative example calculations and formulas. For detailed proofs and complete technical statements of the algebraic background we refer the reader to [17], [19], [30] and the references therein.

II. Technical background

Consider a set Z, henceforth called *alphabet*. Elements $a \in Z$ are called letters. For the purpose of this note assume Z is finite, $|Z| = m$, and *indexes* the set of vector fields f_a and controls u_a in (1). For $k \in \mathbb{Z}_0^+$ let Z^k denote the set of all sequences of length k with values in Z. The empty word is denoted $e \in Z^0$. Concatenation of sequences endows the sets $Z^* = \cup_{k=0}^{\infty} Z^k$ and $Z^+ = \cup_{k=1}^{\infty} Z^k$ with a natural (noncommutative, associative) product structure. We use juxtaposition $w = a_1 a_2 \ldots a_k \in Z^k$ to denote both this product and sequences of letters, and henceforth call such w a *word*.

The product extends linearly to the algebra $\mathcal{A}$ of all finite linear combinations of words in Z^* with coefficients in a field $\mathbf{k}$, here taken to be $\mathbb{R}$. Declaring $Z^* \subseteq \mathcal{A}$ to be an orthonormal basis equips $\mathcal{A}$ with an inner product. Write $\mathcal{A}^+$ for the subalgebra spanned by Z^+.

Given a collection of m analytic vector fields $\mathcal{F} = \{f_a : a \in Z\}$ on $\mathbb{R}^n$, the map $\mathrm{Ev}_{\mathcal{F}} : a \mapsto f_a$ naturally extends to an associative algebra homomorphism from $\mathcal{A}$ to the algebra of partial differential operators under compositions. For a polynomial $p \in A$ write f_p for the corresponding partial differential operator. In the sequel we will refine the view of this map, and introduce corresponding maps for the controls, actually for functionals on the space of controls.

The algebra $\mathcal{A}$ contains various subalgebras of interest, and it gives rise to a variety of other product structures. Use $[\cdot,\cdot] : \mathcal{A} \times \mathcal{A} \mapsto \mathcal{A}$ to denote the commutator defined by $[w,z] = wz - zw$. The smallest Lie subalgebra of $\mathcal{A}$ that contains Z and is closed under $[\cdot,\cdot]$ is denoted $L(Z)$ and is isomorphic to the free Lie algebra on m generators. Hence, the map $a \mapsto f_a$ immediately extends to a Lie algebra homomorphism from $L(Z)$ to the Lie algebra of analytic vector fields on $\mathbb{R}^n$.

Define a (right) Zinbiel algebraic structure on $\mathcal{A}^+$, compare the identity (3), by defining the product $* : \mathcal{A}^+ \times \mathcal{A} \mapsto \mathcal{A}$ for nonempty words $w, z \in Z^* \setminus \{e\}$, and any letter $a \in Z$ by $e * w = w$, $w * a = wa$,

$$w * (za) = (w * z)a + (z * w)a \tag{4}$$

and extending linearly to $\mathcal{A}^+ \times \mathcal{A}$. Note that $e * e$ remains undefined. The symmetrization of this product

$$w \sqcup\!\sqcup\, z = w * z + z * w \tag{5}$$

extends to $\sqcup\!\sqcup : \mathcal{A} \times \mathcal{A} \mapsto \mathcal{A}$ by setting $e \sqcup\!\sqcup\, e = e$, and equips $\mathcal{A}$ with the familiar (associative) shuffle algebra. To relate these abstract algebras to the control setting, suppose $\mathcal{U} = AC_0([0,T],\mathbb{R})$ is the space of absolutely continuous scalar valued functions defined on an interval $[0,T] \subseteq \mathbb{R}$ that vanish at zero. (Think of their derivatives as the controls u_a in (1)). Consider the set $\mathrm{Map}(\mathcal{U}^Z,\mathcal{U})$ of all mappings from the product $\mathcal{U}^Z$ (of $|Z|$ copies of $\mathcal{U}$ indexed by Z) to $\mathcal{U}$, and define a linear map $\Upsilon : \mathcal{A} \mapsto \mathrm{Map}(\mathcal{U}^Z,\mathcal{U})$ on letters $a \in Z$ for $U \in \mathcal{U}^Z$ by setting $\Upsilon_a(U) = U_a$, and on words $w \in Z^+$ and $a \in Z$ by setting for $U \in \mathcal{U}^Z$ and $t \in [0,T]$

$$\Upsilon_{wa}(U)(t) = \int_0^t \Upsilon_w(U)(s) \cdot U_a'(s)\, ds \tag{6}$$

It is readily seen that this map Υ is indeed a right Zinbiel algebra homomorphism and the image $\mathcal{IIF}(\mathcal{U}) = \Upsilon(\mathcal{A})$ with the product (2) is the right Zinbiel algebra of *iterated integral functionals* on $\mathcal{U}^Z$. Under suitable hypotheses on $\mathcal{U}$ the map Υ is actually injective [17]. Denote by $\hat{\mathcal{A}}, \hat{\mathcal{A}}^+, \hat{L}(Z)$ the completions of the spaces of (Lie) polynomials to (Lie) series with the usual topologies. Most of the maps and products discussed above carry over to series in a natural way. For full technical details see [17].

One recognizes the Fliess series of system (1) essentially as the image of the identity under the map $\Upsilon \otimes \mathrm{Ev}_{\mathcal{F}} : \mathcal{A} \otimes \mathcal{A} \mapsto \mathrm{Map}(U^Z, U)$. Here the identity map $\mathrm{Id}_{\mathcal{A}} : \mathcal{A} \mapsto \mathcal{A}$ is identified with the element $\sum_{w \in Z} w \otimes w \in \mathcal{A} \otimes \hat{\mathcal{A}}$, using the usual identification of $\mathrm{Hom}(V, W) = V^* \otimes W$ for vector spaces V and W (leaving the details about the dual spaces of power series versus polynomials to [17], [30]). More specifically, the Fliess series of an analytic *output map* $\phi\colon \mathbb{R}^n \mapsto \mathbb{R}$ evaluated along a solution curve of (1) starting at $x(0) = 0$ with integrable control $u = U'$ is, for t sufficiently small,

$$\phi(x(t,u)) = \sum_{w\in Z^*} \Upsilon_w(U)(t) \cdot (f_w\phi)(0). \tag{7}$$

Since Υ is a Zinbiel algebra homomorphism and the shuffle product is the symmetrization of the Zinbiel product $w \sqcup\!\sqcup\, z = w * z + z * w$ it is immediate that Υ also maps shuffles to the symmetrization of the Zinbiel product (2). Using elementary calculus this is seen to be the pointwise product in $\mathcal{U}$,

$$\begin{aligned}\Upsilon_{w \sqcup\!\sqcup\, z}(U)(t) &= \Upsilon_{w*z}(U)(t) + \Upsilon_{z*w}(U)(t)\\ &= \int_0^t \left(\Upsilon_w(U)\left(\Upsilon_z(U)\right)' + \Upsilon_z(U)\left(\Upsilon_w(U)\right)'\right)(s)\,ds\\ &= \left(\Upsilon_w(U)\cdot\Upsilon_z(U)\right)(t).\end{aligned} \tag{8}$$

Using the classical language this reestablishes that the coefficients of the Fliess series *satisfy the shuffle relations*, and hence by Ree's theorem [29] the underlying series is the exponential of a Lie series. Thus it is clear that for any set $\mathcal{H}$ that encodes a basis for the free Lie algebra over Z there exist well-defined functions $\zeta,\ \xi\colon \mathcal{H} \mapsto \mathcal{A}$, called coordinates of the first and second kind, respectively, such that in the complete tensor algebra $\mathcal{A}\tilde{\otimes}\hat{\mathcal{A}}$ with the shuffle product on the left and the concatenation product on the right

$$\sum_{w\in Z^*} w\otimes w = \exp\left(\sum_{H\in\mathcal{H}} \zeta_H \otimes [H]\right) \tag{9}$$

$$\sum_{w\in Z^*} w\otimes w = \overleftarrow{\prod_{H\in\mathcal{H}}} \exp\left(\xi_H\otimes[H]\right). \tag{10}$$

Here $[H]$ denotes the Lie polynomial corresponding to the tree H, see below for details. The images of these expansions under the map $\Upsilon\otimes \mathrm{Ev}_{\mathcal{F}}$ are effectively a continuous version of the Campbell-Baker-Hausdorff formula, and an exponential product expansion of the Chen-Fliess series.

The utility of these expansions hinges on the knowledge of explicit bases $\mathcal{H}$ of the free Lie algebra, and on explicit formulas for the maps ξ and ζ which ultimately encode the iterated integral functionals. Several bases for free Lie algebras were introduced in the past, but shown by Viennot [35] to effectively all be special cases of (generalized) Hall sets. A Hall set over an alphabet Z is a strictly ordered subset $\tilde{\mathcal{H}} \subseteq \mathcal{T}(Z)$ of the set of binary trees labeled by Z that satisfies

- $Z \subseteq \tilde{\mathcal{H}}$,
- for $w \in Z^*$ and $a \in Z$, $(w, a) \in \tilde{\mathcal{H}}$ if and only if ($w \in \tilde{\mathcal{H}}$, $w \prec a$ and $a \prec (w, a)$), and (11)
- for $u, v, w, (u, v) \in \tilde{\mathcal{H}}$ $(u,(v,w)) \in \tilde{\mathcal{H}}$ if and only if ($v \preceq u \preceq (v, w)$ and $u \prec (u,(v,w))$).

There is a natural map, with slight abuse of notation conveniently also denoted by the symbol for the commutator, $[\cdot]\colon \mathcal{M}(Z) \mapsto L(Z)$ that maps binary trees $a \in Z$ and (t, t') labeled by Z to the Lie polynomials $[a] = a$ and $[(t, t')] = [t][t'] - [t'][t]$. Due to properties of the Lazard elimination process the image $[\tilde{\mathcal{H}}] \subseteq L(Z)$ of a Hall set forms a basis for the free Lie algebra $L(Z)$.

The foliage map $\beta\colon \mathcal{M}(Z) \mapsto Z^*$ sends a labeled tree to the word (sequence) that labels its leafs, i.e. for $a \in Z$, $\beta(a) = a$, and recursively $\beta(t, t') = \beta(t)\beta(t')$.

A characteristic property of Hall sets is that the restriction of this map β to a Hall-set is injective [35]. This allows one to work with the words in the image $\mathcal{H} = \beta(\tilde{\mathcal{H}})$ rather than the binary trees themselves in the sequel. But care has to be taken here as e.g. $[a,[b,[a,b]]] = [b,[a,[a,b]]]$ as Lie elements, but only the word $baab = \beta((b(a(ab))))$ is in the image of the standard Hall set – i.e. Lie products do not have well defined left and right factors, whereas nontrivial binary trees have, as do the words that are the foliage of Hall trees. As a consequence, the functions ζ and ξ are defined not on a Hall basis $[\tilde{\mathcal{H}}] \subseteq L(Z)$, but rather on the Hall-set $\tilde{\mathcal{H}}$, or by convenience on the image $\mathcal{H} = \beta(\tilde{\mathcal{H}}) \subseteq Z^*$.

Explicit formulas for the coordinates of the second kind ξ for Hall sets have been discovered by various authors in different settings – including Schützenberger [32], Sussmann [34], Grossman [12], and for the special case of Lyndon bases by Melançon and Reutenauer [24]. We now understand that the elegance and simplicity of the formula for the function ξ results from a perfect match of the Zinbiel product with the Lazard elimination process which yields both Hall sets and underlies the convergence of solutions of differential equations by iteration or by recursive variation of parameters. For letters $a \in Z$ and Hall words $H, K, HK \in \mathcal{H}$ one has

$$\begin{aligned}\xi_a &= a\\ \xi_{HK} &= \mu_{HK}\cdot\xi_H * \xi_K\end{aligned} \tag{12}$$

where μ is a simple normalization factor in terms of multifactorials that depends on the number of repetitions of subtrees in the tree $\beta^{-1}(HK)$.

An immediate application is a generalization of the controllable normal form of linear systems to maximally accessible nilpotent systems – now indexed by a finite subset of a Hall set, e.g. all Hall words of length less than r.

$$\begin{aligned}\dot{x}_a &= u_a\\ \dot{x}_{HK} &= x_H\cdot\dot{x}_K\end{aligned} \tag{13}$$

It is straightforward to recursively expand the equations to rewrite it in the standard form (1). The Lie algebra generated by the corresponding fields f_a will be nilpotent of order r and span the tangent space at every point, i.e. the system is accessible. For details see e.g. [19].

The quest for similarly simple and elegant formulas for the function ζ is continuing. Rocha [31] and Murua [28] have proposed algorithms and formulas that should map to formulas for ζ, but they are considerably more complicated than (12). Using a brute force computer algebra calculation formulas for ζ_h for the first 14 Hall words h over a two-letter alphabet were computed in [18]. These may be enough for some numerical integration algorithms, but do not satisfy the needs for deeper analytic studies and for tracking and path planning algorithms, or to study e.g. high order conditions for optimal controls. In the next section we provide a simple formula in terms of natural maps in Hopf algebras that allows one to quickly calculate ζ_h from the known ξ_h.

III. Maps in the Hopf algebra

Hopf algebraic structures of trees and differential operators are well studied, see e.g. [13]. More recently [26] uses them for closely related geometric integration algorithms, and Murua [28] uses a Hopf algebra of labeled rooted trees for computations of exponentials of Lie series and to provide formulas for a continuous Baker-Campbell-Hausdorff formula. Here we follow the terminology and notation introduced in [30] for studying free Lie algebras.

The algebra $\mathcal{A}$ is endowed with two Hopf algebra structures, that is, two matching triples each consisting of a product, co-product, and an antipode. The first uses the concatenation product as the product and uses as co-product the concatenation-algebra homomorphism $\Delta\colon \mathcal{A} \mapsto \mathcal{A}\otimes\mathcal{A}$ defined on letters $a \in Z$ by

$$\Delta(a) = e\otimes a + a\otimes e. \tag{14}$$

For example, for letters $a, b \in Z$, one has

$$\Delta(ab) = e\otimes ab + a\otimes b + b\otimes a + ab\otimes e. \tag{15}$$

This coproduct is just the transpose of the shuffle product, i.e., for all $u, v, w \in \mathbb{Z}^*$,

$$\langle u \sqcup\!\sqcup\, v, w\rangle = \langle u\otimes v, \Delta(w)\rangle. \tag{16}$$

A second Hopf algebra structure uses the shuffle product as its product and uses as its co-product the shuffle-algebra homomorphism $\Delta'\colon \mathcal{A}\mapsto\mathcal{A}\otimes\mathcal{A}$ that satisfies

$$\Delta'(w) = \sum_{u,v\in Z^*} \langle w, uv\rangle\; u\otimes v. \tag{17}$$

Both Hopf algebra structures use the same antipode $\alpha\colon A\mapsto A$, the linear map that maps any word $w = a_1a_2\ldots a_k \in Z^k$ to its signed reversal

$$\alpha(w) = (-1)^k\, a_k\ldots a_2a_1. \tag{18}$$

Each Hopf algebra structure on gives rise to an associative convolution product of linear endomorphism of the algebra $\mathcal{A}$. For linear endomorphisms $f, g\colon \mathcal{A}\mapsto\mathcal{A}$ define

$$\begin{aligned} f\star g &= \operatorname{conc}\circ(f\otimes g)\circ\Delta, \quad\text{and}\\ f\star' g &= \operatorname{shu}\circ(f\otimes g)\circ\Delta', \end{aligned} \tag{19}$$

i. e., for any word $w\in\mathcal{A}$

$$(f\star g)(w) = \sum_{u,v\in Z^+} \langle w, u \sqcup\!\sqcup\, v\rangle\; f(u)g(v) \quad\text{and} \tag{20}$$

$$\begin{aligned} (f\star' g)(w) &= \sum_{u,v\in Z^+} \langle w, uv\rangle\; f(u) \sqcup\!\sqcup\, g(v)\\ &= \sum_{\substack{uv=w\\ u,v\in Z^+}} f(u) \sqcup\!\sqcup\, g(v). \end{aligned} \tag{21}$$

It appears that confusion of these two different maps has led to incorrect formulas in some of the existing literature. Note that while the sum in the first convolution product $\star$ is over all pairs of words u, v such that the given word w appears with nonzero coefficient in the shuffle of u and v, the second product is conceptually much easier to evaluate. Fortunately, it is the latter which we need in our eventual computations of the map ζ.

However, the first convolution product is required for the projections onto the fundamental subspaces. Write $I\colon\mathcal{A}\mapsto\mathcal{A}^+$ for the projection that maps the empty word to zero and that is the identity on Z^+. The key map is the logarithm of the identity in the $\star$ convolution algebra of endomorphisms on $\mathcal{A}$. Using iterated convolution products, this is

$$\pi_1 = \sum_{k\geq 1} (-1)^{k-1}\tfrac{1}{k} I^{\star k}. \tag{22}$$

Using (20) this may also expressed in terms of higher k-ary analogues of the coproduct and concatenation product, which is the form commonly employed in example calculations

$$\pi_1 = \sum_{k\geq 1} (-1)^{k-1}\tfrac{1}{k}\operatorname{conc}^k\circ I^{\otimes k}\circ\Delta^k. \tag{23}$$

It is not hard to see that the image of $\mathcal{A}$ under π_1 is contained in $L(Z)$, and that the restriction of π_1 to $L(Z)$ is the identity on $L(Z)$. However, π_1 is *not* an orthogonal projection with respect to the standard inner product. To see the connection between the kernel of π_1 and its higher powers w.r.t. the convolution product, first consider the symmetrization of the concatenation product on $\mathcal{A}$. More specifically, the symmetric product of any finite collection $p_1,\ldots p_k\in\mathcal{A}$ of polynomials in $\mathcal{A}$ is the usual multi-linear map defined by

$$\operatorname{Sym}(w_1,\ldots w_k) = \tfrac{1}{k!}\sum_{\sigma\in S_k} w_{\sigma(1)}w_{\sigma(2)}\ldots w_{\sigma(k)} \tag{24}$$

where the sum is taken over the symmetric group of order k. Specializing to symmetric products of Lie polynomials $\ell\in L(Z)$, define the fundamental subspaces U_k, $k\in\mathbb{Z}^+$, as the linear subspaces spanned by all symmetric products of k Lie polynomials. In particular, $U_1 = L(Z)$ and U_2 is the linear span of all polynomials of the form $\ell_1\ell_2+\ell_2\ell_1$ where ℓ_1, $\ell_2\in L(Z)$. These fundamental subspaces provide a direct sum composition of the algebra $\mathcal{A}$

$$\mathcal{A} = \bigoplus_{k=1}^{\infty} U_k \tag{25}$$

and it turns out that the associated projections $\pi_k\colon\mathcal{A}\mapsto U_k$ are nothing else than the normalized k-fold convolutions of π_1 with itself,

$$\pi_k = \tfrac{1}{k!}\pi_1^{\star k}. \tag{26}$$

IV. The main result and examples

The definition (9) makes clear that the desired function ζ takes value on *the other* side, i.e., whereas π_k projects onto the fundamental subspace U_k, ζ takes values in the duals.

Thus define the map $\pi_1' \colon \mathcal{A} \mapsto \mathcal{A}$ as the adjoint map of π_1, i.e. the linear map for all $w, z \in \mathbb{Z}$

$$\langle w, \pi_1(z) \rangle = \langle \pi_1'(w), z \rangle. \tag{27}$$

From the definition (22) of π_1 in terms of the convolution product $\star$, and that I is selfadjoint, conc and Δ' are adjoints of each other, as are $\sqcup\!\sqcup$ and Δ, one immediately obtains

$$\pi_1' = \sum_{k \geq 1} (-1)^{k-1} \tfrac{1}{k} I^{\star' k}. \tag{28}$$

Note that this formula uses the second convolution product $\star'$, and on words $w \in Z$ one readily calculates

$$\pi_1'(w) = \sum_{k \geq 1} \tfrac{1}{k} (-1)^{k-1} \sum_{u_1 \ldots u_k = w} u_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup u_k$$

As a simple example calculate

$$\pi_1'(\xi_{ab}) = \pi_1'(ab) = ab - \tfrac{1}{2}\, a \sqcup\!\sqcup b = \tfrac{1}{2}\, (ab - ba). \tag{29}$$

It is easy to see that ζ and π_1' both take values in the same subspace U_1', but it takes work, for details see [10], to see that π_1' actually maps ξ_h elementwise onto ζ_h for every Hall word $h \in \mathcal{H}$.

Theorem 1: For every Hall set $\mathcal{H} \subseteq Z^*$, for all $h \in \mathcal{H}$

$$\zeta_h = \pi_1'\, (\xi_h). \tag{30}$$

Or, using function notation, simply $\zeta = \pi_1' \circ \xi$.

Recall the characteristic property of Hall sets that every word $w \in Z^*$ factors uniquely into a decreasing product of Hall words, i.e. there exists Hall words $h_1 \geq h_2 \geq \ldots \geq h_k$ such that $w = h_1 h_2 \cdots h_k$. Such factorizations play a fundamental role in the Poincaré-Birckhoff-Witt basis of $\mathcal{A}$ when regraded as the enveloping algebra of $L(Z)$. Indeed, it is common to extend the domains of ξ and ζ to all of $\mathcal{A}$, and one finds that if $w = h_1^{r_1} h_2^{r_2} \cdots h_k^{r_k}$ for a strictly decreasing sequence $h_1 > h_2 > \ldots > h_k$ of Hall words then

$$\xi_w = \tfrac{1}{r_1! \cdots r_k!} \xi_{h_1}^{\sqcup\!\sqcup r_1} \sqcup\!\sqcup \xi_{h_2}^{\sqcup\!\sqcup r_2} \sqcup\!\sqcup \ldots \sqcup\!\sqcup \xi_{h_k}^{\sqcup\!\sqcup r_k} \tag{31}$$

provides the correct extension. In general the maps π_k' do not commute with shuffle products. But surprisingly they do on products of this special form. Indeed, if $|r| = r_1 + \ldots + r_k$ is the number of factors in the factorization of w, then

$$\zeta_w = \pi_{|r|}'(\xi_w) = \tfrac{1}{r_1! \cdots r_k!} \zeta_{h_1}^{\sqcup\!\sqcup r_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup \zeta_{h_k}^{\sqcup\!\sqcup r_k} \tag{32}$$

For complete proofs, details, and studies of the geometry of the subspaces U_k and U_k' we refer the reader to [10].

With the characterization (27) of π_1' and formulas for it, it now is a straightforward calculation to obtain explicit formulas for the ζ_h. As an example first consider the projection of an individual word

$$\begin{aligned} \pi_1'(aaabb) = aaabb &- \tfrac{1}{2}\, (aaab \sqcup\!\sqcup b + aaa \sqcup\!\sqcup bb + aa \sqcup\!\sqcup aab \\ &+ a \sqcup\!\sqcup aabb) + \tfrac{1}{3}(aaa \sqcup\!\sqcup b \sqcup\!\sqcup b + aa \sqcup\!\sqcup ab \sqcup\!\sqcup b \\ &+ a \sqcup\!\sqcup aabb) + \tfrac{1}{3}(aaa \sqcup\!\sqcup b \sqcup\!\sqcup b + aa \sqcup\!\sqcup ab \sqcup\!\sqcup b \\ &+ 2aa \sqcup\!\sqcup a \sqcup\!\sqcup bb + a \sqcup\!\sqcup aab \sqcup\!\sqcup b + a \sqcup\!\sqcup a \sqcup\!\sqcup abb) \\ &- \tfrac{1}{4}(aa \sqcup\!\sqcup a \sqcup\!\sqcup b \sqcup\!\sqcup b + a \sqcup\!\sqcup aa \sqcup\!\sqcup b \sqcup\!\sqcup b + a \sqcup\!\sqcup a \sqcup\!\sqcup ab \sqcup\!\sqcup b \\ &+ a \sqcup\!\sqcup a \sqcup\!\sqcup a \sqcup\!\sqcup bb) + \tfrac{1}{5} a \sqcup\!\sqcup a \sqcup\!\sqcup a \sqcup\!\sqcup b \sqcup\!\sqcup b. \end{aligned} \tag{33}$$

Using the above, and similar straightforward manipulations, starting from $\xi_{abaab} = \frac{1}{2}(ab \sqcup\!\sqcup a \sqcup\!\sqcup a)b = 3aaabb + 2aabab + abaab$, the following calculation yields ζ_{abaab}.

$$\begin{aligned} \zeta_{abaab} &= \pi_1'(\xi_{abaab}) = \pi_1'(3aaabb + 2aabab + abaab) \\ &= \tfrac{1}{10} aaabb + \tfrac{1}{15} aabab + \tfrac{1}{15} aabba + \tfrac{1}{15} abaab \\ &\quad - \tfrac{1}{10} ababa + \tfrac{1}{15} abbaa - \tfrac{1}{10} baaab + \tfrac{1}{15} baaba \\ &\quad + \tfrac{1}{15} babaa - \tfrac{1}{10} bbaaa. \end{aligned} \tag{34}$$

In a similar fashion one calculates the following, reaffirming the formulas first presented in [18], then obtained by a brute force linear algebra using computer algebra system.

$$\begin{aligned} \zeta_a &= a \qquad \zeta_b = b \\ \zeta_{ab} &= \tfrac{1}{2}(ab - ba) = \tfrac{1}{2}[a, b] \\ \zeta_{aab} &= \tfrac{1}{6}\, (\, aab - 2aba + baa) = \tfrac{1}{6}\, [a, [a, b]] \\ \zeta_{aab} &= \tfrac{1}{6}\, (-abb + 2bab - bba) = \tfrac{1}{6}\, [b, [a, b]] \\ \zeta_{aaab} &= \tfrac{1}{6}(abaa - aaba) \\ \zeta_{baab} &= \tfrac{1}{6}(abab - aabb + bbaa - baba) \\ \zeta_{bbab} &= \tfrac{1}{6}(bbab - babb) \\ \zeta_{aaaab} &= -\tfrac{1}{30}(aaaab + aaaba - 4aabaa + abaaa + baaaa) \\ \zeta_{baaab} &= \tfrac{1}{30}(-2aaabb + 3aabab + 3aabba - 2abaab \\ &\qquad - 2ababa + 3abbaa - 2baaab \\ &\qquad - 2baaba + 3babaa - 2bbaaa) \\ \zeta_{abaab} &= \tfrac{1}{30}(-3aaabb + 2aabab + 2aabba + 2abaab \\ &\qquad - 3ababa + 2abbaa - 3baaab \\ &\qquad + 2baaba + 2babaa - 3bbaaa) \\ \zeta_{bbaab} &= \tfrac{1}{30}(2aabbb - 3ababb + 2abbab + 2abbba \\ &\qquad - 3baabb + 2babab + 2babba \\ &\qquad - 3bbaba - 3bbaab + 2bbbaa) \\ \zeta_{abbab} &= \tfrac{1}{30}(-aabbb - ababb + 4abbab - abbba \\ &\qquad - baabb - babab + 4babba \\ &\qquad - bbaba - bbaab - bbbaa) \\ \zeta_{bbbab} &= \tfrac{1}{30}(abbbb + babbb - 4bbabb + bbbab + bbbba) \end{aligned} \tag{35}$$

The direct implementation of the main formula (30) expresses each of the ζ_h, $h \in \mathcal{H}$, as a linear combination of the standard basis vectors $w \in Z^*$. These immediately translate into iterated integrals of a plain left to right structure as in the original Chen-Fliess series – albeit summed over a nontrivial subset of words (or multi-indices) with rational coefficients whose values are not easily predicted.

However, it is relatively straightforward to rewrite these results to again yield recursive formulas, but in terms of nontrivially structured iterated integrals. For the sake of clarity, rather than integral signs, similar to (13) we again write a nilpotent cascade system of differential equations satisfied by the iterated integrals $z_h(t, U') = \mu_h \Upsilon(\zeta_h)(U)(t)$, compare [18]. Suppressing the normalization factors μ_h we obtain

$$
\begin{aligned}
\dot{z}_a &= u_a \qquad \dot{z}_b = u_b \\
\dot{z}_{ab} &= -\tfrac{1}{6} z_b u_a + \tfrac{1}{6} z_a u_b \\
\dot{z}_{aab} &= \left(-\tfrac{1}{2} z_{ab} - \tfrac{1}{12} z_a z_b\right) u_a + \tfrac{1}{12} z_a^2 u_b \\
\dot{z}_{bab} &= -\tfrac{1}{12} z_b^2 u_a + \left(-\tfrac{1}{2} z_{ab} + \tfrac{1}{12} z_a z_b\right) u_b \\
\dot{z}_{aaab} &= \left(-\tfrac{1}{2} z_{aab} - \tfrac{1}{12} z_{ab} z_a\right) u_a \\
\dot{z}_{baab} &= \left(-\tfrac{1}{2} z_{bab} - \tfrac{1}{12} z_{ab} z_b\right) u_a - \left(\tfrac{1}{2} z_{aab} + \tfrac{1}{12} z_{ab} z_a\right) u_b \\
\dot{z}_{bbab} &= \left(-\tfrac{1}{2} z_{bab} - \tfrac{1}{12} z_{ab} z_b\right) u_b \\
\dot{z}_{aaaab} &= \left(-\tfrac{1}{2} z_{aaab} - \tfrac{1}{12} z_a z_{aab} - \tfrac{1}{720} z_a^3 z_b\right) u_a \\
&\quad -\tfrac{1}{720} z_a^4 u_b \\
\dot{z}_{bbbab} &= \tfrac{1}{720} z_b^4 u_a - \left(\tfrac{1}{2} z_{bbab} + \tfrac{1}{12} z_b z_{bab} + \tfrac{1}{720} z_a z_b^3\right) u_b \\
\dot{z}_{baaab} &= \left(\tfrac{1}{240} z_a^2 y_b^2 - \tfrac{1}{2} z_{baab} - \tfrac{1}{12} z_b z_{aab} - \tfrac{1}{12} z_a z_{bab}\right) u_a \\
&\quad - \left(\tfrac{1}{2} z_{aaab} + \tfrac{1}{12} z_a z_{aab} + \tfrac{1}{240} z_a^3 z_b\right) u_b \\
\dot{z}_{abaab} &= \left(-\tfrac{1}{2} z_{baab} + \tfrac{1}{12} z_b z_{aab} - \tfrac{1}{12} z_a z_{bab} - \tfrac{1}{12} y_{ab}^2\right. \\
&\quad \left. + \tfrac{1}{360} z_a^2 y_b^2\right) u_a + \left(-\tfrac{1}{6} z_a z_{aab} - \tfrac{1}{360} z_a^3 z_b\right) u_b \\
\dot{z}_{abbab} &= \left(-\tfrac{1}{2} z_{bbab} \tfrac{1}{12} z_b z_{bab} + \tfrac{1}{720} z_a z_b^3\right) u_a \\
&\quad - \left(\tfrac{1}{6} z_a z_{bab} + \tfrac{1}{12} y_{ab}^2 + \tfrac{1}{720} z_a^2 y_b^2\right) u_b \\
\dot{z}_{bbaab} &= \left(-\tfrac{1}{2} z_{bbab} - \tfrac{1}{12} z_b z_{bab} + \tfrac{1}{240} z_a z_b^3\right) u_a \\
&- \left(\tfrac{1}{2} z_{baab} + \tfrac{1}{12} z_b z_{aab} + \tfrac{1}{12} z_a z_{bab} + \tfrac{1}{240} z_a^2 y_b^2\right) u_b
\end{aligned} \tag{36}
$$

V. Summary and Conclusion

Using basic maps that utilize Hopf algebra structures connected to the free Lie algebra, we presented a direct formula for obtaining the coordinates of the first kind ζ_h from the respective coordinates of the second kind ξ_h. Elegant formula for the latter are known in terms of the Zinbiel product. While the new formulas do not exhibit as elegant and as simple a product structure as the latter – and thus do not immediately give insight to geometric features, controllability and optimality properties, they are nonetheless explicitly and easily computable. The given format is amenable for direct implementation into e.g. controls for tracking, path planning, or for geometric integration procedures.

The main contribution is not the existence of such a map – which was clear a priori – but rather expressing this map in terms of elementary objects in the Hopf algebra. The quest for a different way of expressing the ζ_h in terms of some new product, or as a one-line recursive description such as (12), continues.

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