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Impulsive Consensus Control for Complex Dynamical Networks with Non-identical Nodes and Coupling Time-Delays

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Abstract— **This paper investigates the problem of global consensus between a complex dynamical network (CDN) and a known goal signal by designing an impulsive consensus control scheme. The dynamical network is complex with respect to the uncertainties, non-identical nodes and coupling time-delays. The goal signal can be a measurable vector function or a solution of a dynamical system. By utilizing Lyapunov function and Lyapunov-Krasovskii functional methods, robust global exponential stability (RGES) criteria are derived for the error system, under which global exponential consensus is achieved for the complex dynamical networks. These criteria are expressed in terms of LMIs and algebraic inequalities. Thus, the impulsive controller can be easily designed by solving the derived inequalities. Meanwhile, the estimations of the consensus speed rate for global exponential consensus is also obtained. One example with numerical simulations is worked out for illustration.**

I. INTRODUCTION

Synchronization of chaotic systems and its potential applications to secure communication has been an active research area since the 1990's. Numerous methods have been developed for chaos synchronization (see, for example, [1-11]). Recently, synchronization of complex dynamical networks (CDNs) is also reported in the literature (see, for example, [12-27]). The complex dynamical network consists of coupled nodes, which are usually dynamical systems. There have been proposed several approaches for synchronization of a CDN, for example, feedback control synchronization (see, for example, [22-23]), adaptive synchronization [20], synchronization based on the invariance principle [10], state-observer-based approach [26], and impulsive synchronization [34], etc.

It has been noticed that the complex dynamical networks (CDNs) studied in the literature have the following limitations: (i) the coupled nodes have the same dynamics; (ii) by using the linearization technique and matrix eigenvalue method, most synchronization criteria are local; (iii) uncertainty and time delays which are common in practical CDNs have not been studied fully, although there are published results ([8, 15, 20, 34]) which study the robust synchronization problem with respect to uncertainties, and some ([21-24]) deal with a single constant time delay. Uncertainties often occur due to the uncertainties of parameters, modeling mismatches, measurement errors, approximations, channel noises etc. And time delays occur commonly due to the congestion of the network traffic and the fact that the switching and spreading speed of the hardware and circuit implementation is finite. Moreover, the time delays presented in many real synchronization schemes are difficult to know *a priori* and they are in the form of multiple time delays and time-varying.

The CDNs with non-identical nodes represent more general and practical networks than the models typically studied in the literature. Moreover, to the best of our knowledge, no literature has been published for the consensus problem between CDNs and a known goal function. However, allowing different dynamics of nodes in a CDN brings difficulties in achieving consensus. If the uncertainty and time delays occur simultaneously in a CDN with non-identical nodes, and consensus is to be achieved to a known goal function, it will be much more difficult to use previous synchronization control schemes, specially for the global consensus problem. Hence, there is a need to study new consensus control schemes which can achieve the objective.

In this paper, we propose an impulsive consensus control scheme for the consensus problem between CDNs and the known signal. In this control scheme, the control signal is designed to input into the CDN as follows: at impulsive instances, the impulse signal is input into the nodes, and at other times, the signal containing the goal signal is input into the nodes. Hence, this control scheme is a type of impulsive control scheme. Impulsive control arises naturally from a wide variety of applications, such as drug administration, spacecraft control, inspection processes in operations research, native forest ecosystems management, just to name a few. Based on the stability theory of impulsive systems (see [28-30, 35-38], and references therein), the impulsive control method (see [31-33, 39] and references therein) provides greater prospect for solving many problems that are basically defined by continuous dynamical systems, but on which only discrete-time actions are exercised. An essential benefit of the impulsive control approach may be derived from the fact that such controls are typically simpler and cheaper to implement. In [9], impulsive control was first introduced to synchronize chaotic systems. Since then, significant progress has been made in impulsive synchronization of chaotic systems, see [11] and the references therein. Recently, impulsive synchronization for CDNs was also reported in [34]. The theory and experiments have proved that the impulsive synchronization scheme for chaotic systems or CDNs (with identical nodes) has good robustness against uncertainties and can achieve

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global synchronization.

The aim of this paper is to study the global exponential consensus problem for CDNs and a known goal function by using an impulsive consensus control scheme. The model of CDNs consists of non-identical nodes, uncertainties, and coupling time delays. Here, the "uncertainties" means the uncertainty of parameters, which take values in some intervals. The exponential consensus scheme has an obvious advantage over other consensus schemes, in which the consensus speed and consensus time can be estimated easily. By utilizing the Lyapunov function and Lyapunov-Krasovskii functional ([40-41]) methods, robust global exponential stability (RGES) results for delay error systems shall be established, and then we shall derive several criteria under which the global exponential impulsive consensus (GEIC) is achieved for the CDNs. These criteria are expressed in terms of LMIs and algebraic inequalities. Thus, the conditions of consensus are easy to be tested. Moreover, the solutions of the LMIs and algebraic inequalities give rise directly to impulsive controllers for a CDN under which GEIC is achieved.

II. Preliminaries and Problem Formulation

Let R^n denote the n-dimensional Euclidean space. Let $R_+ = [0,+\infty)$, $\mathbb{Z} = \{0,1,2,\cdots,\}$, and $\|\cdot\|$ be the Euclidean norm in R^n. Let I be the identity matrix, and matrix $X > (\geq,<,\leq)\ 0$ means that X is a symmetric positive definite (positive semi-definite, negative definite, negative semi-definite) matrix. Denote by $\lambda_{\max}(\cdot)\big(\lambda_{\min}(\cdot)\big)$ the maximum (minimum) eigenvalue of matrix $(\cdot)$.

Consider the uncertain CDN consisting of N non-identical nodes ($n-$dimensional dynamical systems) with coupling time-delays:

$$\dot{x}_i = f_i(t,x_i) + g_i(x_1(t-h_i),\cdots,x_N(t-h_i)), i = 1,\cdots,N, \quad (1)$$

where $x = (x_1^T,x_2^T,\cdots,x_n^T)^T \in R^{nN}$, $x_i = (x_{i1},x_{i2},\cdots,x_{in})^T \in R^n$ represents the state vector of the ith node; $f_i : R_+ \times R^n \to R^n$ is a smooth vector function; $g_i : R^{nN} \to R^n$ is a smooth vector function representing the coupling of the ith node; h_i is some nonnegative constant which represents the time-delay of the signal transmitted from the network to the ith node, where the coupling time-delay h_i satisfies $0 \leq h_i \leq \tau$ for some constant $\tau > 0$ and $i = 1,2,\cdots,N$. We call the system (1) an uncertain network based on the fact that there are uncertainties in functions f_i, g_i, $i = 1,2,\cdots,N$. In this paper, we make the following assumptions:

Assumption 2.1. Assume that $f_i(t,x_i) = A_i x_i + \tilde{f}_i(t,x_i)$, where $A_i \in R^{n\times n}$ is an interval matrix with $A_i \in N[A_{i_1},A_{i_2}]$, where $N[A_{i_1},A_{i_2}] = \big\{(a_{ij}) \in R^{n\times n} : u_{ij} \leq a_{ij} \leq v_{ij}\big\}$ for known matrices $A_{i_1} = (u_{ij})_{n\times n}$ and $A_{i_2} = (v_{ij})_{n\times n}$, and function $\tilde{f}_i$ satisfies $\|\tilde{f}_i(t,s_1) - \tilde{f}_i(t,s_2)\| \leq L_i\|s_1 - s_2\|$, for some positive constant $L_i > 0$ and for all $t \in R_+$.

It should be noticed that when the network (1) achieves consensus, namely, when the states $x_1(t) = \cdots = x_N(t) = s(t)$ as $t \to \infty$, the coupling terms should vanish: i.e., $g_i(s,s,\cdots,s) = 0$, $i = 1,2,\cdots,N$. Thus, we give the following assumption on function g_i:

Assumption 2.2. Assume that $g_i(x_1,\cdots,x_N) = \sum_{j=1}^N B_{ij}x_j + \tilde{g}_i(x_1,\cdots,x_N)$, where $B_{ij} \in R^{n\times n}$ is an interval matrix with $B_{ij} \in N[B_{ij_1},B_{ij_2}]$, where B_{ij_1},B_{ij_2} are known matrices, and matrices B_{ij} and function $\tilde{g}_i$ satisfy:

$$\sum_{j=1}^N B_{ij} = 0, \quad i = 1,2,\cdots,N; \quad (2)$$

and $\tilde{g}_i(s,s,\cdots,s) = 0$, and for some constants $M_{ij} \geq 0$,

$$\|\tilde{g}_i(x_1,\cdots,x_N) - \tilde{g}_i(y_1,\cdots,y_N)\| \leq \sum_{j=1}^N M_{ij}\|x_j - y_j\|. \quad (3)$$

Problem formulation: Let $s(t)$ be a given measurable smooth vector function satisfying $s(t) \in R^n$ for any $t \in R_+$. The aim of this paper is to design an impulsive hybrid control scheme for the CDN (1) such that the consensus among the node states $x_i(t)$ $(i = 1,2,\cdots,N)$ and goal $s(t)$ can be achieved.

Consider the CDN (1) under impulsive consensus control:

$$\dot{x}_i = f_i(t,x_i) + g_i + u_i(t,x_i,s), \quad i = 1,2,\cdots,N, \quad (4)$$

where $\{u_i(t,x_i,s), i = 1,2,\cdots,N\}$ is the *impulsive hybrid controller* as shown in Fig.1, where $C_{i_k} \in R^{n\times n}, k \in \mathbb{Z}$, are *impulsive control gain matrices* to be designed, and $\{t_k, k \in \mathbb{Z}\}$ are the *impulsive instances* satisfying $0 \leq t_0 < t_1 < t_2 < \cdots$, with $\sup_{k\in\mathbb{Z}}\{t_{k+1} - t_k\} < \infty$ and $\lim_{k\to\infty} t_k = \infty$.

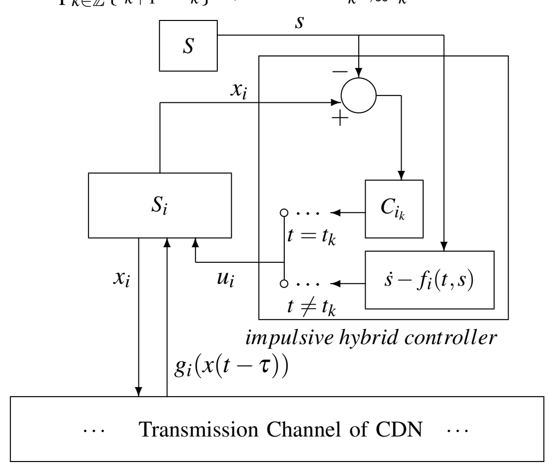

Fig.1. Impulsive consensus scheme of S_i.

Fig.1 depicts the entire impulsive control scheme for the consensus ("*impulsive consensus scheme*" for short) between the known signal $s(t)$ and the CDN (1) with coupling time-delays, where S_i stands for the i-th node, S is objective vector function $s(t)$, and g_i is the delay network coupling of the i-th node, $i = 1,2,\cdots,N$. In this control scheme, the control signal is input into the CDN as: at impulsive instance t_k, the impulse signal $(C_{i_k} - I)(x_i(t_k) - s(t_k))$ is input into the node S_i, and at other non-impulse times $t \neq t_k$, the signal $\dot{s} - f_i(t,s)$ containing the goal signal is input into the node S_i of CDN.

By Fig.1, (4) is equivalent to the following system:

$$\begin{aligned} &\dot{x}_i = f_i(t,x_i) + g_i + \dot{s} - f_i(t,s), \quad t \in [t_k,t_{k+1}), \\ &\Delta x_i(t_k) = x(t_k) - x(t_k^-) = (C_{i_k} - I)(x_i(t_k) - s(t_k)), \\ &\qquad t = t_k, k \in \mathbb{Z}, i = 1,2,\cdots,N. \end{aligned} \quad (5)$$

Define the consensus errors as $e_i(t) := x_i(t) - s(t)$, then, by Assumptions 2.1-2.2, one has an error dynamical system:

$$\dot{e}_i = A_i e_i + \hat{f}_i + \sum_{j=1}^{N} B_{ij} e_j(t-h_i) + \hat{g}_i, t \in [t_k, t_{k+1}),$$
$$\Delta e_i = (C_{i_k} - I)e_i(t), t = t_k, k \in \mathbb{Z}, i = 1, \cdots, N, \quad (6)$$

where $\hat{f}_i = \hat{f}_i(t,x_i,s) = \tilde{f}_i(t,x_i) - \tilde{f}_i(t,s)$, $\hat{g}_i = \hat{g}_i(x(t-h_i),s) = \tilde{g}_i((x_1(t-h_i),\cdots,x_N(t-h_i)) - \tilde{g}_i(s(t-h_i),\cdots,s(t-h_i))$.

Remark 2.1. It should be noticed that (1) represents a more general dynamical network than that considered in the literature in the following senses:

(i) The nodes in network (1) are non-identical, i.e., functions f_i $(i = 1,2,\cdots,N)$ can be different. Moreover, if the given function $s(t)$ satisfies $\dot{s} = f_i(t,s)$, $i = 1,2,\cdots,N$, i.e., $s(t)$ is the same solution of single node, then, the impulsive consensus scheme is to make all states of the non-identical nodes approach the same solution $s(t)$. If all the functions f_i are the same: $f_i = f$, and the given vector function $s(t)$ is a solution of system $\dot{y} = f(t,y)$, then the consensus problem is the synchronization problem discussed in the literature, for examples, see [20-27].

(ii) In the literature for study of complex dynamical networks, see examples [20-27], the coupling coefficient matrices have the form $B_{ij} = cC_{ij}\Gamma$, where $c > 0$ denotes the coupling strength, $\Gamma = \mathrm{diag}\{r_1,r_2,\cdots,r_n\}$ and $C = (C_{ij})_{N\times N}$, are the linking matrices of network nodes. However, only a single time delay τ is considered in the literature. In this paper, the multi time delays is involved.

(iii) The matrices and functions are considered as interval matrices and uncertain functions satisfying some conditions.

By [34], for any $X \in N[X_1,X_2]$, it can be formulated as:

$$X = X_0 + \Delta X = X_0 + E\Sigma F, \quad (7)$$

where $X_0 = \frac{1}{2}(X_1 + X_2), H = \frac{1}{2}(X_2 - X_1) = (h_{ij})_{n\times n}$, $E \cdot E^T = \mathrm{diag}\{\sum_{j=1}^n h_{1j},\cdots,\sum_{j=1}^n h_{nj}\}$, $F^T \cdot F = \mathrm{diag}\{\sum_{j=1}^n h_{j1},\cdots,\sum_{j=1}^n h_{jn}\}$, $\Sigma \in \Sigma^* = \{\Sigma \in R^{n^2\times n^2} : \Sigma = \mathrm{diag}(\varepsilon_{11},\cdots,\varepsilon_{nn}), |\varepsilon_{ij}| \le 1; i,j = 1,2,\cdots,n\}$.

Assumption 2.3. For interval matrices A_i, B_{ij}, there exist known matrices E, F_{A_i}, F_{ij}, such that for any $\Sigma \in \Sigma^*$,

$$\begin{bmatrix}\Delta A_i & \Delta B_{ij}\end{bmatrix} = E\Sigma\begin{bmatrix}F_{A_i} & F_{ij}\end{bmatrix}, i,j = 1,2,\cdots,N. \quad (8)$$

Definition 2.1. The error system (6) is said to be robustly globally asymptotically stable (RGAS) if, for any initial condition: $\phi \in C[[t_0-\tau,t_0],R^{nN}]$, for any $A_i \in N[A_{i_1},A_{i_2}], B_{ij} \in N[B_{ij_1},B_{ij_2}]$, and for any h_j with $0 \le h_j \le \tau$, the trivial solution of (6) is globally asymptotically stable (GAS).

Definition 2.2. The error system (6) is said to be robustly globally exponentially stable (RGES) with decay rate α if, for any initial condition: $\phi \in C[[t_0-\tau,t_0],R^{nN}]$, for any $A_i \in N[A_{i_1},A_{i_2}], B_{ij} \in N[B_{ij_1},B_{ij_2}]$, and for any h_j with $0 \le h_j \le \tau$, the trivial solution is globally exponentially stable (GES), i.e., for some positive numbers $\alpha > 0, K > 0$,

$$\|e(t)\| \le K\|\phi\|_\tau e^{-\alpha(t-t_0)}, \quad t \ge t_0, \quad (9)$$

where $\phi(t) = (\phi_1^T(t),\cdots,\phi_N^T(t))^T \in R^{nN}, \phi_i(t) \in R^n$, and $\|\phi\|_\tau^2 = \sum_{i=1}^N \|\phi_i\|_\tau^2$, with $\|\phi_i\|_\tau = \sup_{t_0-\tau\le t\le t_0}\{\|\phi_i(t)\|\}$.

Definition 2.3. The error system (6) is said to be quasi-RGES with decay rate α if, for any initial condition: $\phi \in C[[t_0-\tau,t_0],R^{nN}]$, for any $A_i \in N[A_{i_1},A_{i_2}], B_{ij} \in N[B_{ij_1},B_{ij_2}]$, and for any time-delays h_j with $0 \le h_j \le \tau$, there exist two positive numbers: $\alpha > 0, K > 0$, such that

$$\|e(t)\| \le K\|\phi\|_\tau e^{-\alpha(k-t_0)}, \quad t \in [t_k,t_{k+1}), k \in \mathbb{Z}. \quad (10)$$

Definition 2.4. The impulsive consensus scheme (5) is said to achieve global exponential impulsive consensus (GEIC) with speed rate α if, for any initial condition ϕ, the error system (6) is RGES with decay rate α. If the error system (6) is quasi-RGES with decay rate α, then the network (1) is said to achieve quasi-GEIC with speed rate α. If the system (6) is RGAS, then we say the network (1) can achieve global impulsive consensus (GIC).

Remark 2.2. (i) For the impulsive instances $\{t_k, k \in \mathbb{Z}\}$, $t_k \to \infty$ if and only if $k \to \infty$; thus, Definition 2.3 is well-defined. (ii) In the GEIC or quasi-GEIC scheme, the consensus speed or consensus time can be estimated by using the speed rate, while in the GIC scheme it fails to do the estimation.

III. GEIC CRITERIA

In this section, GEIC criteria for network (1) will be established. Due to space limitations, we include only an outline for the results. The details are presented in [39].

By Assumption 2.3, we denote: $A = A_0 + E_A\Sigma F_A$, and $B_{ij} = B_{ij_0} + E\Sigma F_{ij}$, where $\Sigma \in \Sigma^*$.

Theorem 3.1. Let Assumptions 2.1-2.3 be satisfied. Suppose $\Delta_{sup} \triangleq \sup_{k\in\mathbb{Z}}\{t_k - t_{k-1}\} < \infty$ and that there exist positive definite matrices $P_i \in R^{n\times n}$ and constants $\varepsilon_{ij} > 0, \tilde{\varepsilon}_{ij} > 0, \varepsilon_i > 0$, $\hat{\alpha}_i > 0$, $i,j = 1,2,\cdots,N$, such that

(i) there exist positive constants $\nu_i > 0, \mu_i > 0$ such that

$$\nu_i I \le P_i \le \mu_i I, \quad i = 1,2,\cdots,N; \quad (11)$$

(ii) for $k \in \mathbb{Z}$, $i = 1,2,\cdots,N$, the following LMIs hold:

$$\begin{pmatrix}
\Psi_i(A_0) - \hat{\alpha}_i P_i & P_iB_{i1_0} & \cdots & P_iB_{iN_0} & P_iE & F_{A_i}^T \\
B_{i1_0}^T P_i & -\varepsilon_{i1} I & \cdots & 0 & 0 & F_{i1}^T \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
B_{iN_0}^T P_i & 0 & \cdots & -\varepsilon_{iN} I & 0 & F_{iN}^T \\
E^T P_i & 0 & \cdots & 0 & -\varepsilon_i^{-1} I & 0 \\
F_{A_i} & F_{i1} & \cdots & F_{iN} & 0 & -\varepsilon_i I
\end{pmatrix} < 0$$

where $\Psi_i(A_0) = P_iA_0 + A_0^T P_i + 2L_i\sqrt{\frac{\mu_i}{\nu_i}}P_i + \sum_{j=1}^N M_{ij}\tilde{\varepsilon}_{ij}^{-1}\|P_i\| I$;

(iii) for any $k \in \mathbb{Z}$, the following inequality holds:

$$\beta_k \triangleq \max_{1\le i\le N}\left\{\lambda_{\max}\left(P_i^{-1}C_{i_k}^T P_i C_{i_k}\right)\right\} < 1; \quad (12)$$

(iv) there exists a positive integer $m \in \mathbb{Z}$ such that $t_{k-m} \le t_k - \tau < t_{k+1-m}$ for any $k \ge m, k \in \mathbb{Z}$, and the discrete system:

$$z(k+1) = J_k(m)z(k), \quad k \in \mathbb{Z}, \quad (13)$$

is GES with decay rate $\sigma > 0$, where

$$J_k(m) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \alpha_{k+1-m} & \alpha_{k+2-m} & \alpha_{k+3-m} & \cdots & \alpha_{k-1} & \tilde{\alpha}_k \end{pmatrix},$$

where $\tilde{\alpha}_{k-1} = \beta_k e^{p\Delta_k} + \alpha_{k-1}$, $\alpha_{k-j} = p_2 \Delta_{k-j+1} e^{p\Delta_{k-j+1}}$, $j = 1, 2, \cdots, m$, and $p = p_1 + p_2$, $p_1 = \max_{1 \le i \le N}\{\hat{\alpha}_i\}$, $p_2 = \max_{1 \le i \le N}\left\{\sum_{j=1}^N \frac{\hat{\varepsilon}_{ji}}{v_i}\right\}$, where $\hat{\varepsilon}_{ij} = \varepsilon_{ij} + M_{ij}\tilde{\varepsilon}_{ij}\|P_i\|$.
Then, the error system (6) is quasi-RGES with decay rate $\alpha \triangleq \frac{1}{2}\sigma$. Moreover, if there exist $k_1 \ge k_0, k_1 \in \mathbb{Z}$ and positive constant $\gamma > 0$ such that $\sup_{k \ge k_1}\{\frac{t_k}{k}\} \le \gamma$, then (6) is RGES with decay rate $\alpha \triangleq \frac{\sigma}{2\gamma}$, and hence the CDN (1) can achieve GEIC with the given state $s(t)$ with speed rate α.

Proof. Let Lyapunov-Krasovskii functional V be: $V(e(t)) = V_1(t) + V_2(t)$, where $V_1(t) = \sum_{i=1}^N e_i^T(t) P_i e_i(t)$, $V_2(t) = \sum_{i=1}^N \sum_{j=1}^N \hat{\varepsilon}_{ji} \int_{t-h_j}^t e_i^T(s) e_i(s) ds$, for some constants $\lambda_{ij} > 0, i, j = 1, 2, \cdots, N$.

Claim 1: For $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, by condition (i) and Schur Complement Theorem [43], we claim

$$V(e(t)) \le V(e(t_k)) e^{p(t-t_k)}. \tag{14}$$

Claim 2: Let: $W(k) = (w_1(k), w_2(k), \cdots, w_m(k))^T$, $k \in \mathbb{Z}$, where $w_1(k) = V(e(t_{k+1})), w_2(k) = V(e(t_{k+2})), \cdots, w_m(k) = V(e(t_{k+m}))$. Then, we claim $W(k-m) \le z(k), \quad k \ge m-1$.

Claim 3: If the comparison system (13) is GES with decay rate $\sigma > 0$, then we claim $\|e(t)\| \le \hat{K}\|\phi\|_\tau e^{-\frac{\sigma}{2\gamma}(t-t_0)}, t \ge t_0$, where $\hat{K} > 0$ is some constant.

By Claims 1-3, we conclude that the result is true. □

Corollary 3.1. Suppose that $\tau \le t_k - t_{k-1}$ for any $k \in \mathbb{Z}$, i.e., $m = 1$ in Theorem 3.1, and that there exist positive definite matrices $P_i \in R^{n \times n}$ and constants $\varepsilon_{ij} > 0, \tilde{\varepsilon}_{ij} > 0, \varepsilon_i > 0$, $\hat{\alpha}_i > 0$, $i, j = 1, 2, \cdots, N$, $k \in \mathbb{Z}$, such that (i)-(iii) of Theorem 3.1 hold, while (iv) of Theorem 3.1 is replaced by:

(iv*) there exists a positive constant $\sigma > 0$ such that

$$\ln(\beta_k + p_2\tau) + (p + \sigma)(t_k - t_{k-1}) \le 0, \tag{15}$$

where β_k, p_1, p_2, p are defined in Theorem 3.1.
Then, the error system (6) is RGES with decay rate $\alpha \triangleq \frac{\sigma}{2}$ and thus the CDN (1) can achieve GEIC with the given state $s(t)$ with speed rate α.

Remark 3.1. From Corollary 3.1, if $\tau \le t_k - t_{k-1}$ for any $k \in \mathbb{Z}$, we get the maximum estimations of time-delay τ^* and the interval of impulses as follows:

$$\tau^* \le \sup_{k \in \mathbb{Z}}\left\{\frac{e^{-(p+\sigma)(t_k - t_{k-1})} - \beta_k}{p}\right\}, \tag{16}$$

$$\Delta_{sup} \le \sup_{k \in \mathbb{Z}}\left\{\frac{-\ln(\beta_k + p_2\tau)}{p + \sigma}\right\}. \tag{17}$$

Corollary 3.2. Suppose $\Delta_{sup} \triangleq \sup_{k \in \mathbb{Z}}\{t_k - t_{k-1}\} < \infty$ and that there exist positive definite matrices $P_i \in R^{n \times n}$ and constants $\varepsilon_{ij} > 0, \hat{\varepsilon}_{ij} > 0, \varepsilon_i > 0$, $\hat{\alpha}_i > 0$, $i, j = 1, 2, \cdots, N$, such that (i)-(iii) of Theorem 3.1 hold, while (iv) of Theorem 3.1 is replaced by

(iv**) there exists a positive integer $m > 1$ such that $t_{k-m} \le t_k - \tau < t_{k+1-m}$ for any $k \ge m, k \in \mathbb{Z}$, and

$$J(m) \triangleq \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ a & a & a & \cdots & a & a+b \end{pmatrix}. \tag{18}$$

satisfies the spectrum radius of matrix $J(m)$ conditions: $\rho(J(m)) < e^{-\sigma}$, where $\sigma > 0$ is some positive constant, $a = p_2 \Delta_{sup} e^{p\Delta_{sup}}$, $b = \beta e^{p\Delta_{sup}}$, $\beta = \sup_{k \in \mathbb{Z}}\left\{\lambda_{\max}\left(P_i^{-1} C_{i_k}^T P_i C_{i_k}\right)\right\}$, and p, p_1, p_2 are defined in Theorem 3.1.
Then, the error system (6) is quasi-RGES with decay rate $\alpha \triangleq \frac{1}{2}\sigma$. Moreover, if there exist $k_1 \ge k_0, k_1 \in \mathbb{Z}$ and positive constant $\gamma > 0$ such that $\sup_{k \ge k_1}\{\frac{t_k}{k}\} \le \gamma$, then (6) is RGES with decay rate $\alpha \triangleq \frac{\sigma}{2\gamma}$, and hence the CDN (1) can achieve GEIC with the given state $s(t)$ with speed rate α.

IV. IMPULSIVE CONSENSUS CONTROL DESIGN

In this section, by using the obtained results, we design impulsive control gain matrices for the CDN (1) such that GEIC can be achieved.

Theorem 4.1. Assume $\Delta_{sup} \triangleq \sup_{k \in \mathbb{Z}}\{t_k - t_{k-1}\} < \infty$ and that conditions (i)-(ii) of Theorem 3.1 still hold, while conditions (iii)-(iv) are changed to the following (iii′)-(iv′):

(iii′) there exist positive constants $0 < \beta_i < 1$, $i = 1, 2, \cdots, N$, such that the following LMIs hold:

$$\begin{pmatrix} \Omega_1 & \Omega_2 & Y_i^T \\ \Omega_2^T & -I & 0 \\ Y_i & 0 & -P_i \end{pmatrix} \le 0, \tag{19}$$

where $\Omega_1 = P_i + Y_i^T + Y_i - \beta_i P_i$, $\Omega_2 = P_i E + Y_i^T E$;

(iv′) let $\beta_k = \max_{1 \le i \le N}\{\beta_i\}$, $k \in \mathbb{Z}$, then the condition (iv) of Theorem 3.1 holds.

Then, under impulsive control gain matrices $\left\{C_{i_k} = I + P_i^{-1} Y_i, i = 1, 2, \cdots, N, k \in \mathbb{Z}\right\}$, the error system (6) is quasi-RGES with decay rate $\alpha = \frac{1}{2}\sigma$. Moreover, if there exist $k_1 \ge k_0, k_1 \in \mathbb{Z}$ and positive constant $\gamma > 0$ such that $\sup_{k \ge k_1}\{\frac{t_k}{k}\} \le \gamma$, then system (6) is RGES with decay rate $\alpha \triangleq \frac{\sigma}{2\gamma}$, and thus the CDN (1) can be achieved GEIC with the given state $s(t)$ with speed rate α.

Corollary 4.1. Assume the conditions (i)-(ii) of Theorem 3.1 and condition (iii*) of Theorem 4.1 hold. Then, under impulsive control gain matrices $\left\{C_{i_k} = I + P_i^{-1} Y_i, i = 1, 2, \cdots, N, k \in \mathbb{Z}\right\}$, condition (iv*) implies that the CDN (1) can achieve GEIC with speed rate $\frac{\sigma}{2}$; condition (iv**) implies that the CDN (1) can achieve GEIC with speed rate $\frac{\sigma}{2}$; and (iv***) implies that (6) is RGAS and the CDN (1) can achieve GIC.

In the following, one example is given for illustration.

Example 4.1. Use the chaotic Colpitts' oscillator as nodes of the CDN. The Colpitts' oscillator is described by:

$$\dot{y} = A_0 y + \varphi(y) \tag{20}$$

where $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, $A_0 = \begin{pmatrix} 0 & \alpha & 0 \\ -\sigma & -\gamma\sigma & -\sigma \\ a_1\beta & \beta & 0 \end{pmatrix}$, and $\varphi(y) = (0, 0, a_3\beta y_1^3)^T$, in which $\alpha, \beta, \sigma, a_1, \gamma, a_3 \in R$. It is known that with parameters $\alpha = 2.4$, $\beta = 2.2$, $\sigma = 1$, $\gamma = 0.252$, $a_1 = 1$, and $a_3 = -0.2$, the oscillator (20) is chaotic.

Suppose that the CDN (1) is given by

$$\dot{x}_i = A_i x_i + \varphi(x_i) + \sum_{j=1}^{N} B_{ij} x_j(t-h_i),\ i = 1,\cdots,N, \quad (21)$$

where $x_i = (x_{i1}, x_{i2}, x_{i3})^T$, with matrix $A_i \in N[\underline{A}, \overline{A}]$, where

$$\underline{A} = (\underline{a}_{ij})_{3\times 3} = \begin{pmatrix} -0.5 & \alpha - 0.5 & -0.5 \\ -\sigma - 0.5 & -\gamma\sigma - 0.5 & -\sigma - 0.5 \\ a_1\beta - 0.5 & \beta - 0.5 & -0.5 \end{pmatrix},$$

$\overline{A} = (\underline{a}_{ij} + 1)_{3\times 3}$; and B_{ij} satisfy: $B_{ij} = B_{ij_1} = B_{ij_2}$, $i,j = 1,2,\cdots,N$, $B_{ii} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.2 \\ 0 & 0 & -0.5 \end{pmatrix}$, $B_{i,i+1} = \begin{pmatrix} -1.0 & -0.3 & 0 \\ 0 & 0.25 & -0.1 \\ 0 & 0 & 1.0 \end{pmatrix}$, $B_{i,i+2} = \begin{pmatrix} 0.5 & -0.2 & 0 \\ 0 & -0.75 & -0.1 \\ 0 & 0 & -0.5 \end{pmatrix}$, $B_{N-1,N+1} = B_{N-1,1}$, $B_{N-1,N+2} = N_{N-1,2}$, $B_{N,N+1} = B_{N1}$, $B_{N,N+2} = B_{N2}$.

Suppose the goal $s(t)$ is the solution of Lorenz system:

$$\dot{s} = Ls + \tilde{\varphi}(s) \quad (22)$$

where $s = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$, $L = \begin{pmatrix} -b_1 & b_1 & 0 \\ b_2 & -1 & 0 \\ 0 & 0 & -b_3 \end{pmatrix}$, $\varphi(s) = (0, -s_1 s_3, s_1 s_2)^T$, in which $b_1, b_2, b_3 \in R$. It is well-known that with parameters $b_1 = 10, b_2 = 28, b_3 = \frac{8}{3}$, the Lorenz system (22) is chaotic.

It is easy to show for any matrix $A \in N[\underline{A}, \overline{A}]$, A is not a stable matrix. In the following, by using Theorem 3.1, we design impulsive control gain matrices $K_{i_k} - I, i = 1,\cdots,10, k \in \mathbb{Z}$ and the impulsive instances $t_k, k \in \mathbb{Z}$ such that the system in form of (4) can achieve GEIC.

By [42], we get that $\left\|\frac{\partial\varphi(y)}{\partial y}\right\| \le 5.28$, which implies that $L_i = 5.28$, $i = 1,2,\cdots,10$. Choosing $\varepsilon_{ij} = 1, \varepsilon_i = 1$, $\nu_i = 1, \mu_i = 2$, we solve the LMIs in (ii) of Theorem 3.1, for $i = 1,2,\cdots,10$, getting $\hat{\alpha}_i = 6$, and $P_i = \begin{pmatrix} 1.5437 & 0.3646 & 0.1371 \\ 0.3646 & 1.4298 & 0.1290 \\ 0.1371 & 0.1290 & 1.1520 \end{pmatrix}$. Then, by solving the LMIs in (iii′) of Theorem 4.1, we get that $\beta_k = 0.01$, and $Y_i = \begin{pmatrix} -1.5328 & -0.3546 & -0.1330 \\ -0.3546 & -1.4215 & -0.1264 \\ -0.1330 & -0.1264 & -1.1502 \end{pmatrix}$. Thus, we get $p_1 = 6, p_2 = 10, \beta_k = 0.01$, $K_{i_k} - I = P_i^{-1} Y_i = \begin{pmatrix} -0.9944 & 0.0054 & 0.0024 \\ 0.0054 & -0.9957 & 0.0012 \\ 0.0023 & 0.0011 & -1.0004 \end{pmatrix}$. Moreover, there exists $\sigma = 0.01$ such that (19) in Corollary 3.1 holds. By Remark 3.2, we get $\tau^* < 0.0619$ and $\Delta_{sup} \le 0.1107$. Thus, we can design the impulsive controllers $\{K_{i_k} x_i, t_k\}$ as:

Case 1: If $\tau \le t_k - t_{k-1}, k \in \mathbb{Z}$, then, let: $t_0 = 0$, $t_k - t_{k-1} = 0.1$, $k \in \mathbb{Z}$, and K_{i_k}, $i = 1,2,\cdots,10$, are chosen as above. Therefore, by Theorem 4.1, the impulsive controllers $\{K_{i_k} e_i, t_k\}$ can achieve GEIC for all node states $x_i(t)$ $(i = 1,2,\cdots,10)$ and the dynamical goal $s(t)$. Moreover, the consensus speed rate is $\alpha = \frac{1}{2}\sigma = 0.005$.

Case 2: If there is $m > 1$ such that $t_{k-m} \le t_k - \tau < t_{k+1-m}, k \in \mathbb{Z}$, then, by Corollary 3.2, we design the impulsive controllers. For example, let $\tau = h_i = 0.05$, and $m = 2$, then by Corollary 3.2, the impulsive instances can be set as: $t_0 = 0$, $t_k - t_{k-1} = 0.03$, $k \in \mathbb{Z}$. Therefore, by Corollary 3.2, the impulsive controllers $\{K_{i_k} e_i, t_k\}$ can achieve GEIC for all node states $x_i(t)$ $(i = 1,2,\cdots,10)$ and the dynamical goal $s(t)$. Moreover, the consensus speed rate is $\alpha = \frac{1}{2\gamma}\sigma = 0.0045$.

In simulation, let $\tau = h_i = 0.05$, $t_{k+1} - t_k = 0.03, k \in \mathbb{Z}$, and, without loss of generality, $x_i(t) = 0$ whenever $t < 0$, and $x_i(0) \ne 0, i = 1,2,\cdots,N$. The matrices $A_i \in N[\underline{A}, \overline{A}]$ are set as: $A_1 = A_2 = A_3 = \tilde{A}_1$, $A_4 = A_5 = A_6 = \tilde{A}_2$, $A_7 = A_8 = \tilde{A}_3$, and $A_9 = A_{10} = \tilde{A}_4$, where $\tilde{A}_2 = A_{i_0}, \tilde{A}_3 = \underline{A}$, $\tilde{A}_4 = \overline{A}$, where $\tilde{A}_1 = \underline{A} + \text{Rand}(3,3), \tilde{A}_{i_0} = A_0$.

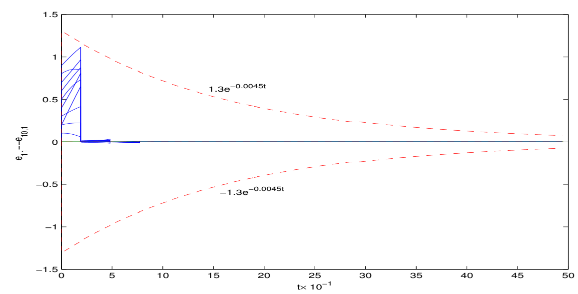

Fig.2. Exponential consensus error properties (with speed rate 0.0045) of $e_{k1}(t), k = 1,2,\cdots,10$.

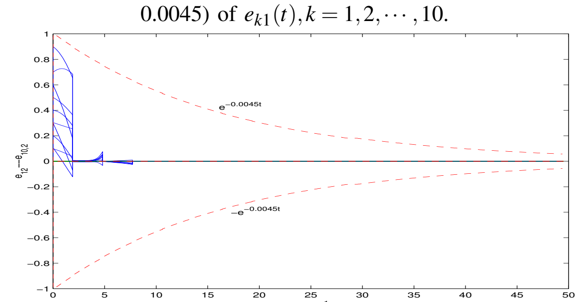

Fig.3. Exponential consensus error properties (with speed rate 0.0045) of $e_{k2}(t), k = 1,2,\cdots,10$.

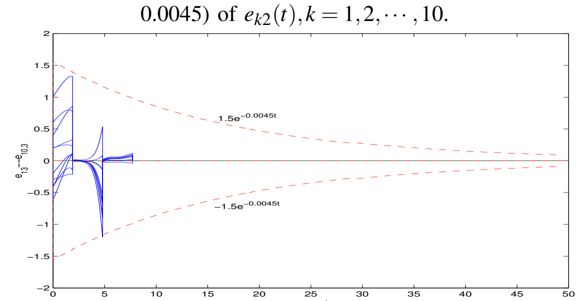

Fig.4. Exponential consensus error properties (with speed rate 0.0045) of $e_{k3}(t), k = 1,2,\cdots,10$.

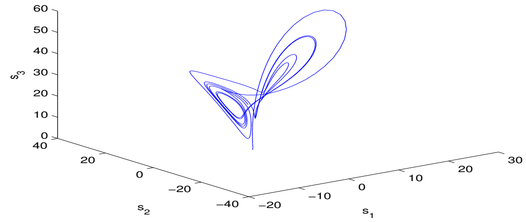

Fig.5. The phase figure of Lorenz system.

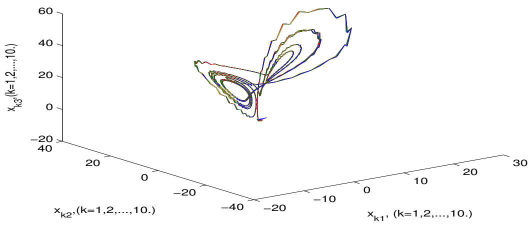

Fig.6. The phase figure of the network under impulsive control.

In Figs. 2-4, one can see the consensus properties of the goal state s_i of the Lorenz system (22) and node states x_{li}, $l = 1,2,\cdots,10$, $i = 1,2,3$, of the network. In Figs.5-6, one can see the whole consensus properties of Lorenz system (22) and the network with coupling delay τ.

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