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Control of Underactuated Mechanical Systems: Observer Design and Position Feedback Stabilization

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***Abstract*— We identify a class of mechanical systems for which a globally exponentially stable reduced order observer can be designed. The class is characterized by (the solvability of) a set of partial differential equations and contains all systems that can be rendered linear in (the unmeasurable) momenta via a (partial) change of coordinates. It is shown that this class is larger than the one reported in the literature of observer design and linearization. We also prove that, under very weak assumptions, the observer can be used in conjunction with an asymptotically stabilizing full state–feedback Interconnection and Damping Assignment Passivity–Based Controller, preserving stability.**

Caveat Emptor: This paper is a shortened version of the technical note [1] which can be obtained upon request from the authors.

I. Introduction

In this paper, we are interested in the problems of observation and output feedback control of n degree of freedom underactuated mechanical systems modeled in Hamiltonian form as

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} + \begin{pmatrix} 0 \\ G(q) \end{pmatrix} u, \quad (1)$$

where $q \in \mathbb{R}^{\mathrm{n}}$, $p \in \mathbb{R}^{\mathrm{n}}$ are the generalized positions and momenta respectively, $u \in \mathbb{R}^m$ is the input, G is an $\mathrm{n} \times \mathrm{m}$ full rank matrix with $\mathrm{m} \leq \mathrm{n}$. Further, the Hamiltonian function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the total energy of the system given as

$$H(q,p) = \frac{1}{2} p^\top M^{-1}(q) p + V(q), \quad (2)$$

where $M = M^\top > 0$ is the mass matrix and V is the potential energy function. We consider q to be measurable, p to be unmeasurable and assume that there exists a full state feedback controller that stabilizes a desired equilibrium point $(q_\star, 0)$.

The problems of velocity reconstruction and position feedback stabilization (either regulation or tracking) of mechanical systems are of great practical interest and have henceforth been extensively studied in the literature. Since the publication of the first result in the fundamental paper [2] in 1990, many interesting solutions have been reported—we refer the reader to the recent books [3], [4], [5] for an exhaustive list of references.

The contributions of this paper are:

- Identification, in terms of two sets of partial differential equations (PDEs) depending on the inertia matrix M, of the class of systems for which we can construct a globally (exponentially) convergent reduced order observer for p.
- Proof that solvability of the first set of PDEs is equivalent to the existence of a change of coordinates of the form $(q, P) = (q, \mathcal{T}^\top(q)p)$, with $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ full rank, that renders the system linear in the unmeasurable states. We also prove that the results reported in the control literature on linearization, either in the context of observer design or not, are particular cases of our result and that the new characterization covers a larger class of practical examples.
- Proof of a separation principle for the proposed observer when used in conjunction with a full state feedback regulator designed following the Interconnection and Damping Assignment Passivity–Based Control methodology [6], [7].

The remaining part of the paper is organized as follows. In Section II we present the observer design methodology and identify—in terms of two key assumptions that yield the two sets of PDEs—the class of systems for which we can generate a stable observer error dynamics. In section III, we discuss the system theoretic interpretation of the first assumption, that turns out to be equivalent to the aforementioned "partial" linearization (via change of coordinates) of the dynamics. In Section IV, we consider some well known physical examples for which the PDE's occurring in the first and second assumptions are solvable and hence construct reduced-order observers for them. In section V, we present a separation principle for IDA–PBC with the proposed observer. We wrap up the paper with some concluding remarks and future work in Section VI.

II. Immersion and Invariance Observers: General Constructive Procedure

A. Problem Formulation and Proposed Approach

In this note we adopt the observer design framework proposed in [8], which follows the Immersion and Invariance (I&I) principles first articulated in [9]—see [5] for a tutorial

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account of this method and its applications. In the context of observer design the objective of I&I is to generate an attractive invariant manifold, defined in the extended state-space of the plant and the observer. This manifold is defined by an invertible function in such a way that the unmeasurable part of the state can be reconstructed by inversion of this function. We thus introduce the definition of an I&I observer for the system (1), which is a particular case of the one given in [8], see also [10].

Definition 1: The dynamical system

$$\dot{\eta} = \alpha(q, \eta), \tag{3}$$

with $\eta \in \mathbb{R}^n$, is called an I&I observer for the system (1) if there exists a full rank matrix $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and a vector function $\beta : \mathbb{R}^n \to \mathbb{R}^n$, such that the manifold

$$\mathcal{M} := \{(\eta, q, p) : \beta(q) = \eta + \mathcal{T}^\top(q)p\} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \tag{4}$$

is invariant and attractive.[1] In this way, an asymptotic estimate of p, which we will denote by $\hat{p}$, is given by

$$\hat{p} = \mathcal{T}^{-\top}(\beta - \eta).$$

B. Definition of the Class of Mechanical Systems

Given a inertia matrix M, we introduce the following assumptions.

Assumption 1: There exists a full rank matrix $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that, for $i \in \bar{n} := \{1, \dots, n\}$,

$$\mathcal{B}_i(q) + \mathcal{B}_i^\top(q) = 0,$$

where the matrices $\mathcal{B}_i : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ are defined as

$$\begin{aligned}\mathcal{B}_i := & \sum_{j=1}^{n} [\mathcal{T}_i, \mathcal{T}_j]\mathcal{T}_j^\top (\mathcal{T}\mathcal{T}^\top)^{-1} M^{-1} \\ & + \frac{1}{2}\mathcal{T}_{ij}^\top \mathcal{T} \frac{\partial}{\partial q_j}(\mathcal{T}^{-1} M^{-1} \mathcal{T}^{-\top})\mathcal{T}^\top,\end{aligned} \tag{5}$$

where $\mathcal{T}_i = \mathcal{T} e_i$ and $\mathcal{T}_{ij} = e_i^\top \mathcal{T} e_j$, with e_i, $i \in \bar{n}$ being the Euclidean basis vector and $[\mathcal{T}_i, \mathcal{T}_j]$ is the standard Lie bracket.[2]

Assumption 2: There exists a matrix $\mathcal{P} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ satisfying the following two conditions:

(i) The matrix inequality

$$\mathcal{A}(q) + \mathcal{A}^\top(q) \geq \epsilon I_n, \tag{6}$$

holds, uniformly in q, for some $\epsilon > 0$, where

$$\mathcal{A}(q) := \mathcal{P}(q)[\mathcal{T}^\top(q) M(q)]^{-1}. \tag{7}$$

(ii) The rows of $\mathcal{P}$, denoted $\mathcal{P}^j$, satisfy the integrability condition

$$\frac{\partial \mathcal{P}^j}{\partial q} = \left(\frac{\partial \mathcal{P}^j}{\partial q}\right)^\top, \quad j \in \bar{n}. \tag{8}$$

Assumption 1 defines a set of PDEs given by (5), that have to be solved for the unknown $\mathcal{T}$. Further, for a given $\mathcal{T}$, the matrix $\mathcal{P}$ of Assumption 2 can be computed from the solution of the PDEs (8), subject to the inequality constraint (6). Although the assumptions look quite technical and cryptic, we will show in the sequel that Assumption 1 is equivalent to the well–known property of linearizability (via partial change of coordinates) of the system dynamics.

C. I&I Observer

Proposition 1: If the matrices $\mathcal{T}$ and $\mathcal{P}$ satisfy Assumptions 1 and 2, the dynamical system

$$\begin{aligned}\dot{\eta} &= \mathcal{P}(\mathcal{T}^\top M)^{-1}(\beta - \eta) + \mathcal{T}^\top(\frac{\partial V}{\partial q} - Gu) &(9)\\ \hat{p} &= \mathcal{T}^{-\top}(\beta - \eta) &(10)\end{aligned}$$

with

$$\frac{\partial \beta}{\partial q} = \mathcal{P}, \tag{11}$$

is a globally exponentially convergent reduced order observer for (1)—with the estimation error verifying

$$|\hat{p}(t) - p(t)|^2 \leq \exp^{-\epsilon t} |\hat{p}(0) - p(0)|^2,$$

where $|\cdot|$ is the Euclidean norm.

Proof: We consider the manifold $\mathcal{M}$ and differentiate its off–the–manifold coordinate $z = \beta - \eta - \mathcal{T}^\top p$ to obtain

$$\begin{aligned}\dot{z} &= \dot{\beta} - \dot{\eta} - \mathcal{T}^\top \dot{p} - \dot{\mathcal{T}}^\top p \\ &= -\mathcal{P}(\mathcal{T}^\top M)^{-1} z + \mathcal{T}^\top \frac{\partial}{\partial q}(\frac{1}{2}p^\top M^{-1} p) - \dot{\mathcal{T}}^\top p\end{aligned}$$

where we have made use of (1), (9), (10) and (11). We now define

$$\mathbf{D}_\mathcal{T}(q, p) := \mathcal{T}^\top \frac{\partial}{\partial q}(\frac{1}{2}p^\top M^{-1} p) - \dot{\mathcal{T}}^\top p \tag{12}$$

and shall prove that Assumption 1 is equivalent to condition, $\mathbf{D}_\mathcal{T} = 0$. We first see that, $\frac{\partial}{\partial q}(\frac{1}{2}p^\top M^{-1}p) = \frac{\partial}{\partial q}(\frac{1}{2}p^\top \mathcal{T}\mathcal{T}^{-1} M^{-1} \mathcal{T}^{-\top}\mathcal{T}^\top p)$, which further equals

$$\sum_{i=1}^{n} e_i \{p^\top [\frac{\partial \mathcal{T}}{\partial q_i}\mathcal{T}^{-1} M^{-1} + \frac{1}{2}\mathcal{T}\frac{\partial}{\partial q_i}(\mathcal{T}^{-1} M^{-1} \mathcal{T}^{-\top})\mathcal{T}^\top] p\}. \tag{13}$$

We now compute

$$\dot{\mathcal{T}}^\top p = \sum_{i=1}^{n} (\frac{\partial \mathcal{T}^\top}{\partial q_i} p)(e_i^\top \mathcal{T})\mathcal{T}^{-1} M^{-1} p. \tag{14}$$

We next note that, if we define

$$\mathcal{J} := \sum_{i=1}^{n} \{(\mathcal{T}^\top e_i)(p^\top \frac{\partial \mathcal{T}}{\partial q_i}) - (p^\top \frac{\partial \mathcal{T}}{\partial q_i})^\top (e_i^\top \mathcal{T})\}, \tag{15}$$

[1] We recall that the set $\mathcal{M}$ is invariant if $(\eta(0), q(0), p(0)) \in \mathcal{M} \Rightarrow (\eta(t), q(t), p(t)) \in \mathcal{M}$ for all $t \geq 0$. It is said to be globally attractive if, for all $(\eta(0), q(0), p(0))$, the distance of the state vector to the manifold asymptotically goes to zero, i.e., $\lim_{t\to\infty} \mathrm{dist}\{(\eta(t), q(t), p(t)), \mathcal{M}\} = 0$.

[2] The standard Lie bracket of two vector fields $\mathcal{T}_i$, $\mathcal{T}_j$ is defined as

$$[\mathcal{T}_i, \mathcal{T}_j] = \frac{\partial \mathcal{T}_j}{\partial q}\mathcal{T}_i - \frac{\partial \mathcal{T}_i}{\partial q}\mathcal{T}_j$$

then some simple computations leads to

$$e_{\rm j}^\top \mathcal{J} e_{\rm k} = p^\top [\mathcal{T}_{\rm j}, \mathcal{T}_{\rm k}]. \quad (16)$$

Finally, substituting (13), (14), (16) in (12) and performing some simplifications leads to

$$\mathbf{D}_\mathcal{T} = \sum_{\rm i=1}^{\rm n} e_{\rm i} p^\top \mathcal{B}_i p, \quad (17)$$

where we have invoked the definition of $\mathcal{B}_i$ given in (5). Hence, each element of the vector $\mathbf{D}_\mathcal{T}$ is a quadratic form in p, which becomes zero for all p if and only if Assumption 1 is satisfied.

Now, with the integrability condition (8) of Assumption 2 and (11), the error dynamics reduces to

$$\dot{z} = -\mathcal{P}(\mathcal{T}^\top M)^{-1} z. \quad (18)$$

The manifold $\mathcal{M}$ is clearly positively invariant. To establish global exponential attractivity of $\mathcal{M}$ consider the Lyapunov function $V(z) = \frac{1}{2}|z|^2$. Condition (6) ensures that $\dot{V} \leq -\epsilon V$, which proves the global exponential convergence to zero of z, hence of $\hat{p} - p$—with exponential rate ϵ. ■

Remark 1: It is clear that, if $\mathcal{T}^\top M + M\mathcal{T} > 0$, Assumption 2 is satisfied with $\mathcal{P} = I_n$. The design parameter $\mathcal{P}$ gives us an extra degree of freedom when this is not the case. Also, it is obvious from the proof above that we can replace (6) by $Q\mathcal{A} + \mathcal{A}^\top Q \geq \epsilon I_n$, for some constant matrix $Q \in \mathbb{R}^{n\times n}$, $Q = Q^\top > 0$. In this case, we should take as Lyapunov function for the observer error dynamics $\tilde{V}(z) = \frac{1}{2} z^\top Q z$.

III. System Theoretic Interpretation of Assumption 1

As shown in the proof of Proposition 1, the role of Assumptions 1 and 2 in the stability analysis of the observer error dynamics is clear: they ensure, respectively, that the disturbance term $\mathbf{D}_\mathcal{T}$ identically vanishes and that the dynamics of z is stable. However, both assumptions seem to be only motivated by the chosen (I&I) framework and the (Lyapunov) analysis technique and are, furthermore, quite cryptic—that stymies the physical interpretation of the class. Nevertheless, in this section we will show that Assumption 1 is precisely identifying the class of mechanical systems for which a change of coordinates of the form $(q, P) = (q, \mathcal{T}^\top(q)p)$, renders the system linear in the unmeasurable states. We also discuss some particular selections of $\mathcal{T}$ that, either have been been reported in the literature, or are useful to verify Assumption 2.

A. Assumption 1 is Equivalent to (Partial) Linearization

Proposition 2: The dynamics of the system (1) expressed in the coordinates (q, P), where $P = \mathcal{T}^\top(q)p$, is linear in P if and only if Assumption 1 holds, in which case, the dynamics becomes

$$\begin{aligned} \dot{q} &= M^{-1}\mathcal{T}^{-\top} P, \\ \dot{P} &= -\mathcal{T}^\top \left(\frac{\partial V}{\partial q} - Gu\right). \end{aligned} \quad (19)$$

Proof: The equation for $\dot{q}$ follows trivially from the definition of P. Now, $\dot{P}$ can be expressed as

$$\begin{aligned} \dot{P} &= \dot{\mathcal{T}}^\top p + \mathcal{T}^\top \dot{p} \\ &= -\mathbf{D}_\mathcal{T} - \mathcal{T}^\top (\frac{\partial V}{\partial q} - Gu), \end{aligned} \quad (20)$$

where we used (12) to get the second equation. From (20) we see that the dynamics is linear in P,[3] if and only if Assumption 1 holds or equivalently $\mathbf{D}_\mathcal{T} = 0$. Further, under Assumption 1, the dynamics expressed in the coordinates (q, P) takes the form (19). ■

To streamline the presentation in the sequel we find it convenient at this point to recall the Lagrangian model of the mechanical system (1)

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + \frac{\partial V}{\partial q} = G(q)u, \quad (21)$$

where $C(q, \dot{q})\dot{q}$ is the vector of Coriolis and centrifugal forces with the ik–th element of the matrix $C : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ being defined by $C_{ik}(q, \dot{q}) = \sum_{j=1}^n C_{ij}^k(q)\dot{q}_j$. Further, $C_{ij}^k : \mathbb{R}^n \to \mathbb{R}$ are the Christoffel symbols of the second kind of the inertia matrix M given by

$$C_{ij}^k(q) := \frac{1}{2}[\frac{\partial M_{\rm ik}}{\partial q_{\rm j}} + \frac{\partial M_{\rm jk}}{\partial q_{\rm i}} - \frac{\partial M_{\rm ij}}{\partial q_{\rm k}}], \quad \forall\, i, j, k \in \bar{n}, \quad (22)$$

where M_{ij} is the ij–th element of M. We further recall the well–known fact that

$$\frac{\partial}{\partial q}(\frac{1}{2}\dot{q}^\top M \dot{q}) = (C - \dot{M})\dot{q}. \quad (23)$$

See [11] for other important properties of mechanical systems that are relevant in control applications.

B. $\mathcal{T} = M^{-1}$: A Strong Condition for (Partial) Linearizability

Proposition 3: Consider the parameterized vector $\mathbf{D}_\mathcal{T}$ defined in (12). The following statements are equivalent:

(i) Assumption 1 holds with $\mathcal{T} = M^{-1}$, that is, $\mathbf{D}_{M^{-1}} = 0$.

(ii) The Christoffel symbols of the second kind of the inertia matrix M, defined in (22), are all equal to zero.

(iii) The Coriolis and centrifugal forces $C(q, \dot{q})\dot{q}$ are equal to zero.

Proof: Define the vector function $\tilde{\mathbf{D}}_\mathcal{T}(q, \dot{q}) := \mathbf{D}_\mathcal{T}(q, M(q)\dot{q})$. Proceeding from (12), we will now express this function in terms of the matrices C and M

$$\begin{aligned} \tilde{\mathbf{D}}_\mathcal{T} &= \mathcal{T}^\top \frac{\partial}{\partial q}(\frac{1}{2}\dot{q}^{\rm T} M \dot{q}) - \dot{\mathcal{T}}^\top M \dot{q}, \\ &= [\mathcal{T}^\top C - \frac{d}{dt}(\mathcal{T}^\top M)]\dot{q}, \end{aligned} \quad (24)$$

where, to obtain the second identity, we have used (23). Hence, $\tilde{\mathbf{D}}_{M^{-1}} = M^{-1}C\dot{q}$, which is zero iff $C\dot{q} = 0$. ■

Remark 2: The choice $\mathcal{T} = M^{-1}$ or equivalently the case where there are no Coriolis or centrifugal forces acting on the

[3] We recall that, as shown by (17), $\mathbf{D}_\mathcal{T}$ is quadratic in p—hence also quadratic in P.

system,[4] is clearly of limited practical interest and is given here only to illustrate one particular physical interpretation of Assumption 1.

C. $\mathcal{T}\mathcal{T}^\top = M^{-1}$: A Weaker Condition for (Partial) Linearizability

In this subsection we propose—as suggested in [8], [12]—to take $\mathcal{T} = T$ where, $T : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ satisfies $T(q)T^\top(q) = M^{-1}(q)$. We now prove that Assumption 1, in this case, is strictly weaker than the absence of Coriolis and centrifugal forces and, furthermore, has a nice geometric interpretation.

Proposition 4: Consider the factorization $TT^\top = M^{-1}$ and the parameterized vector $\mathbf{D}_\mathcal{T}$ defined in (12). The following statements are equivalent:

(i) Assumption 1 holds with $\mathcal{T} = T$, that is, $\mathbf{D}_T = 0$.
(ii) For all $i \in \bar{n}$, the $n \times n$ matrices $\sum_{\mathrm{j}=1}^{\mathrm{n}}[T_\mathrm{i}, T_\mathrm{j}]T_\mathrm{j}^\top$ are skew symmetric, where $T_i := Te_i$.

Proof: Evaluating the matrices $\mathcal{B}_i$ defined in (5) for $\mathcal{T} = T$ and noting that the second right term vanishes, we get

$$\mathcal{B}_i = \sum_{\mathrm{j}=1}^{\mathrm{n}}[T_\mathrm{i}, T_\mathrm{j}]T_\mathrm{j}^\top \tag{25}$$

Referring to (17) we easily see the equivalence between (i) and (ii). ■

D. Condition (ii) of Proposition 4 is Strictly Weaker than Commutativity of the Columns of T

A sufficient condition for (ii) to hold is clearly that, for all $i, j \in \bar{n}$, $[T_\mathrm{i}, T_\mathrm{j}] = 0$—when it is said that the columns of T commute. However, we show in the next subsection that (for $n \geq 3$) this condition is not necessary.

The case when the columns of T commute has been extensively studied in analytical mechanics and has a deep geometric significance—stemming from Theorem 2.36 in [13]. It is widely accepted that this condition is quite restrictive and a natural question is whether the skew–symmetry condition (ii) of Proposition 4 is strictly weaker than commutativity. In this subsection we show that this is indeed the case for $n \geq 3$.

Before presenting the result we find it convenient to recall the following well–known fact of Riemannian geometry that has been exploited, in the context of linearization, in the control literature in [14], [15].

Fact 1: Given an inertia matrix M. The following statements are equivalent:

i) There exists a matrix T verifying $TT^\top = M^{-1}$ and such that $[T_\mathrm{i}, T_\mathrm{j}] = 0$, for all $i, j \in \bar{n}$.
ii) There exists a vector function $Q : \mathbb{R}^n \to \mathbb{R}^n$ such that
$$\frac{\partial Q}{\partial q} = T^{-1}(q). \tag{26}$$
iii) The Riemann symbols (that can be computed directly from M with the formulas given on page (4D-7) of [16]) vanish identically.

If the conditions of Fact 1 are satisfied, the system is said to be *Euclidean* [14], where the qualifier stems from the fact that the dynamics expressed in the coordinates (Q, P) reduces to a "linear double integrator" of the form

$$\dot{Q} = P,\ \dot{P} = -\frac{\partial \tilde{V}}{\partial Q} + T^\top G u,$$

where $\tilde{V}(Q) := V(Q^I(Q))$, with $Q^I : \mathbb{R}^n \to \mathbb{R}^n$ a right inverse of $Q(q)$, that is, $Q(Q^I(x)) = x$ for all $x \in \mathbb{R}^n$. We next state the following proposition.

Proposition 5: For a given inertia matrix M, the fact that there exists a factorization $TT^\top = M^{-1}$ such that the matrices $\mathcal{B}_i$ defined in (25) are skew–symmetric *does not imply* that the system is Euclidean for $n \geq 3$. On the other hand, for $n \leq 2$ both conditions are equivalent.

Proof: First, we prove that for $n \leq 2$ commutativity is equivalent to skew–symmetry. For $n = 1$ the equivalence is, of course, trivial. For $n = 2$ this can be easily shown using the fact that all 2×2 skew–symmetric matrices are of the form $\begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}$, $\alpha \in \mathbb{R}$.

The first claim of the proposition will be established constructing an inertia matrix whose Riemann symbols are not all zero, but for which we can find a factorization that satisfies the skew–symmetry condition. Towards this end, set $n = 3$ and consider

$$M^{-1} = \begin{bmatrix} 1+q_2^2 & 0 & q_2\sqrt{1+q_2^2} \\ 0 & (1+q_2^2)^2 & 0 \\ q_2\sqrt{1+q_2^2} & 0 & 1+q_2^2 \end{bmatrix}. \tag{27}$$

We now compute the Riemann symbols, defined in page (4D-7) of [16] as

$$\begin{aligned} R_{ijlk} := \frac{1}{2}\left[\frac{\partial^2 M_{\mathrm{ik}}}{\partial q_\mathrm{j}\partial q_\mathrm{l}} + \frac{\partial^2 M_{\mathrm{jl}}}{\partial q_\mathrm{i}\partial q_\mathrm{k}} - \frac{\partial^2 M_{\mathrm{il}}}{\partial q_\mathrm{j}\partial q_\mathrm{k}} - \frac{\partial^2 M_{\mathrm{jk}}}{\partial q_\mathrm{i}\partial q_\mathrm{l}}\right] \\ + \sum_{a,b=1}^{\mathrm{n}}(M^{-1})_{ab}\left[C_{jl}^a C_{ik}^b - C_{il}^a C_{jk}^b\right] \end{aligned} \tag{28}$$

where C_{ij}^k are the Christoffel symbols of the second kind as defined in (22) and $(M^{-1})_{\mathrm{ij}}$ is the ij-th element of the inertia matrix inverse. After some computations we verify that $R_{1212}, R_{1323}, R_{2323} \neq 0$ for all q and $R_{1223} \neq 0$ for $q_2 \neq 0$, and hence we conclude from Fact 1 that the system is not Euclidean.

On the other hand, it can be easily verified that the matrix M^{-1} admits a factorization $TT^\top = M^{-1}$ with

$$T = \begin{bmatrix} \sin(q_1)q_2 & \cos(q_1)q_2 & 1 \\ (1+q_2^2)\cos(q_1) & -(1+q_2^2)\sin(q_1) & 0 \\ \sqrt{1+q_2^2}\sin(q_1) & \sqrt{1+q_2^2}\cos(q_1) & 0 \end{bmatrix}. \tag{29}$$

Computing the Lie brackets with the vectors T_i we obtain

$$[T_1, T_2] = T_3, \quad [T_2, T_3] = T_1, \quad [T_3, T_1] = T_2. \tag{30}$$

Hence, each of the matrices $\mathcal{B}_1 = [T_1, T_2]T_2^\top + [T_1, T_3]T_3^\top$, $\mathcal{B}_2 = [T_2, T_1]T_1^\top + [T_2, T_3]T_3^\top$, $\mathcal{B}_3 = [T_3, T_1]T_1^\top +$

[4]Note, however, that $C\dot{q} = 0$ does not imply that the inertia matrix M is constant.

$[T_2, T_3]T_3^\top$ are skew symmetric as desired. This completes the proof. ■

Remark 3: The non-Euclidean system presented in the previous section is a mathematical example used to show that the condition (ii) of Proposition 4 is strictly weaker than commutativity of the columns of T. The kind of physical systems that fall under such class is currently under investigation.

Remark 4: It is important to underscore the limited applicability of the the "linearization" procedure for Euclidean systems, which requires the solution of the PDE (26). Indeed, in contrast with (some of) the PDEs that we encounter in the current paper, this PDE has no *free parameters* and its explicit solution may be even impossible. This happens in the case of the classical cart–pole system which is Euclidean but, as shown in [14], equation (26) leads to an elliptic integral of the second kind that does not admit a closed form solution.

IV. Physical Examples

The technical note [1] contains a constructive algorithm for computing $\mathcal{P}$ from (6), (8) for some special choices of $\mathcal{T}$. We now illustrate the observer design on three physical systems which are Euclidean where, in each case, the computation of $\mathcal{P}$ has been done by following the algorithmic procedure.

A. Inverted Pendulum on a Cart [6]

The inertia matrix M of the well–known inverted pendulum on a cart system and its corresponding lower triangular Cholesky factorization, $TT^\top = M^{-1}$ are given as,

$$M = \begin{bmatrix} 1 & * \\ b\cos q_1 & m_3 \end{bmatrix},$$

$$T = \begin{bmatrix} \frac{\sqrt{m_3}}{\sqrt{m_3 - b^2\cos^2 q_1}} & 0 \\ \frac{-b\cos q_1}{\sqrt{m_3}\sqrt{m_3 - b^2\cos^2 q_1}} & \frac{1}{\sqrt{m_3}} \end{bmatrix}.$$

We can easily check that the columns of T commute thus satisfying Assumption 1. We now set $\mathcal{P}$ as

$$\mathcal{P} = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{\partial \phi_2}{\partial q_1} & 0 \end{bmatrix},$$

where $\Lambda_{ii} > 0$ and ϕ_2 is a smooth function of q_1. We notice that $\mathcal{P}$ trivially satisfies (ii) of Assumption 2, and $\mathcal{A} = \mathcal{P}T$ is lower triangular with strictly positive diagonal entries. Our strategy is to make the off-diagonal terms in $\mathcal{A}$ to equal zero, which ensures that equation (6) is satisfied. We accordingly solve, $\mathcal{A}_{21} = 0$ to obtain, $\phi_2 = \frac{\Lambda_{22} b}{m_3}\sin(q_1)$. Thus, we get

$$\beta = \begin{bmatrix} \Lambda_{11} q_1 \\ \Lambda_{22}(q_2 + \frac{b}{m_3}\sin(q_1)) \end{bmatrix}. \tag{31}$$

B. 3-Link Underactuated Planar Manipulator [17]

This is a 3-DOF underactuated mechanical system with inertia matrix M and lower triangular Cholesky factorization, $TT^\top = M^{-1}$ given as,

$$M = \begin{bmatrix} 1 & * & * \\ -m_3 L sin q_1 & m_x & * \\ m_3 L\cos q_1 & 0 & m_y \end{bmatrix},$$

$$T = \begin{bmatrix} \frac{1}{F} & 0 & 0 \\ \frac{m_3 L}{m_x F}\sin q_1 & \frac{1}{\sqrt{m_x}} & 0 \\ -\frac{m_3 L}{m_y F}\cos q_1 & 0 & \frac{1}{\sqrt{m_y}} \end{bmatrix},$$

where $F(q) := \sqrt{1 - \frac{m_3^2 L^2}{m_y}\cos^2 q_1 - \frac{m_3^2 L^2}{m_x}\sin^2 q_1}$. We can easily check that the columns of T commute and thus the system is Euclidean. We set $\mathcal{P}$ as

$$\mathcal{P} = \begin{bmatrix} \Lambda_{11} & 0 & 0 \\ \frac{\partial \phi_2}{\partial q_1} & \Lambda_{22} & 0 \\ \frac{\partial \phi_3}{\partial q_1} + \frac{\partial \psi_{32}}{\partial q_1} q_2 & \psi_{32} & \Lambda_{33} \end{bmatrix},$$

where $\Lambda_{ii} > 0$ and ϕ_1, ϕ_2, ψ_{32} are smooth functions of q_1. We notice that $\mathcal{P}$ satisfies (ii) of Assumption 2 and $\mathcal{A} = \mathcal{P}T$ is lower triangular with strictly positive diagonal entries. We now proceed to make the off-diagonal terms in $\mathcal{A}$ to equal zero in order to satisfy (6). We accordingly solve in the order $\mathcal{A}_{32} = 0$, $\mathcal{A}_{31} = 0$, $\mathcal{A}_{21} = 0$ to obtain $\psi_{32} = 0$, $\phi_3 = \frac{\Lambda_{33} m_3 L}{m_y}\sin q_1$ and $\phi_2 = \frac{\Lambda_{22} m_3 L}{m_x}\cos q_1$ respectively. We then integrate $\mathcal{P}$ to obtain

$$\beta = \begin{bmatrix} \Lambda_{11} q_1 \\ \Lambda_{22}(q_2 + \frac{m_3 L}{m_x}\cos q_1) \\ \Lambda_{33}(q_3 + \frac{m_3 L}{m_y}\sin q_1) \end{bmatrix}.$$

C. Planar Redundant Manipulator with one elastic degree of freedom [18]

This is a 4-DOF underactuated mechanical system whose inertia matrix depends on two coordinates and is given as,

$$\begin{bmatrix} I + \bar{M} & * & * & * \\ \bar{M} & \bar{M} & * & * \\ \frac{\bar{M}}{L} s(q_1 + q_2) & \frac{\bar{M}}{L} s(q_1 + q_2) & \tilde{M} + m & * \\ -\frac{\bar{M}}{L} c(q_1 + q_2) & -\frac{\bar{M}}{L} c(q_1 + q_2) & 0 & \tilde{M} + m \end{bmatrix},$$

where $s(\cdot) = \sin(\cdot)$, $c(\cdot) = \cos(\cdot)$ and $\bar{M} = \tilde{M} L^2$. We now compute the lower triangular cholesky factorization, $TT^\top = M^{-1}(q)$ as

$$\begin{bmatrix} \frac{1}{\sqrt{I}} & 0 & 0 & 0 \\ -\frac{1}{\sqrt{I}} & \frac{\sqrt{\tilde{M}+m}}{\sqrt{\tilde{M} m L}} & 0 & 0 \\ 0 & -\sqrt{\frac{\tilde{M}}{m}}\frac{1}{\sqrt{\tilde{M}+m}} s(q_1 + q_2) & \frac{1}{\sqrt{\tilde{M}+m}} & 0 \\ 0 & \sqrt{\frac{\tilde{M}}{m}}\frac{1}{\sqrt{\tilde{M}+m}} c(q_1 + q_2) & 0 & \frac{1}{\sqrt{\tilde{M}+m}} \end{bmatrix}$$

We can check that the columns of T commute among each other thus satisfying Assumption 1. We let the matrix $\mathcal{P}$ be given as,

$$\mathcal{P} = \begin{bmatrix} \Lambda_{11} & 0 & 0 & 0 \\ \frac{\partial \phi_2}{\partial q_1} & \frac{\partial \phi_2}{\partial q_2} + \Lambda_{22} & 0 & 0 \\ \frac{\partial \phi_3}{\partial q_1} & \frac{\partial \phi_3}{\partial q_2} & \Lambda_{33} & 0 \\ \frac{\partial \phi_4}{\partial q_1} + \frac{\partial \psi_{43}}{\partial q_1} q_3 & \frac{\partial \psi_{43}}{\partial q_2} q_3 + \frac{\partial \phi_4}{\partial q_1} & \psi_{43} & \Lambda_{44} \end{bmatrix},$$

where $\Lambda_{ii} > 0$ and each of the functions ϕ_2, ϕ_3, ϕ_4, ψ_{43} depend smoothly on both q_1 and q_2. We note that $\mathcal{P}$ satisfies (ii) of Assumption 2 and $\mathcal{A} = \mathcal{P}T$ is lower triangular. We now proceed to make the off diagonal terms in $\mathcal{A}$ equal zero and the diagonal entries as strictly positive. From $\mathcal{A}_{21} = 0$, we get $\frac{\partial\phi_2}{\partial q_1} = \frac{\partial\phi_2}{\partial q_2}$ and from $\mathcal{A}_{22} > 0$, we get $\frac{\partial\phi_2}{\partial q_1} > 0$. Thus, we let $\phi_2 = k(q_1 + q_2)$ where $k > 0$. We now solve $\mathcal{A}_{43} = 0$ to obtain $\psi_{43} = 0$. We then solve $\mathcal{A}_{42} = 0$ to get $\phi_4 = -\frac{\tilde{M}L\Lambda_{44}}{\tilde{M}+m}\sin(q_1 + q_2) + g(q_1)$. Finally, from $\mathcal{A}_{41} = 0$, we get $\frac{\partial\phi_4}{\partial q_1} = \frac{\partial\phi_4}{\partial q_2}$ and hence we can set $g = 0$. We next solve $\mathcal{A}_{32} = 0$ to obtain $\phi_3 = -\frac{\tilde{M}L\Lambda_{33}}{\tilde{M}+m}\cos(q_1+q_2)+f(q_1)$. Next, from $\mathcal{A}_{31} = 0$, we get $\frac{\partial\phi_3}{\partial q_1} = \frac{\partial\phi_3}{\partial q_2}$ and hence we can set $f = 0$. We finally get

$$\beta = \begin{bmatrix} \Lambda_{11}q_1 \\ \Lambda_{22}q_2 + k(q_1 + q_2) \\ \Lambda_{33}(q_3 - \frac{\tilde{M}L}{\tilde{M}+m}\cos(q_1 + q_2)) \\ \Lambda_{44}(q_4 - \frac{\tilde{M}L}{\tilde{M}+m}\sin(q_1 + q_2)) \end{bmatrix}.$$

V. A Separation Principle for IDA–PBC Designs with I&I Observers

In this section we establish a separation principle for the combination of the IDA-PBC proposed in [7] (see also [6]), with the I&I observer derived in Section 2. In particular, we prove that under very weak conditions, the measurement of momenta, p, required in IDA-PBC, can be replaced by its observed signal, $\hat{p}$, preserving asymptotic stability of the desired equilibrium.

For the sake of brevity we do not review here the IDA–PBC methodology, but only give the key equations needed for our analysis. We refer the reader to [6] and [7] for additional details. The objective in IDA–PBC is to assign to the closed–loop an energy function of the form

$$H_{\rm d}(q, p) = \frac{1}{2}p^\top M_{\rm d}^{-1}(q)p + V_{\rm d}(q) - V_d(q_\star)$$

where $M_{\rm d} = M_d^\top > 0$, V_d are the desired inertia matrix and potential energy function, respectively, and $q_\star$ is the desired position. This is achieved imposing the closed–loop dynamics

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{bmatrix} 0 & M^{-1}M_{\rm d} \\ -M_{\rm d}M^{-1} & J_2 - GK_vG^\top \end{bmatrix} \begin{pmatrix} \frac{\partial H_{\rm d}}{\partial q} \\ \frac{\partial H_{\rm d}}{\partial p} \end{pmatrix}, \tag{32}$$

where $K_v = K_v^\top > 0$ is a damping injection matrix and $J_2(q, p)$ is a skew–symmetric matrix having each element of the form $p^\top\alpha_i(q)$ where, $\alpha_i : \mathbb{R}^n \to \mathbb{R}^n,\ i = 1,\ldots,\frac{n}{2}(n-1)$, are free functions.

If $q_\star = \arg\min V_d(q)$ then $(q_\star, 0)$ is a stable equilibrium of the closed loop with Lyapunov function H_d clearly verifying $\dot{H}_d = -p^\top M_d^{-1}GK_vG^\top M_d^{-1}p \leq -c_1|\bar{p}|^2$, where, to simplify the notation in the sequel, we define the function $\bar{p}(q, p) := G^\top(q)M_d^{-1}(q)p$ and use the convention of denoting with c_i a (often unspecified) positive constant—in this case $c_1 := \lambda_{min}\{K_v\}$. Stability will be asymptotic if $\bar{p}$ is a *detectable* output for the closed–loop system (32).

The full–state measurement IDA–PBC law is given by

$$u = (G^\top G)^{-1}G^\top\left(\frac{\partial H}{\partial q} - M_dM^{-1}\frac{\partial H_d}{\partial q} + J_2M_d^{-1}p\right) - K_v\bar{p}, \tag{33}$$

which, as shown in [6], may be written in the form

$$u(q, p) = u_0(q) + \begin{bmatrix} p^\top A_1(q)p \\ \vdots \\ p^\top A_m(q)p \end{bmatrix} - K_v\bar{p}, \tag{34}$$

where the vector $u_0 : \mathbb{R}^n \to \mathbb{R}^n$ and the matrices $A_i : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ are functions of q. As will be shown below, establishing boundedness of $A_i,\ i = 1,...m$, will be critical for our analysis. We next introduce the following assumption.

Assumption 3: The matrices $\frac{\partial M}{\partial q_i}$, $\frac{\partial M_d}{\partial q_i}$ and G are bounded.

Proposition 6: Consider the system (1) and define the *position feedback* controller as $u = u(q, \hat{p})$ with $\hat{p}$ an estimate of p generated by the I&I observer (10). Assume $\bar{p}(q, p)$ is a *detectable* output for the closed–loop system (32) and that Assumptions 1 and 2 are satisfied. Then there exists a neighborhood of the point $(q^\star, 0, \beta(q^\star))$ such that all trajectories of the closed–loop system starting in this neighborhood are bounded and satisfy

$$\lim_{t\to\infty}(q(t), p(t), \eta(t)) = (q^\star, 0, \beta(q^\star)).$$

Furthermore, if Assumption 3 holds and the full state–feedback controller (34) ensures global asymptotic stability then the neighborhood is the whole space $\mathbb{R}^{3n}$, thus boundedness and convergence are *global*.

Proof: To carry out the proof we will write the overall system as a cascade connection of the observer error subsystem $\dot{z} = -\mathcal{A}z$ and the full state–feedback dynamics (32). For, we notice that $u(q, \hat{p}) = u(q, p) + \chi(q, p, z)$ where we have defined

$$\chi := \sum_{i=1}^{m}\left[z^\top\mathcal{T}^{-1}A_i\mathcal{T}^{-\top}z + z^\top\mathcal{T}^{-1}(A_i + A_i^\top)p\right]e_i - K_vG^\top M_d^{-1}\mathcal{T}^{-\top}z. \tag{35}$$

The overall system can be written in the cascaded form

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{bmatrix} 0 & M^{-1}M_{\rm d} \\ -M_{\rm d}M^{-1} & J_2 - GK_vG^\top \end{bmatrix} \begin{pmatrix} \frac{\partial H_{\rm d}}{\partial q} \\ \frac{\partial H_{\rm d}}{\partial p} \end{pmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix}\chi$$

$$\dot{z} = -\mathcal{A}z \tag{36}$$

From the discussion above we have that the system with $\chi = 0$ is asymptotically stable. Furthermore, the disturbance term is such that $G(q)\chi(q, p, 0) = 0$. Invoking well–known results of asymptotic stability of cascaded systems [19] completes the proof of local asymptotic stability.

To complete the global claim we invoke the fundamental result of [20], see also [21], and see that the proof will be completed if we can establish boundedness of the trajectories $(q(t), p(t))$. Computing the time derivative of H_d along the trajectories of (36) we get the bound

$$\dot{H}_d \leq -c_1|\bar{p}|^2 + |\bar{p}||G\chi|. \tag{37}$$

Comparing (33) with (34), we observe that the matrices A_i will be bounded if Assumption 3 holds. Further, from the

IDA–PBC procedure we know that J_2 satisfies the so–called kinetic energy PDE

$$G^{\perp}[M_d M^{-1}\frac{\partial}{\partial q}(p^\top M_d^{-1}p) - \frac{\partial}{\partial q}(p^\top M^{-1}p)] = 2G^{\perp} J_2 M_d^{-1} p, \tag{38}$$

and hence by comparing in this equation the terms which are quadratic in p and the form of J_2, we can obtain, under Assumption 3, the bound $\|J_2\| \leq c_2|p|$, where $\|\cdot\|$ is the matrix norm induced by the Euclidean norm.

From the previous discussion we get the bound $|G\chi| \leq |z|(c_2 + c_3|p|)$, which replaced in (37) yields

$$\dot{H}_d \leq -c_1|\bar{p}|^2 + |\bar{p}||z|(c_2 + c_3|p|). \tag{39}$$

Now, invoking standard Young's inequality arguments we get $|\bar{p}||z| \leq \frac{c_1}{c_2}|\bar{p}|^2 + \frac{c_2}{4c_1}|z|^2$, which upon replacement in (39) yields $\dot{H}_d \leq \frac{c_2^2}{4c_1}|z|^2 + c_5|z||p|^2$, where we have used the bound of $|\bar{p}| \leq c_4|p|$ to define $c_5 := c_3c_4$. Now, let us consider the non-negative function

$$W(q,p,z) := H_d(q,p) + \frac{c_2^2}{4c_1\epsilon}V(z),$$

where $V(z) = \frac{1}{2}|z|^2$, which as shown in the proof of Proposition 1 verifies $\dot{V} \leq -\epsilon V$. Evaluating the derivative we get

$$\dot{W} \leq c_5|z||p|^2 \leq c_6|z|W, \tag{40}$$

where we have used the bounds $W \geq H_d \geq \frac{1}{2}\lambda_{\max}(M_{\rm d})|p|^2$ to obtain the last inequality. Since z is clearly an *integrable* function, invoking the Comparison Lemma [22], we immediately conclude boundedness of W and, consequently, boundedness of the trajectories $(q(t), p(t))$ and complete the proof. ■

VI. Concluding Remarks and Future Work

We have identified in this paper a class of mechanical systems for which a globally exponentially stable reduced order observer can be designed. The class is characterized by (the solvability of) a set of PDEs and contains all systems that can be rendered linear in (the unmeasurable) momenta via a (partial) change of coordinates $P = \mathcal{T}(q)p$. It is also shown that this class is larger than the one reported in the literature of observer design and linearization. We also prove that, under a very weak assumption, the observer can be used in conjunction with an asymptotically stabilizing full state–feedback IDA–PBC preserving stability. To the best of our knowledge, this is the strongest, and more general, result of position feedback stabilization of mechanical systems reported to date.

Several open questions are currently under investigation:

- Similar to the well–known characterization of Euclidean systems in terms of the Riemann symbols, it would be interesting to derive necessary and sufficient conditions on M to verify the skew-symmetry assumption of Proposition 4.
- It is possible to show that the skew-symmetry condition of Proposition 4, using the Cholesky factorization, is not verified for manipulators with more than one rotational joint. However, it is not clear whether other factorizations may exist of it or whether they can be handled imposing the weaker Assumption 1, that is, by considering a general $\mathcal{T}$.
- The solvability of the PDEs arising in Assumption 1 for a general $\mathcal{T}$ is a widely open question. These PDEs are, in general, nonlinear and quite involved.

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