

Feedback Control and the Arrow of Time

Tryphon T. Georgiou and Malcolm C. Smith

Abstract—The purpose of this paper is to highlight the central role that the time asymmetry of stability plays in feedback control. We show that this provides a new perspective on the use of doubly-infinite or semi-infinite time axes for signal spaces in control theory. We then focus on the implication of this time asymmetry in modeling uncertainty, regulation and robust control. We point out that modeling uncertainty and the ease of control depend critically on the direction of time. We finally discuss the relationship of this control-based time arrow with the well-known arrows of time in physics.

I. INTRODUCTION

The origin and implications of the “arrow of time” is one of the deepest and least understood subjects of physics. The “arrow” dictates physics and life as we know it. Yet its emergence in thermodynamics and cosmology, from physical laws which are apparently impervious to it, remains a controversial subject. At first sight, this subject may seem unconnected with the theory of feedback control. However, starting from the very basic fact that our notion of stability in the sense of Lyapunov is time-asymmetric, we argue that the “arrow of time” does have important implications on modeling and uncertainty, robustness of stability, as well as on the topology for the study of the dynamics of feedback interconnections.

The circle of ideas that gave rise to this paper began in a short note published by the authors thirteen years ago [5]. In it, it was pointed out that the doubly-infinite time axis presents some intrinsic difficulties for developing a suitable input-output systems theory—difficulties that are not present in the semi-infinite time axis setting. These difficulties are not mere mathematical technicalities. Rather, they relate fundamentally to the consistency of the theory of stabilizability across different frameworks. Subsequently, a number of papers were written which shed light on the problem [21], [22], [23], [12], [13], [14]. The present paper takes a fresh look and traces the origin of the “puzzle” to the arrow of Lyapunov stability, and then, explores the relevance of this arrow to the topology of dynamical systems and feedback theory.

The relationship of the modern theory of dynamical systems with classical physics and thermodynamics is a developing one. A classical contribution by Nyquist and Johnson [18], [15] is a derivation of the electromotive force due to thermal agitation in conductors. In [3] the issue of irreversibility is treated from the point of view of stochastic control theory. More recently [10] has sought to formalize classical thermodynamics in the mathematical language of modern dynamical systems (see also [27]). In [25] information flow and entropy have been studied in the context of the

Kalman filter. In [26] it is shown that a linear macroscopic dissipative system can be approximated by a linear lossless microscopic system over arbitrary long time intervals. Our point of view here is influenced by [19] and is somewhat different to the above references in that our main goal is to highlight a time-asymmetry, point out its implications, and discuss its relationship to other well-known asymmetries.

The paper begins by providing a new explanation of the issues raised in [5] with regard to an input-output theory for the doubly-infinite time axis. In Section III we introduce the time-conjugation operator in order to discuss and underscore implications of the time-arrow in optimal control problems. In Section IV we analyse the effect of the time-arrow on modelling uncertainty. In particular, we explore the fact that dynamical systems which are close in the usual sense, that a common controller can stabilise and give similar closed-loop responses for either, may not be close when the time-arrow is reversed. Then, in Section V, we further expose the inherent time-asymmetry in our ability to control a dynamical system with two specific examples. These can be thought of as examples of time irreversible feedback phenomena (see Section V-B). We conclude with some thoughts on the “arrow of time” and its various manifestations across physical disciplines.

II. TIME-ASYMMETRY AND STABILITY

A. Input-output and Lyapunov stability

We focus on finite-dimensional linear dynamical systems which, for the most part, are assumed to be time-invariant. The dimensions of input, state and output (column) vectors, as well as the consistent sizes of transformation matrices in state-space models, are suppressed for notational simplicity. The following result is basic and well-known, cf. [31, p. 52-53], [8, p. 82].

Proposition 1: Let \mathbf{P} be a linear time-invariant system which is controllable and observable and is specified by

$$\dot{x} = Ax + Bu, \quad (1)$$

$$y = Cx + Du, \quad (2)$$

and an initial condition $x(0) = 0$. Then $y \in \mathcal{L}_2[0, \infty)$ for all $u \in \mathcal{L}_2[0, \infty)$ if and only if the matrix A is Hurwitz. Moreover, if this condition holds, y is determined uniquely by $\hat{y}(s) = (C(sI - A)^{-1}B + D)\hat{u}(s)$, where $\hat{\cdot}$ denotes the Fourier transform.

Many variants and extensions of the result are familiar: signal spaces on $[0, \infty)$ with different norms can also be used; there is a finite-gain property relating $\|y\|_2$ to $\|u\|_2$ (the \mathcal{L}_2 -norms of output and input); even with $x(0) \neq 0$ the main equivalence in the proposition still holds. Here we would like to highlight the fact that the result establishes an equivalence between stability defined in terms of the *forced response* and stability defined in terms of the *free response*, i.e. an equivalence between bounded-input/bounded-output (BIBO)

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T.T. Georgiou is with Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis, MN 55455; tryphon@ece.umn.edu
M.C. Smith is with the Department of Engineering, University of Cambridge, Cambridge, CB2 1PZ, U.K.; mcs@eng.cam.ac.uk

stability and Lyapunov stability for a system operating on the positive time-axis. Asymptotic stability in the sense of Lyapunov is obviously a time-asymmetric concept since convergence of the state vector is required as t tends to PLUS infinity, starting from an arbitrary initial condition at $t = 0$. In itself, BIBO stability does not appear to have this asymmetry, yet it is implicit in the formulation of Proposition 1.

To further illustrate the point we can write down the following obvious corollary of Proposition 1, obtained by running time backwards from 0 to $-\infty$. By changing the support of the signal spaces from the positive half-line to the negative half-line stability defined through the forced response (BIBO stability) becomes equivalent to asymptotic stability in the sense of Lyapunov for the reversed time direction as t tends to MINUS infinity.

Proposition 2: Let \mathbf{P} be a linear system as in Proposition 1. Then $y \in \mathcal{L}_2(-\infty, 0]$ for all $u \in \mathcal{L}_2(-\infty, 0]$ if and only if the matrix $-A$ is Hurwitz.

We now turn to the situation where inputs and outputs may have support on the doubly-infinite time-axis. In this case the following holds.

Proposition 3: Let \mathbf{P} be a linear system as in Proposition 1. Then there exists $y \in \mathcal{L}_2(-\infty, \infty)$ for all $u \in \mathcal{L}_2(-\infty, \infty)$ if and only if A has no imaginary-axis eigenvalues. Moreover, if this condition holds, y is determined uniquely by $\hat{y}(s) = (C(sI - A)^{-1}B + D)\hat{u}(s)$.

We remark that Proposition 3 is the natural generalisation of Proposition 1 when systems are viewed as operators. A linear system in Proposition 1 becomes a multiplication operator on the Fourier transformed spaces. The operator is bounded if and only if the “symbol” (the transfer-function) belongs to H_∞ , which under the controllability and observability assumption is equivalent to A being Hurwitz. On the double-axis a multiplication operator on the Fourier transformed spaces is bounded if and only if the symbol belongs to L_∞ —which for rational symbols excludes only poles on the imaginary axis.

In Proposition 3 there is no longer any relationship between a notion of BIBO stability and Lyapunov stability (in either time-direction). Clearly, both A and $-A$ may fail to be Hurwitz. Since only the existence of some $y \in \mathcal{L}_2(-\infty, \infty)$ is required for a given $u \in \mathcal{L}_2(-\infty, \infty)$, and the free motion solutions of (1) are ignored, this is not surprising. Propositions 1 and 2, by contrast, establish a connection between BIBO stability and Lyapunov stability as $t \rightarrow +\infty$ (respectively, $t \rightarrow -\infty$) without putting in explicit requirements on the free motion solutions.

We now consider the feedback interconnection in the form of Fig. 1 where \mathbf{P} and \mathbf{C} are linear systems. The existence of signals u_i, y_j ($i, j \in \{1, 2\}$) in $\mathcal{L}_2[0, \infty)$ which satisfy the feedback equations for a given pair of external inputs u_0, y_0 in $\mathcal{L}_2[0, \infty)$, for any set of initial conditions, is a well-known and natural definition of stability in terms of the forced response. From Proposition 1 stability in this sense is equivalent to asymptotic stability in the sense of Lyapunov of the combined state-space (assuming minimal realizations for \mathbf{P} and \mathbf{C} and well-posedness). Again, BIBO stability inherits the required time-asymmetry from the asymmetry of the support interval $[0, \infty)$.

It is apparent that the corresponding definition of BIBO stability for this feedback interconnection with $\mathcal{L}_2(-\infty, \infty)$

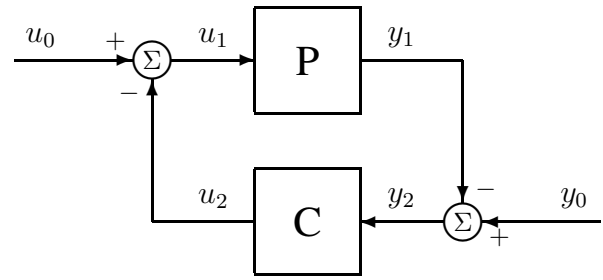


Fig. 1. Standard feedback configuration.

signals, generalising Proposition 3, will not correspond to a sensible notion of closed-loop stability. Indeed, we can easily check that a system \mathbf{P} with transfer function $P(s) = 1/(s - 1)$ is “stabilised” by any of the controllers with $C(s) = 2$, $C(s) = 0$, or $C(s) = -0.5/(s + 1)$. (In conventional terms the controllers give closed-loop poles which are in the open left-half plane (LHP), the open right-half plane (RHP), and in both half planes, respectively.)

B. The two-sided time axis and causality

The fact that the doubly-infinite time axis causes problems for the analysis of stability and of stabilisation was first pointed out in [5]. The explanation given there is consistent with that of Section II, but the overall argument was somewhat different. We now summarize the reasoning of [5].

Two systems \mathbf{P}_i ($i = 1, 2$) defined by convolution operators were considered:

$$y(t) = \int_{-\infty}^{\infty} h_i(t - \tau)u(\tau)d\tau = h_i * u$$

where $h_1(t) = e^t$ for $t \geq 0$ and zero otherwise, and $h_2(t) = -e^t$ for $t \leq 0$ and zero otherwise, respectively. Each system has Laplace transfer function equal to $1/(s - 1)$, but with differing regions of convergence. The first system is unstable and causal and the second is stable and non-causal (in fact anticausal) according to the usual definitions.

When viewed on $\mathcal{L}_2(-\infty, \infty)$, \mathbf{P}_2 is a bounded operator and hence is a stable system in an input-output sense. On the other hand, it was shown in [5] that \mathbf{P}_1 fails to be stabilisable on $\mathcal{L}_2(-\infty, \infty)$. This is a counterintuitive result since \mathbf{P}_1 is stabilisable in the ordinary way on any positive half-line. The proof that \mathbf{P}_1 fails to be stabilisable on the doubly-infinite time-axis reduces to the observation that the graph of \mathbf{P}_1 fails to be closed.

It was also pointed out in [5] that the closure of the graph of \mathbf{P}_1 coincides with the graph of \mathbf{P}_2 . Once the graph is closed there is no problem with stabilisation. But in closing the graph “anti-causal” trajectories are brought in which are inconsistent with the convolution representation of the system, so this was considered inadmissible.

Another possible remedy discussed in [5] was to consider the underlying differential equation representations rather than the convolution representations. In fact both systems are defined by the same differential equation

$$\dot{y} = y + u. \quad (3)$$

More precisely, the trajectories of both \mathbf{P}_1 and \mathbf{P}_2 satisfy this equation. In terms of “flow of time” thinking, \mathbf{P}_1 appears

to arise by solving this equation forwards in time while \mathbf{P}_2 is obtained by solving it backwards. This suggestion seems to make stronger the argument to consider \mathbf{P}_1 and \mathbf{P}_2 to be the same system. But this was considered unnatural in [5] on the grounds that it appears to abandon any notion of causality, or that it leaves the direction of time undefined.

The discussion of Section II allows the difficulties pointed out in [5] to be explained in a new way. Let us suppose we are willing to accept the closure of the graph of \mathbf{P}_1 which makes it “stabilizable” on the double-axis. As explained, \mathbf{P}_1 and \mathbf{P}_2 can now be thought of as one and the same system defined by (3)—a state-space description as in (1-2) solved forwards or backwards as desired. Does the closure of the graph resolve the stabilizability problem? The answer is no, since the notion of stability does not correspond to the usual notions. As is made clear by Proposition 3, the feedback system may turn out to be stable in a conventional sense in the forward, backward or neither time-directions.

A number of interesting observations and contributions have followed from [5] in the work of Mäkilä, Partington and Jacob [12], [13], [14], [21], [22], [23], [24]. These contributions are discussed in detail in [6].

III. TIME-ASYMMETRY AND OPTIMAL REGULATION

This section focusses on the time-asymmetry of the definition of stability and its implications in the context of optimal regulation. Specifically, we point that that the ease of (asymptotic) optimal quadratic regulation as time tends to PLUS infinity and MINUS infinity, respectively, can be vastly different. In fact, the optimal cost can be expressed by the two extremal solutions of the appropriate algebraic Riccati equation.

A. The time-conjugation operator, f -stability and b -stability

Let \mathbf{P} denote a dynamical system described by a differential equation as in (1-2), initialized at time zero and running forwards in time. Let \mathfrak{J} denote the operation on \mathbf{P} which corresponds to solving (1-2) backwards from $t = 0$ followed by a flip of the time axis (so the new system runs forward again). Equivalently we set $t_1 = -t$, so that $\frac{d}{dt} = -\frac{d}{dt_1}$ which results in

$$\begin{aligned} -\dot{x} &= Ax + Bu, \text{ with } x(0) = x_0 \\ y &= Cx + Du \end{aligned}$$

for the system $\mathfrak{J}(\mathbf{P})$. The effect on the transfer function is as follows: if \mathbf{P} has transfer function $P(s)$, then $\mathfrak{J}(\mathbf{P})$ has transfer function $P(-s)$.

Define the system \mathbf{P} to be f -stable if A is Hurwitz, and define \mathbf{P} to be b -stable if $\mathfrak{J}(\mathbf{P})$ is f -stable, or equivalently, if $-A$ is Hurwitz. It is immediately obvious that a linear time-invariant system of the type (1-2) can never be both f -stable and b -stable. Similarly, a controller which makes (1-2) f -stable cannot make it b -stable as well.

B. The classical linear-quadratic regulator

Again let \mathbf{P} be a linear time-invariant system which is controllable and observable and described by (1-2) with $D = 0$ and $x(0) = x_0$. It is well-known [1] that

$$J = \int_0^\infty (y(t)'Qy(t) + u(t)'Ru(t))dt \quad (4)$$

has a minimum given by $J_+^* = x_0'S_+x_0$ where S_+ is the unique positive-definite solution to the algebraic Riccati equation

$$A'S + SA - SBR^{-1}B'S + C'QC = 0. \quad (5)$$

It is also well-known that S_+ is the unique solution of (5) for which $A - BR^{-1}B'S$ has all its eigenvalues in the open LHP. In the language of the present paper we can say that S_+ is the unique solution of (5) which makes the system (1-2) f -stable with the controller $u^* = -R^{-1}B'Sx$.

What happens if we require the minimisation of

$$J = \int_{-\infty}^0 (y(t)'Qy(t) + u(t)'Ru(t))dt \quad (6)$$

for (1-2) running backwards in time? This is the same as the conventional problem for the system $\mathfrak{J}(\mathbf{P})$. It is easy to see that the minimum is given by $J_-^* = -x_0'S_-x_0$ where S_- is the unique negative-definite solution to (5). It is also well-known that S_- is the unique solution of (5) for which $A - BR^{-1}B'S$ has all its eigenvalues in the open RHP [33]. In the language of the present paper we can say that S_- is the unique solution of (5) which makes the system (1-2) b -stable with the controller $u^* = -R^{-1}B'Sx$.

In general $J_+^* = x_0'S_+x_0$ and $J_-^* = -x_0'S_-x_0$ are different. This shows that “difficulty of control” is time-asymmetric for the standard linear-quadratic regulator on the infinite horizon. The difference can be significant, e.g. if $A = 1$, $B = \epsilon$, $C = 1$, $Q = 1$ and $R = 1$ then $S_+ \approx 2/\epsilon^2 + 1/2$ and $S_- \approx -1/2$ for ϵ small.

IV. TIME-ASYMMETRY AND MODELLING UNCERTAINTY

In this section we look at the topology for uncertainty in feedback control and how this is affected by the time arrow. We will see that dynamical systems which are close in the usual sense, that a common controller can stabilise them and give a similar closed-loop behaviour, may not be close if time is reversed.

A. The gap metric and robustness of stability

Zames and El-Sakkary [32] introduced a metric on dynamical systems for the purpose of assessing robustness. This was based on the gap metric used in functional analysis to study invertibility of operators [16], [28]. Specifically, systems are considered to be operators on $\mathcal{L}_2[0, \infty)$ with a graph which is a closed subspace of $\mathcal{L}_2[0, \infty)$. Consider two linear systems \mathbf{P}_i ($i = 1, 2$) with transfer functions $P_i(s) = n_i(s)(m_i(s))^{-1}$ where $n_i(s)$ and $m_i(s)$ are right-coprime polynomial matrices, and let $\mathcal{G}_{\mathbf{P}_i, \mathcal{H}_2} = G_i(s)\mathcal{H}_2$ denote (the Fourier transform of) the graph of \mathbf{P}_i where $G_i(s)$ is the *graph symbol*. (See [4] for further details.) Then the gap between \mathbf{P}_1 and \mathbf{P}_2 is defined to be $\delta_{\mathcal{H}_2}(\mathbf{P}_1, \mathbf{P}_2) := \|\Pi_{\mathcal{G}_{\mathbf{P}_1, \mathcal{H}_2}} - \Pi_{\mathcal{G}_{\mathbf{P}_2, \mathcal{H}_2}}\|$ where $\Pi_{\mathcal{K}}$ denotes orthogonal projection onto a closed subspace \mathcal{K} .

Let the feedback configuration of Fig. 1 be denoted by $[\mathbf{P}, \mathbf{C}]$, where \mathbf{P} and \mathbf{C} are linear systems defined as operators on $\mathcal{L}_2[0, \infty)$ which may possibly be unbounded. Define $\mathbf{H}_{\mathbf{P}, \mathbf{C}}$ to be the transfer function of the operator mapping $(u_0^T \ y_0^T)^T$ to $(u_1^T \ y_1^T)^T$. The following are basic robustness results for gap metric uncertainty.

Proposition 4: [4] Assume that the closed-loop system $[\mathbf{P}, \mathbf{C}]$ is f-stable. Then, $[\mathbf{P}_1, \mathbf{C}]$ is f-stable for all \mathbf{P}_1 such that $\delta_{\mathcal{H}_2}(\mathbf{P}, \mathbf{P}_1) \leq b$ if and only if $b < b_{\mathbf{P}, \mathbf{C}}$ where

$$b_{\mathbf{P}, \mathbf{C}} := \|\mathbf{H}_{\mathbf{P}, \mathbf{C}}\|_{\infty}^{-1}.$$

Proposition 5: [32] Assume that the closed-loop system $[\mathbf{P}, \mathbf{C}]$ is f-stable. Then, the following are equivalent:

- (i) $\delta_{\mathcal{H}_2}(\mathbf{P}_n, \mathbf{P}) \rightarrow 0$ as $n \rightarrow \infty$.
- (ii) $\mathbf{H}_{\mathbf{P}_n, \mathbf{C}}$ is f-stable for sufficiently large n and $\|\mathbf{H}_{\mathbf{P}_n, \mathbf{C}} - \mathbf{H}_{\mathbf{P}, \mathbf{C}}\|_{\infty} \rightarrow 0$ as $n \rightarrow \infty$.

Proposition 5 was the primary justification for the claim in [32] that the gap metric defines the “correct” topology for robustness of feedback systems. In the present context, it can be seen that the choice of a signal space with support on the positive half-line is essential in achieving an appropriate topology. To emphasize the point, if $\mathcal{L}_2[0, \infty)$ were replaced by $\mathcal{L}_2(-\infty, 0]$ then the above proposition would hold with f-stability replaced by b-stability.

Let us consider the case where systems are defined on $\mathcal{L}_2(-\infty, \infty)$. Then we define

$$\delta_{\mathcal{L}_2}(\mathbf{P}_1, \mathbf{P}_2) := \|\Pi_{\mathcal{G}_{\mathbf{P}_1, \mathcal{L}_2}} - \Pi_{\mathcal{G}_{\mathbf{P}_2, \mathcal{L}_2}}\|$$

where

$$\mathcal{G}_{\mathbf{P}_i, \mathcal{L}_2} := G_i(s)\mathcal{L}_2$$

and $\mathcal{L}_2 := \mathcal{L}_2(-j\infty, j\infty)$. With this definition, $\mathcal{G}_{\mathbf{P}, \mathcal{L}_2}$ is always closed, but may contain “non-causal” input-output pairs (as pointed out in [5]—see Section II-B). It is easy to construct examples to demonstrate that convergence of $\delta_{\mathcal{L}_2}(\mathbf{P}_i, \mathbf{P})$ to zero does not allow any closed-loop stability prediction, e.g., $[\mathbf{P}, \mathbf{C}]$ f-stable does not imply $[\mathbf{P}_n, \mathbf{C}]$ f-stable in the limit.

In [29], [30] Vinnicombe introduced a new metric $\delta_v(\cdot, \cdot)$ on dynamical systems which defines the same topology as $\delta_{\mathcal{H}_2}(\cdot, \cdot)$, and which satisfies the following inequality:

$$\delta_{\mathcal{L}_2}(\cdot, \cdot) \leq \delta_v(\cdot, \cdot) \leq \delta_{\mathcal{H}_2}(\cdot, \cdot).$$

The v-gap between \mathbf{P}_1 and \mathbf{P}_2 is defined as follows:

$$\delta_v(\mathbf{P}_1, \mathbf{P}_2) := \begin{cases} \delta_{\mathcal{L}_2}(\mathbf{P}_1, \mathbf{P}_2) & \text{if} \\ & \text{wno}(\det(G_2(-s)^T G_1(s))) = 0, \\ 1 & \text{otherwise,} \end{cases} \quad (7)$$

where $\text{wno}(g(s))$ denotes the winding number of $g(s)$ about the origin, as s traces the standard Nyquist D-contour [29].

It turns out that Propositions 4 and 5 both hold with $\delta_{\mathcal{H}_2}$ replaced by δ_v (see [29]). Since $\delta_{\mathcal{L}_2} = \delta_v$ when

$$\text{wno}(\det(G_2(-s)^T G_1(s))) = 0 \quad (8)$$

holds, this condition effectively imposes a positive time-arrow on the double-axis graph which forces f-stability to be retained under small perturbations in $\delta_v(\cdot, \cdot)$. This is illustrated by the following result (which can be readily derived from [29, Theorem 4.2]; see also [7]).

Proposition 6: Let $[\mathbf{P}, \mathbf{C}]$ be f-stable and suppose $\delta_{\mathcal{L}_2}(\mathbf{P}_n, \mathbf{P}) \rightarrow 0$ as $n \rightarrow \infty$. Then $[\mathbf{P}_n, \mathbf{C}]$ is f-stable for all sufficiently large n if and only if $\text{wno}(\det(G_n(-s)^T G(s))) = 0$ for all sufficiently large n .

B. The effect of the time-arrow on gap distances

We define a forward and a backward v-gap as follows,

$$\begin{aligned} \delta_{v,f}(\mathbf{P}_1, \mathbf{P}_2) &:= \delta_v(\mathbf{P}_1, \mathbf{P}_2) \\ \delta_{v,b}(\mathbf{P}_1, \mathbf{P}_2) &:= \delta_v(\mathfrak{J}(\mathbf{P}_1), \mathfrak{J}(\mathbf{P}_2)). \end{aligned}$$

It is straightforward to see that

$$\delta_{\mathcal{L}_2}(\mathbf{P}_1, \mathbf{P}_2) = \delta_{\mathcal{L}_2}(\mathfrak{J}(\mathbf{P}_1), \mathfrak{J}(\mathbf{P}_2)),$$

so any difference between $\delta_{v,f}(\mathbf{P}_1, \mathbf{P}_2)$ and $\delta_{v,b}(\mathbf{P}_1, \mathbf{P}_2)$ lies in the winding number condition in (7). The following result can be established [6].

Proposition 7: Let $P_i(s)$ ($i = 1, 2$) be the rational transfer functions of linear time-invariant dynamical systems as above, with McMillan degrees μ_i , and assume that $\delta_{\mathcal{L}_2}(\mathbf{P}_1, \mathbf{P}_2) < 1$. Define

$$h(s) := \det(m_2(-s)^T m_1(s) + n_2(-s)^T n_1(s)).$$

Then $h(s)$ admits a canonical factorization $h(s) = h_+(s)h_-(s)$ where $h_+(s)$ and $h_-(-s)$ are Hurwitz polynomials.

1) The following are equivalent:

- a) $\delta_{v,f}(\mathbf{P}_1, \mathbf{P}_2) < 1$,
- b) $\deg(h_+(s)) = \mu_1$,
- c) $\deg(h_-(s)) = \mu_2$.

2) The following are equivalent:

- a) $\delta_{v,b}(\mathbf{P}_1, \mathbf{P}_2) < 1$,
- b) $\deg(h_-(s)) = \mu_1$,
- c) $\deg(h_+(s)) = \mu_2$.

3) The following are equivalent:

- a) $\delta_{v,f}(\mathbf{P}_1, \mathbf{P}_2) = \delta_{v,b}(\mathbf{P}_1, \mathbf{P}_2) = \delta_{\mathcal{L}_2}(\mathbf{P}_1, \mathbf{P}_2) < 1$,
- b) $\mu_1 = \mu_2 = \deg(h_+(s)) = \deg(h_-(s))$.

In the above proposition, 1) expresses the zero winding number condition in (7) in an equivalent form, while 2) does the same for $\delta_{v,b}(\mathbf{P}_1, \mathbf{P}_2)$. It is interesting that when the two conditions are combined as in 3) the result is a very stringent requirement which includes the necessity that $P_1(s)$ and $P_2(s)$ have the same McMillan degree. This serves to highlight the fact that “unmodelled dynamics” which may account for a small error in $\delta_{v,f}$ (and which may be neglected in the design of a robust controller) will inevitably account for a substantial error in $\delta_{f,b}$.

V. TIME-ASYMMETRY AND ROBUST CONTROL

This section addresses the implications of the time-asymmetry in the theory of robust control. In particular, we will also see that a system which is “easy” to control in one direction of time may be far from easy to control in the opposite direction.

A. Optimal robustness and difficulty of control

In [9] it was shown that $b_{\mathbf{P}, \mathbf{C}}$ could be supremised over all stabilising \mathbf{C} and that this amounts to solving a Nehari problem. This optimum value, which we denote by $b_{\text{opt},f}(\mathbf{P})$, can be interpreted as a measure of ease/difficulty of control, where a value near to 1 means the plant is “easy to control” and a value near 0 means the plant is “hard to control”.

With the understanding that $b_{\text{opt},f}(\mathbf{P})$ has the meaning of ease of control with respect to the forward time-arrow for stability, it is interesting to define $b_{\text{opt},b}(\mathbf{P}) := b_{\text{opt},f}(\mathfrak{J}(\mathbf{P}))$,

which represents ease of control with respect to the backwards time-arrow. Our main purpose in defining $b_{\text{opt},b}(\mathbf{P})$ is to highlight the influence of the usual time arrow in feedback regulation. In the following two examples we will see situations where $b_{\text{opt},f}(\cdot)$ and $b_{\text{opt},b}(\cdot)$ are very different.

Example 8: Near pole-zero cancellations.

Consider $P(s) = 1 + \frac{\epsilon}{s+1}$. It can be checked that $b_{\text{opt},f}(\mathbf{P}) \rightarrow 1$ as $\epsilon \rightarrow 0$ while $b_{\text{opt},b}(\mathbf{P}) \rightarrow 0$ as $\epsilon \rightarrow 0$. This is accounted for by the fact that $P(s)$ has a near pole-zero cancellation in the LHP, which is innocuous for f-stabilisation, but highly challenging for b-stabilisation. The latter is equivalent to f-stabilisation of $P(-s)$, which has a troublesome near pole-zero cancellation in the RHP.

Example 9: Riding Bicycles.

A feedback stability problem in everyday experience is bicycle riding. An elementary model to study rider-bicycle stability is given in [2] which gives the following transfer function from steering angle input to tilt angle:

$$\alpha V \frac{s + \beta V}{s^2 - \gamma} \quad (9)$$

where α, β, γ are positive constants and V is the forward speed. This model has one RHP pole, but the zero is in the LHP. As such, this plant is not too difficult to control.

Let us consider what happens if we try to ride the bicycle *backwards in time*. This corresponds to trying to stabilise the plant $P(-s)$ *forwards in time*. The model still has one RHP pole, but the zero is also in the RHP, which makes stabilisation much more difficult. Indeed if $V\beta = \sqrt{\gamma}$ the plant is technically not stabilisable. It is interesting to note that an experimental bicycle with the steered wheel at the rear instead of the front has a transfer function from steering angle input to tilt angle given by [2] (see also [20])

$$\alpha V \frac{-s + \beta V}{s^2 - \gamma}. \quad (10)$$

This is exactly the transfer function for the conventional bicycle ridden *backwards in time*.

Figure 2 shows the value of $b_{\text{opt},f}$ and $b_{\text{opt},b}$ versus V with parameter values $\alpha = 1/3$, $\beta = 2$ and $\gamma = 9$ (which are deemed reasonably realistic). Recall that $b_{\text{opt},b}$ is the same as $b_{\text{opt},f}$ for the rear-wheel steered bicycle model (10) at the same V . It can be observed that $b_{\text{opt},b}$ is less than $b_{\text{opt},f}$ for any V . Also, $b_{\text{opt},b}$ is very small for low V , indicating difficulty of control, and zero at $V = 1.5$ m/s. For larger V , $b_{\text{opt},b}$ increases, indicating that control becomes easier. These results are equivalent to the rear-wheeled steered bicycle being more difficult to ride than the front-wheel steered one, but still being reasonably controllable at higher speeds [2].

B. Time irreversible feedback phenomena

The concept of ease or difficulty of control gives a thought-provoking perspective on reversibility. Systems which in a limiting situation are very difficult to control (in the sense that $b_{\text{opt},f}(\mathbf{P})$ tends to zero) are unlikely to be observed in nature or technology. Nevertheless, such a system may be easy to control in the time-reversed direction (see Examples 8 and 9). This is independent of the fact that the underlying differential equation can be integrated equally well in either time direction. This is reminiscent of phenomena (such as a bottle falling from the table and shattering into many pieces)

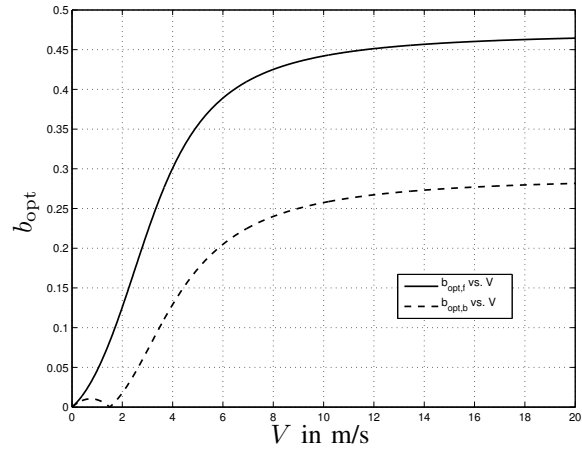


Fig. 2. $b_{\text{opt},f}$ and $b_{\text{opt},b}$ versus V for the bicycle model of (9) with $\alpha = 1/3$, $\beta = 2$, and $\gamma = 9$.

that appear to be associated with an intrinsic direction of time even though classical physics would also allow the reversed motion as a solution (see Section VI for a further discussion).

We expand this point in the context of Example 9. The loss of stabilizability of the rear-wheeled steered bicycle at $V = \sqrt{\gamma/\beta}$ has the following interesting consequence. Imagine a video of a rear-wheeled steered bicycle being ridden stably at this critical speed. Let us assume that it is possible to verify from the video the actual speed (e.g., by knowing the frame-rate and observing markings on the ground). An observer with a good grounding in control theory would be led to the inescapable conclusion that the video had been made when the said bicycle was actually being ridden backwards in space (i.e., with a negative V) and then played backwards in time as well, giving the impression of a forward motion.

VI. THE ARROW OF TIME IN PHYSICS

The subject of the “arrow of time” is a well-known conundrum in physics. The second law of thermodynamics states that the entropy of a system increases with time. It is the time-asymmetry in this law which gives rise to the notion of the “thermodynamic arrow of time”. The classical derivation of the second law in statistical mechanics due to Boltzmann is connected with a famous puzzle known as Loschmidt’s paradox. This essentially points out that the laws of mechanics used in the derivation of the second law are *time-symmetric* whereas the conclusion is not. Evidently the time-asymmetry creeps in through the statistical assumptions. An illuminating discussion of this issue is given in [19]. Other arrows of time have also been defined, for example (i) the “psychological arrow”—the direction in which time passes as perceived by a sentient being, (ii) the “cosmological arrow”—the direction of time in which the universe is expanding. Hawking [11] argues that the thermodynamic and psychological arrows are always aligned with each other but these need not always be aligned with the cosmological arrow (though they are at present).

In this paper we have described the time-asymmetry in the definition of control systems stability as a time-arrow. In the theory of dynamical systems there is also the notion of passivity, which again defines a time-arrow. For electrical circuits the time-arrow of passivity can be seen in the behaviour

of the resistor, in contrast to the inductor and capacitor which are time-symmetric in their operation. If the electrical resistor were to operate backwards in time one would observe a resistor gathering low-grade heat from the environment and charging up a battery. This behaviour would be recognised as a violation of the second law of thermodynamics (see [17, pages 260, 390-2]). In a similar way, an ideal linear damper operating backwards in time extracts low-grade heat from the environment to create mechanical work, in violation of the second law. It seems that the arrow of time in passive systems or circuits coincides with, or is the same as, the thermodynamic arrow.

How does the arrow of time for control system stability relate to other time arrows? It is highly unlikely that a control engineer who is designing a control system for a plant will give even a moment's thought to the preferred time arrow for control. He (or she) will seek decaying free motion solutions in the direction which he (or she) perceives time to be passing. In this way the arrow of time for control could be said to coincide with the psychological arrow. On the other hand, in biological systems, active control is ubiquitous. It is less obvious that, for example, homeostasis in a cell is aligned with the psychological arrow. In purely mathematical terms the arrow of time for control systems stability appears identical with the arrow of time for passivity, and is therefore aligned with the thermodynamic arrow. From all points of view then one reaches the conclusion that the arrow of time for control system stability coincides both with the thermodynamic and psychological arrows.

VII. SYNOPSIS

- 1) Stability is a time-asymmetric concept. The requirement of an asymptotic property as t tends to PLUS infinity defines a time arrow.
- 2) The role of the positive time arrow in the gap metric measure of uncertainty for dynamical systems was highlighted. The usual \mathcal{H}_2 -gap metric inherits the positive time arrow by virtue of systems being defined as operators on the positive half-line. The v -gap metric may be interpreted as the \mathcal{L}_2 -gap with an imposed time-arrow.
- 3) It was seen that closeness of systems in the forward and backwards directions is a strong condition which includes the requirement of equal McMillan degrees.
- 4) It was seen that ease or difficulty of control is a property that depends on the time-arrow.
- 5) The limiting situation of a plant which is easier to control in one time direction but impossible to control in the other shows that irreversibility can be intimately related to control.
- 6) An engineering perspective of control suggests a close link between the control system stability arrow and the psychological arrow. Unified mathematical frameworks for passive circuits and feedback control suggest a close link between the control system stability arrow and the thermodynamic arrow.

VIII. ACKNOWLEDGEMENT

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