

# A Repetitive Process Approach to the Control of Self-servowriting in Disk Drives

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**Abstract**— This paper considers the control problem for self-servowriting in disk drives where information propagation occurs in two independent directions, i.e. time and track number respectively. The resulting state-space model is not of the well known Roesser or Fornasini-Marchesini types and hence cannot be analyzed using the theory associated with these 2D discrete linear systems models. Instead, it is shown here that it can be written as a discrete linear repetitive process state-space model which is an extension of those already encountered and for which it is necessary to develop new tools for stability analysis and controller design. Finally, the application of these is illustrated by a simulation example using industry supplied data.

## I. INTRODUCTION

The unique characteristic of a repetitive, or multipass, process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

To introduce a formal definition, let  $\alpha < +\infty$  denote the pass length (assumed constant). Then in a repetitive process the pass profile  $y_k(p)$ ,  $p = 0, 1, \dots, \alpha-1$ , generated on pass  $k$  acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile  $y_{k+1}(p)$ ,  $p = 0, 1, \dots, \alpha-1$ ,  $k \geq 0$ .

Physical examples of these processes include long-wall coal cutting and metal rolling operations [1]. Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these (the original references are in [1]) include classes of iterative learning control schemes and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle. In this latter case, use of the repetitive process setting provides the basis for the development of highly reliable and efficient solution algorithms and in the former it provides a stability theory which, unlike alternatives, provides information

concerning an absolutely critical problem in this application area, i.e. the trade-off between convergence and the learnt dynamics.

Attempts to control these processes using standard (or 1D) systems theory/algorithms fail (except in a few very restrictive special cases) precisely because such an approach ignores their inherent 2D systems structure. In particular, information propagation occurs from pass-to-pass and along a given pass and also the initial conditions are reset before the start of each new pass.

The majority of the work reported on linear repetitive processes has assumed that at any point on the current pass the only contribution from the previous one is from this same point. In some applications, however, it is more realistic to assume that at any point on the current pass there is a contribution from the complete previous pass profile or portions of it. For example, in long-wall coal cutting the machine rests on the previous pass profile, which is the height of the stone/coal interface as measured relative to some datum line, as it is producing the current one and it is too simplistic in most cases to make the first assumption here. Instead, models for this inter-pass smoothing must be developed as the first step towards stability analysis and control law design.

Some work on modeling and control law design for discrete linear repetitive processes with inter-pass smoothing has been reported [2], where it has been shown that existing algorithms for checking the stability of either 2D linear systems described by Roesser/Fornasini Marchesini state-space models or other discrete linear repetitive process models, cannot be applied (see also [1] and the relevant cited references for the details of the stability analysis only). It was also shown in [2] that control law design is possible using the 1D equivalent model [3] but only for a weak form of stability. Moreover, this route is not applicable to processes whose along the pass dynamics are governed by a linear matrix differential equation, known as differential linear repetitive processes, or to the case when there is uncertainty associated with the dynamics.

In this paper we first draw on preliminary work in [4], [5] to describe a relatively new application of repetitive processes in the modeling and control of self-servowriting in disk drives. This will establish the need to consider a new model for these processes in order to include dynamics which are critical to this application but are not included in any of the models considered previously. Control law design for this new model is then developed based on a Lyapunov function approach where the associated computations can

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be completed using Linear Matrix Inequalities (LMIs). A simulation example to illustrate the design is also given.

Throughout this paper, the null matrix and the identity matrix with appropriate dimensions are denoted by  $0$  and  $I$  respectively. Also  $X \geq Y$  (respectively  $X > Y$ ) means that the matrix  $X - Y$  is positive semi-definite (respectively positive definite). Also we use the notation

$$\bigoplus_{i=1}^k M_i = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_k \end{bmatrix},$$

and

$$\bigoplus_{i=1}^k M = \bigoplus_{i=1}^k M_i,$$

where  $M_i = M, i = 1, 2, \dots, k$ . Finally,  $\star$  is used to denote block entries in the symmetric LMIs.

## II. BACKGROUND

The most basic discrete linear repetitive process state-space model [1] has the following form over  $p = 0, 1, \dots, \alpha - 1, k \geq 0$

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0y_k(p), \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0y_k(p). \end{aligned} \quad (1)$$

Here on pass  $k$ ,  $x_k(p) \in \mathbb{R}^n$  is the state vector,  $y_k(p) \in \mathbb{R}^m$  is the pass profile vector, and  $u_k(p) \in \mathbb{R}^r$  is the vector of control inputs. To complete the process model is necessary to specify the boundary conditions in the form of the state initial vector on each pass and the initial pass profile. The simplest possible are

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \quad k \geq 0, \\ y_0(p) &= f(p), \quad p = 0, 1, \dots, \alpha - 1, \end{aligned} \quad (2)$$

where the  $n \times 1$  vector  $d_{k+1}$  has known constant entries and  $f(p)$  is an  $m \times 1$  vector whose entries are known functions of  $p$ .

One model for inter-pass smoothing in these processes is [2]

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + \sum_{i=0}^{\alpha-1} B_i y_k(i), \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + \sum_{i=0}^{\alpha-1} D_i y_k(i), \end{aligned} \quad (3)$$

with the notation as in (1) and boundary conditions (2). This model assumes that inter-pass smoothing is completely associated with the previous pass profile contribution to the current pass state and pass profile vectors respectively. The application considered in this paper shows that this representation does not contain all possible cases. In particular, we will need to consider a state-space model with inter-pass smoothing of both the previous pass state and profile vectors.

Given the unique control problem, the natural way to formulate a stability theory for these processes is in bounded-input bounded-output terms. The details for processes described by (1) can be found in [1]. This theory can be expressed in Lyapunov function terms, where the function involved is the sum of two quadratic terms, the first of which is formed from the current pass state vector and the second from the previous pass profile. Developing this further leads to an LMI characterization for computation and control law design. In the case of (3) we have to employ the repetitive process interpretation of the well known quadratic stability [6]. This, noting again the unique control problem for these processes, is motivated by the physical argument that the total energy (finite for each) should decrease from pass-to-pass. Next we show how a linear repetitive process state-space model with a form of inter-pass smoothing different from that in (3) arises in a (relatively) new application area.

## III. SELF-SERVOWRITING IN DISK DRIVES — A DISCRETE LINEAR REPETITIVE PROCESS INTERPRETATION

Disk drives are embedded servo systems [4], [5] where the position burst information of the tracks is written onto the surface of the disk by a process known as servo track-writing (or servowriting for short). The burst information is demodulated to determine position off-track information that is then used by the servo system to track follow, or seek, a target track. A typical drive consists of one or more circular platters on which data is stored magnetically in tracks or cylinders.

Self-servowriting is of considerable interest because of the potential savings in the cost of servowriting the drive together with reduced capital expenditure on servowriters etc. One form of self-servowriting is where a few tracks, also known as seeds, are written at a predetermined area on the disk using the servowriter and the remaining tracks are generated without the aid of a servowriter using the seeds as a reference, this is termed seeded self-servowriting.

Despite its advantages, this method suffers from the propagation of errors as successive tracks are written. As the actuator follows on a written track, the read head attempts to follow any deviations from perfect circularity of the written track that are within the bandwidth of the servo control loop. Hence the written track will attempt to duplicate and amplify these errors. Also there are always external disturbances present and hence additional degradation can be expected. The overall effect can be that within a few iterations the errors rapidly build up causing deviation from circularity and variable track spacing. This is termed radial error propagation.

Various ways to control this radial error propagation can be attempted. The simplest of these is to ensure that the magnitude of the closed-loop of the servo control loop is kept below unity at the runout frequencies. This may be achieved by simply shaping the loop frequency response to

give a low open-loop system bandwidth but the quality of the written track could be significantly degraded due to poor rejection capability of the low bandwidth controller. Of the alternatives, one of the most recent is based on a discrete linear repetitive process approach for the servo loop and this is the subject of this paper.

In block diagram terms, the servo loop control scheme is unity negative feedback applied to a forward path consisting of the plant (Voice Coil Meter (VCM) plus actuator) in series with the controller. The major sources of track mis-registration (TMR) on track  $k$  are the spindle Non-Repeatable Run-out (NRRO), the disk flutter  $\zeta_k$ , and the suspension windage and flex bias disturbance  $\varsigma_k$  that act as torque level disturbances. The sensor noise is denoted by  $\omega_k$ . In this setup the read head follows track  $y_k$  as the write head lays down the burst information for the next track  $y_{k+1}$ .

The state-space model for the discretized plant [4], [5] is taken as

$$\begin{aligned} x_k^g(p+1) &= A_g x_k^g(p) + B_g [u_k(p) + \varsigma_k(p)], \\ y_{k+1}(p) &= C_g x_k^g(p) + \zeta_k(p), \end{aligned} \quad (4)$$

where subscript  $k$  here denotes the track number and  $p$  denotes (sampled) time. Here there is information propagated in both directions and also a track is of finite duration. Hence the dynamics here are those of a discrete linear repetitive process where as the measured output we use the position error signal (PES)

$$e_k(p) = \begin{bmatrix} -C_g & 1 \end{bmatrix} \begin{bmatrix} x_k^g(p) \\ y_k(p) \end{bmatrix} + \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} \zeta_k(p) \\ \omega_k(p) \end{bmatrix}. \quad (5)$$

Following [4] we now consider a controller of the form

$$\begin{aligned} \begin{bmatrix} \hat{x}_k(p+1) \\ \hat{y}_{k+1}(p) \end{bmatrix} &= \begin{bmatrix} A_d^{11} & A_d^{12} \\ A_d^{21} & A_d^{22} \end{bmatrix} \begin{bmatrix} \hat{x}_k(p) \\ \hat{y}_k(p) \end{bmatrix} \\ &+ \begin{bmatrix} B_d^1 \\ B_d^2 \end{bmatrix} e_k(p), \\ \hat{y}_{k+1}(p) &= \sum_{i=0}^{\alpha-1} \beta_i \tilde{y}_{k+1}(i), \\ u_k(p) &= \begin{bmatrix} C_d^1 & C_d^2 \end{bmatrix} \begin{bmatrix} \hat{x}_k(p) \\ \hat{y}_k(p) \end{bmatrix} + D_d e_k(p), \end{aligned} \quad (6)$$

where the vectors  $\hat{x}_k(p)$  and  $\hat{y}_k(p)$  are the internal states of the controller (in repetitive process terms the current pass state and pass profile vectors). Also, a first step in analysis, we assume that the disturbance and sensor noise terms can be neglected and hence the controlled process dynamics can be written in state-space form as

$$\begin{aligned} \chi_k(p+1) &= A \chi_k(p) + B^1 y_k(p) + B^2 \hat{y}_k(p), \\ y_{k+1}(p) &= C \chi_k(p), \\ \hat{y}_{k+1}(p) &= \sum_{i=0}^{\alpha-1} C_i \chi_k(i) + \sum_{i=0}^{\alpha-1} D_i y_k(i) + \sum_{i=0}^{\alpha-1} \hat{D}_i \hat{y}_k(i), \end{aligned} \quad (7)$$

where  $\chi_k(p) = \begin{bmatrix} x_k^g(p) \\ \hat{x}_k(p) \end{bmatrix}$  and

$$\begin{aligned} A &= \begin{bmatrix} A_g - B_g D_d C_g & B_g C_d^1 \\ -B_d^1 C_g & A_d^{11} \end{bmatrix}, B^1 = \begin{bmatrix} B_g D_d \\ B_d^1 \end{bmatrix}, B^2 = \begin{bmatrix} B_g C_d^2 \\ A_d^{12} \end{bmatrix}, \\ C &= [C_g \ 0], C_i = \beta_i [-B_d^2 C_g \ A_d^{21}], D_i = \beta_i B_d^2, \hat{D}_i = \beta_i A_d^{22}. \end{aligned} \quad (8)$$

This model describes a discrete linear repetitive process with inter-pass smoothing which differs from that in (3). In particular, inter-pass smoothing here occurs in both the state and pass profile vectors but, as established below, quadratic stability can still be applied.

To define quadratic stability for (7), introduce the Lyapunov "total energy" function associated with pass  $k$  as

$$V(k) = V_1(k) + V_2(k) + V_3(k), \quad (9)$$

where

$$\begin{aligned} V_1(k) &= \sum_{i=0}^{\alpha-1} \chi_{k+1}^T(i) V_1^i \chi_{k+1}(i), \\ V_2(k) &= \sum_{i=0}^{\alpha-1} \tilde{y}_k^T(i) V_2^i \tilde{y}_k(i), \\ V_3(k) &= \sum_{i=0}^{\alpha-1} \hat{y}_k^T(i) V_3^i \hat{y}_k(i), \end{aligned} \quad (10)$$

with  $V_j^i > 0, j = 1, 2, 3, V_2^i, i = 0, 1, \dots, \alpha - 1$ . Here  $V_1(k)$  represents the contribution to the total energy from the state variables,  $V_2(k)$  that from the pass profile vector  $\tilde{y}_k$ , and  $V_3(k)$  that from the pass profile vector  $\hat{y}_k$ . Also introduce

$$V_1'(k) = \sum_{i=1}^{\alpha} \chi_{k+1}^T(i) V_1^{i-1} \chi_{k+1}(i). \quad (11)$$

Then quadratic stability for a process described by (7) is defined as follows.

**Definition 1:** A discrete linear repetitive process described by (7) is said to have the quadratic stability property when

$$V_1'(k) - V_1(k) + V_2(k+1) - V_2(k) + V_3(k+1) - V_3(k) < 0, \quad (12)$$

for all  $\chi_k(p), \tilde{y}_k(p)$  and  $\forall \hat{y}_k(p), k = 0, 1, 2, \dots$  and  $p = 0, 1, \dots, \alpha - 1$ .

The following notation is useful in leading to an LMI based test for this property

$$\begin{aligned} V^j &= \bigoplus_{i=0}^{\alpha-1} V_j^i, \hat{V}^j = \sum_{i=0}^{\alpha-1} V_j^i, j = 1, 2, 3, \\ \tilde{C} &= \begin{bmatrix} C & C & \dots & C \\ C & C & \dots & C \\ \vdots & \vdots & \ddots & \vdots \\ C & C & \dots & C \end{bmatrix}, \hat{D} = \begin{bmatrix} D_0 & D_1 & \dots & D_{\alpha-1} \\ D_0 & D_1 & \dots & D_{\alpha-1} \\ \vdots & \vdots & \ddots & \vdots \\ D_0 & D_1 & \dots & D_{\alpha-1} \end{bmatrix}, \\ \tilde{C} &= C \oplus C \dots \oplus C, \hat{D} = [D_0 \ D_1 \ \dots \ D_{\alpha-1}], \\ \tilde{A} &= A \oplus A \dots \oplus A, \tilde{B}_0 = B_0 \oplus B_0 \dots \oplus B_0, \end{aligned}$$

$$\widehat{B} = \begin{bmatrix} B_0 & B_1 & \cdots & B_{\alpha-1} \\ B_0 & B_1 & \cdots & B_{\alpha-1} \\ \vdots & \vdots & \ddots & \vdots \\ B_0 & B_1 & \cdots & B_{\alpha-1} \end{bmatrix}, \widehat{\beta} = [ B_0 \ B_1 \ \cdots \ B_{\alpha-1} ],$$

$$\widetilde{B}^i = B^i \oplus B^i \cdots \oplus B^i; i = 1, 2$$

$$\widehat{\widehat{C}} = [ C_0 \ C_1 \ \cdots \ C_{\alpha-1} ], \widehat{\widehat{D}} = [ \widehat{D}_0 \ \widehat{D}_1 \ \cdots \ \widehat{D}_{\alpha-1} ].$$

*Theorem 1:* A discrete linear repetitive process described (7) has the quadratic stability property if there exists matrices  $V_j^i > 0$  and  $j = 1, 2, 3; i = 1, 2, \dots, \alpha - 1$ , such that the following LMI holds

$$\Upsilon^4 := \begin{bmatrix} \Upsilon_{11}^4 & \Upsilon_{12}^4 & \Upsilon_{13}^4 \\ * & \Upsilon_{22}^4 & \Upsilon_{23}^4 \\ * & * & \Upsilon_{33}^4 \end{bmatrix} < 0, \quad (13)$$

where

$$\Upsilon_{11}^4 = \widetilde{A}^T V^1 \widetilde{A} + \widetilde{C}^T V^2 \widetilde{C} + \widehat{\widehat{C}}^T \widehat{V}^3 \widehat{\widehat{C}} - V^1,$$

$$\Upsilon_{12}^4 = \widetilde{A}^T V^1 \widetilde{B}^1 + \widehat{\widehat{C}}^T \widehat{V}^3 \widehat{\widehat{D}},$$

$$\Upsilon_{13}^4 = \widetilde{A}^T V^1 \widetilde{B}^2 + \widehat{\widehat{C}}^T \widehat{V}^3 \widehat{\widehat{D}},$$

$$\Upsilon_{22}^4 = \widetilde{B}^1 T V^1 \widetilde{B}^1 + \widehat{\widehat{D}}^T \widehat{V}^3 \widehat{\widehat{D}} - V^2,$$

$$\Upsilon_{23}^4 = \widetilde{B}^1 T V^1 \widetilde{B}^2 + \widehat{\widehat{D}}^T \widehat{V}^3 \widehat{\widehat{D}},$$

$$\Upsilon_{33}^4 = \widetilde{B}^2 T V^1 \widetilde{B}^2 + \widehat{\widehat{D}}^T \widehat{V}^3 \widehat{\widehat{D}} - V^3.$$

*Proof:* This is a straightforward consequence of Definition 1 when written in matrix form and hence the details are omitted. ■

#### IV. CONTROLLER DESIGN

In this section we are interested designing the controller such that the controlled process has the quadratic stability property where, as a first step and to simplify the analysis, it is assumed that

$$V_1^i = V_1, \forall i = 0, \dots, \alpha - 1,$$

$$V_2^i = V_2, \forall i = 0, \dots, \alpha - 1,$$

$$V_3^i = V_3, \forall i = 0, \dots, \alpha - 1.$$

Also we assume that the scalars  $\beta_0, \dots, \beta_{\alpha-1}$  are known and use the following notation

$$\widetilde{\beta} = [ \beta_0 I \ \beta_1 I \ \cdots \ \beta_{\alpha-1} I ],$$

to rewrite the matrices  $\widehat{\widehat{C}}$ ,  $\widehat{\widehat{D}}$ , and  $\widehat{\widehat{D}}$  as

$$\begin{aligned} \widehat{\widehat{C}} &= [ C_0 \ C_1 \ \cdots \ C_{\alpha-1} ] \\ &= [ -\beta_0 B_d^2 C_g \ \beta_0 A_d^{21} \ | \ -\beta_1 B_d^2 C_g \ \beta_1 A_d^{21} \ | \\ &\quad \cdots \ \cdots \ | \ -\beta_{\alpha-1} B_d^2 C_g \ \beta_{\alpha-1} A_d^{21} \ ] \\ &= \widetilde{\beta} \begin{bmatrix} -B_d^2 C_g \ A_d^{21} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & -B_d^2 C_g \ A_d^{21} & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -B_d^2 C_g \ A_d^{21} \end{bmatrix} \\ &= \widetilde{\beta} \bigoplus_{i=0}^{\alpha-1} [ -B_d^2 C_g \ A_d^{21} ], \end{aligned}$$

and

$$\begin{aligned} \widehat{\widehat{D}} &= [ D_0 \ | \ D_1 \ | \ \cdots \ | \ D_{\alpha-1} ] \\ &= [ \beta_0 B_d^2 \ | \ \beta_1 B_d^2 \ | \ \cdots \ | \ \beta_{\alpha-1} B_d^2 ] \\ &= \widetilde{\beta} \begin{bmatrix} B_d^2 & 0 & \cdots & 0 \\ 0 & B_d^2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & B_d^2 \end{bmatrix} = \widetilde{\beta} \bigoplus_{i=0}^{\alpha-1} B_d^2, \end{aligned}$$

$$\begin{aligned} \widehat{\widehat{D}} &= [ \widehat{D}_0 \ | \ \widehat{D}_1 \ | \ \cdots \ | \ \widehat{D}_{\alpha-1} ] \\ &= [ \beta_0 A_d^{22} \ | \ \beta_1 A_d^{22} \ | \ \cdots \ | \ \beta_{\alpha-1} A_d^{22} ] \\ &= \widetilde{\beta} \begin{bmatrix} A_d^{22} & 0 & \cdots & 0 \\ 0 & A_d^{22} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & A_d^{22} \end{bmatrix} = \widetilde{\beta} \bigoplus_{i=0}^{\alpha-1} A_d^{22}. \end{aligned}$$

Suppose that the matrix  $V_1$  and its inverse are partitioned as

$$V_1 = \begin{bmatrix} V_{111} & V_{112} \\ V_{112}^T & V_{122} \end{bmatrix}, V_1^{-1} = \begin{bmatrix} W_{111} & W_{112} \\ W_{112}^T & W_{122} \end{bmatrix}, \quad (14)$$

and set  $V_3^{-1} = W_3$  to define  $\Omega_4 = \bigoplus_{i=0}^{\alpha-1} W_3$ . Finally, to write the resulting design inequality in more compact form, we introduce the following change of variables

$$X_1 = D_d C_g W_{111} + C_d^1 W_{112}^T,$$

$$X_2 = V_{111} B_g D_d + V_{112} B_d^1,$$

$$X_3 = V_{111} B_g C_d^2 W_3 + V_{112} A_d^{12} W_3,$$

$$X_4 = V_{111} A_g W_{111} - V_{111} B_g D_d C_g W_{111} - V_{112} B_d^1 C_g W_{111}, \\ + V_{111} B_g C_d^1 W_{112}^T + V_{112} A_d^{11} W_{112}^T,$$

$$X_5 = C_d^2 W_3,$$

$$X_6 = -B_d^2 C_g W_{111} + A_d^{21} W_{112}^T,$$

$$X_7 = A_d^{22} W_3,$$

(15)

and define the following matrices

$$\begin{aligned} \mathcal{A}^{21} &= \bigoplus_{i=0}^{\alpha-1} \begin{bmatrix} B_d^2 C_g & A_d^{21} \end{bmatrix}, \mathcal{A}^{22} = \bigoplus_{i=0}^{\alpha-1} A_d^{22}, \mathcal{B} = \bigoplus_{i=0}^{\alpha-1} B_d^2, \\ \Delta &= \bigoplus_{i=0}^{\alpha-1} \begin{bmatrix} V_{111} A_g - X_2 C_g & X_4 \\ A_g - B_g D_d C_g & A_g W_{111} + B_g X_1 \end{bmatrix}, \Phi = \bigoplus_{i=0}^{\alpha-1} X_7, \\ \Lambda &= \bigoplus_{i=0}^{\alpha-1} \begin{bmatrix} X_3 \\ B_g X_5 \end{bmatrix}, \Sigma = \bigoplus_{i=0}^{\alpha-1} [C_g \ C_g W_{111}], \Theta = \bigoplus_{i=0}^{\alpha-1} \begin{bmatrix} X_2 \\ B_g D_d \end{bmatrix}, \\ \Psi &= \bigoplus_{i=0}^{\alpha-1} [-B_d^2 C_g \ X_6], \Xi = \bigoplus_{i=0}^{\alpha-1} \begin{bmatrix} V_{111} & I \\ I & W_{111} \end{bmatrix}, \end{aligned} \quad (16)$$

**Theorem 2:** A controlled discrete linear repetitive process of the form (7) has the quadratic stability property if there exist matrices  $V_{111} > 0$ ,  $W_{111} > 0$ ,  $V^2 > 0$ ,  $\Omega_2 > 0$ ,  $\Omega_3 > 0$ ,  $\Omega_4 > 0$ ,  $B_d^2 D_d$ ,  $X_1, \dots, X_7$ , such that the following hold

$$\begin{bmatrix} -\Omega_2 & 0 & \Sigma & 0 & 0 & 0 \\ * & -\Omega_3 & 0 & \tilde{\beta}\Psi & \tilde{\beta}\mathcal{B} & \tilde{\beta}\Phi \\ * & * & -\Xi & \Delta & \Theta & \Lambda \\ * & * & * & -\Xi & 0 & 0 \\ * & * & * & * & -V^2 & 0 \\ * & * & * & * & * & -\Omega_4 \end{bmatrix} < 0, \quad (17)$$

$$\begin{bmatrix} V_{111} & I \\ I & W_{111} \end{bmatrix} > 0, \quad (18)$$

$$V^2 \Omega_2 = I. \quad (19)$$

*Proof:* Noting that  $V_1$  is invertible, and assuming that the matrices  $V_1$  and  $V_1^{-1}$  are partitioned as in (14), it is easy to see that  $V_{111} W_{111} + V_{112} W_{112}^T = I$ . Next, by Theorem 1, there exists a controller such that the controlled process (7) has the quadratic stability property if there exist matrices  $V^1 > 0$ ,  $V^2 > 0$  and  $V^3 > 0$  (which implies  $\tilde{V}^3 > 0$ ) such that the matrix inequality (13) holds. Also, with the notation introduced in (16), (13) can be rewritten as

$$\begin{bmatrix} -V^1 + \tilde{C}^T V^2 \tilde{C} & 0 & 0 \\ 0 & -V^2 & 0 \\ 0 & 0 & -V^3 \end{bmatrix} + \begin{bmatrix} \mathcal{A}^{21T} \tilde{\beta}^T & \tilde{A}^T \\ \mathcal{B}^T \tilde{\beta}^T & \tilde{B}^T \\ \mathcal{A}^{22T} \tilde{\beta}^T & \tilde{B}^{2T} \end{bmatrix} \begin{bmatrix} \tilde{V}^3 & 0 \\ 0 & V^1 \end{bmatrix} \begin{bmatrix} \tilde{\beta} \mathcal{A}^{21} & \tilde{\beta} \mathcal{B} & \tilde{\beta} \mathcal{A}^{22} \\ \tilde{A} & \tilde{B} & \tilde{B}^2 \end{bmatrix} < 0.$$

Application of the Schur's complement formula and an obvious congruence transformation now gives

$$\begin{bmatrix} -V^2 & 0 & V^2 \tilde{C} & 0 & 0 & 0 \\ * & -\tilde{V}^3 & 0 & \tilde{V}^3 \tilde{\beta} \mathcal{A}^{21} & \tilde{V}^3 \tilde{\beta} \mathcal{B} & \tilde{V}^3 \tilde{\beta} \mathcal{A}^{22} \\ * & * & -V^1 & V^1 \tilde{A} & V^1 \tilde{B}^1 & V^1 \tilde{B}^2 \\ * & * & * & -V^1 & 0 & 0 \\ * & * & * & * & -V^2 & 0 \\ * & * & * & * & * & -V^3 \end{bmatrix} < 0. \quad (20)$$

Next, define

$$\Omega_1 = \bigoplus_{i=0}^{\alpha-1} \begin{bmatrix} I & W_{111} \\ 0 & W_{112}^T \end{bmatrix}, \Omega_2 = (V^2)^{-1}, \Omega_3 = (\tilde{V}^3)^{-1},$$

and left and right-multiply (20) by

$$\text{diag}[\Omega_2, \Omega_3, \Omega_1^T, \Omega_1^T, I, \Omega_4^T],$$

and

$$\text{diag}[\Omega_2, \Omega_3, \Omega_1, \Omega_1, I, \Omega_4],$$

respectively to give

$$\begin{bmatrix} -\Omega_2 & 0 & \tilde{C} \Omega_1 \\ * & -\Omega_3 & 0 \\ * & * & -\Omega_1^T V^1 \Omega_1 \\ * & * & * \\ * & * & * \\ * & * & * \\ 0 & 0 & 0 \\ \tilde{\beta} \mathcal{A}^{21} \Omega_1 & \tilde{\beta} \mathcal{B} & \tilde{\beta} \mathcal{A}^{22} \Omega_4 \\ \Omega_1^T V^1 \tilde{A} \Omega_1 & \Omega_1^T V^1 \tilde{B}^1 & \Omega_1^T V^1 \tilde{B}^2 \Omega_4 \\ -\Omega_1^T V^1 \Omega_1 & 0 & 0 \\ * & -V^2 & 0 \\ * & * & -\Omega_4 \end{bmatrix} < 0.$$

Consequently

$$-\Omega_1^T V^1 \Omega_1 = \bigoplus_{i=0}^{\alpha-1} \begin{bmatrix} -V_{111} & -I \\ -I & -W_{111} \end{bmatrix}, \tilde{C} \Omega_1 = \bigoplus_{i=0}^{\alpha-1} [C_g \ C_g W_{111}],$$

$$\Omega_1^T V^1 \tilde{B}^1 = \bigoplus_{i=0}^{\alpha-1} \begin{bmatrix} V_{111} B_g D_d + V_{112} B_d^1 \\ B_g D_d \end{bmatrix}, \mathcal{A}^{22} \Omega_4 = \bigoplus_{i=0}^{\alpha-1} A_d^{22} W_3,$$

$$\Omega_1^T V^1 \tilde{A} \Omega_1 = \bigoplus_{i=0}^{\alpha-1} \begin{bmatrix} V_{111} A_g - V_{111} B_g D_d C_g - V_{112} B_d^1 C_g \\ A_g - B_g D_d C_g \end{bmatrix},$$

$$\Omega_1^T V^1 \tilde{B}^2 \Omega_4 = \bigoplus_{i=0}^{\alpha-1} \begin{bmatrix} \Gamma \\ A_g W_{111} - B_g D_d C_g W_{111} + B_g C_d^1 W_{112}^T \end{bmatrix},$$

$$\Omega_1^T V^1 \tilde{B}^2 \Omega_4 = \bigoplus_{i=0}^{\alpha-1} \begin{bmatrix} V_{111} B_g C_d^2 W_3 + V_{112} A_d^{12} W_3 \\ B_g C_d^2 W_3 \end{bmatrix},$$

$$\mathcal{A}^{21} \Omega_1 = \bigoplus_{i=0}^{\alpha-1} \begin{bmatrix} -B_d^2 C_g & -B_d^2 C_g W_{111} + A_d^{21} W_{112}^T \end{bmatrix},$$

where

$$\begin{aligned} \Gamma &= V_{111} A_g W_{111} - V_{111} B_g D_d C_g W_{111} - V_{112} B_d^1 C_g W_{111} \\ &+ V_{111} B_g C_d^1 W_{112}^T + V_{112} A_d^{11} W_{112}^T. \end{aligned}$$

Finally, introduce the variables (15) to yield the LMI (17). ■

Obviously, the set defined by the inequalities (17) and (18) with constraint (19) is not convex. One method of finding matrices that satisfy this constraint is the Product Reduction Algorithm (PRA) [7]. This is based on the fact that if

$$\begin{bmatrix} V^2 & I \\ I & \Omega_2 \end{bmatrix} > 0 \quad (21)$$

holds for any matrices  $V^2 > 0$  and  $\Omega_2 > 0$  which satisfy  $\text{trace}(V^2 \Omega_2) = n$  (where  $n$  denotes the number of rows of the matrix  $V^2$  (or  $\Omega_2$ )) then (19) holds. The solution to the problem is hence reduced to

$$\begin{aligned} \min_{V^2 > 0, \Omega_2 > 0} & \text{trace}(V^2 \Omega_2) \\ \text{subject to} & (21) \end{aligned}$$

The following procedure can now be applied.

**Step 1.** Compute the singular value decomposition (SVD) of  $I - V_{111}W_{111}$  to obtain matrices  $U_1$ ,  $U_2$  such that  $I - V_{111}W_{111} = U_1\Sigma_1U_2^T$ .

**Step 2.** Choose the matrices  $V_{112}$ ,  $W_{112}$  as

$$V_{112} = U_1\Sigma_1^{\frac{1}{2}}, \quad W_{112} = U_2\Sigma_1^{\frac{1}{2}}.$$

**Step 3.** Perform the following computations to obtain the controller state-space model matrices

$$\begin{aligned} C_d^1 &= (X_1 - D_d C_g W_{111}) W_{112}^{-T}, \\ C_d^2 &= X_5 W_3^{-1}, \quad A_d^{22} = X_7 W_3^{-1}, \\ B_d^1 &= V_{112}^{-1} (X_2 - V_{111} B_g D_d), \\ A_d^{11} &= V_{112}^{-1} (X_4 - V_{111} A_g W_{111} + V_{111} B_g D_d C_g W_{111}, \\ &\quad + V_{112} B_d^1 C_g W_{111} - V_{111} B_g C_d^1 W_{112}^T) W_{112}^{-T}, \\ A_d^{12} &= V_{112}^{-1} (X_3 - V_{111} B_g C_d^2 W_3) W_3^{-1}, \\ A_d^{21} &= (X_6 + B_d^2 C_g W_{111}) W_{112}^{-T}. \end{aligned}$$

## V. NUMERICAL EXAMPLE

Consider the case given in [4] when

$$A_g = \begin{bmatrix} 0.9015 & 0.6242 & 0.1954 & -0.9337 & -0.062 & 0.2627 & -0.0053 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B_g^T = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0],$$

$$C_g = [-0.3283 \ -0.2986 \ 0.8035 \ 0.1024 \ -0.1156 \ -0.1753 \ -0.0053],$$

with  $\alpha = 10$  and  $\beta_0 = \dots = \beta_9 = 0.1$ . Application of the controller design procedure gives the stabilizing controller matrices

$$A_d^{11} = \begin{bmatrix} 0.232 & -0.116 & -0.915 & 0.187 & -0.045 & 0 & 0 \\ -0.130 & 0.041 & -0.186 & -0.881 & -0.040 & 0 & 0 \\ 0.036 & 0.092 & 0.005 & -0.069 & -0.598 & 0 & 0 \\ 0.019 & -0.002 & 0.024 & -0.078 & 0.864 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.369 & 1.902 \\ 0 & 0 & 0 & 0 & 0 & 0.020 & -0.108 \end{bmatrix},$$

$$B_d^{1T} = [0 \ 0 \ 0 \ 0 \ 0 \ 0.027 \ -0.018],$$

$$A_d^{12} = 0, \quad A_d^{21} = 0, \quad A_d^{22} = 0, \quad C_d^2 = 0, \quad D_d = 2.365 \cdot 10^{-7}, \quad B_d^2 = 0,$$

$$C_d^1 = [-0.018 \ 0.152 \ 0.052 \ -0.456 \ -0.230 \ 0 \ 0].$$

To compute the pass profile sequence generated by the controlled processes suppose that the boundary conditions are  $\chi_k(0) = [1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]^T$ ,  $k > 0$ , and  $y_1(p) = \hat{y}_1(p) = 1$ ,  $p = 0, 1, \dots, \alpha - 1$ . Fig. 1 shows the pass profile sequence generated and demonstrates that this controller is also capable of achieving zero error after a relatively small number of tracks have been completed. Further work is required on how to tune this design to best effect.

## VI. CONCLUSIONS

In this paper we have first developed new results on the control of discrete linear repetitive processes with inter-pass smoothing effects. The model considered in this work has arisen from the modeling of self-servowriting for disk drives and has features that are not captured by any of the previously considered models for repetitive processes. This is further evidence of the rich structure of the dynamics of

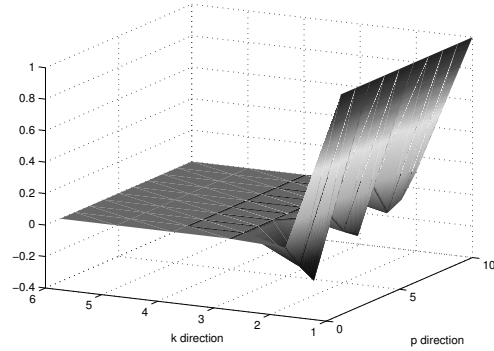


Fig. 1. Pass profile

such processes and of the fact that progress in linear multi-dimensional systems theory is often critically dependent on the particular model structure used.

Given the model, quadratic stability has been defined and characterized in terms of LMIs. This approach has also been extended to the design of a controller proposed within the disk drives industry for self-servowriting. The design objective here has been limited to stability which is the basic requirement in all applications and illustrated by a simulation example using industry supplied data. Further work is clearly required on how to undertake design for stability and pre-specified performance and also in the presence of uncertainty in the model used.

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