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On the Existence and Uniqueness of Solutions for the Concept Values in Fuzzy Cognitive Maps

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***Abstract*— Fuzzy Cognitive Maps (FCM) have been introduced by Kosko to model complex behavioral systems in various scientific areas. One issue that has not been adequately studied so far is the conditions under which they reach a certain equilibrium point after an initial perturbation. This is equivalent to studying the existence and uniqueness of solutions for their concept values. In this paper, we study the existence of solutions by using an appropriately defined contraction mapping theorem. It is proved that when the weight interconnections fulfill certain conditions the concept values will converge to a unique solution regardless the exact values of the initial concept values perturbations. Otherwise the existence or the uniqueness of equilibrium can not be assured. The results are considered very significant because set the basis for the development of reliable system identification and control schemes based on the concept of FCM. In view of these results recently proposed extensions of FCM, the Fuzzy Cognitive Networks are invoked.**

I. INTRODUCTION

Fuzzy Cognitive Maps (FCM) are inference networks using cyclic directed graphs that represent the causal relationships between concepts [1].They use a symbolic representation for the description and modelling of the system. In order to illustrate different aspects in the behavior of the system, a fuzzy cognitive map consists of nodes where each one represents a system characteristic feature. The node interactions represent system dynamics. An FCM integrates the accumulated experience and knowledge on the system operation, as a result of the method by which it is constructed, i.e., by using human experts who know the operation of the system and its behaviour. Different methodologies to develop FCM and extract knowledge from experts have been proposed in [2], [3], [4].

Fuzzy cognitive maps have already been used to model behavioral systems in many different scientific areas. For example, in political science [5], fuzzy cognitive maps were used to represent social scientific knowledge and describe decision-making methods [6], [7], [8]. Kosko enhanced the power of cognitive maps considering fuzzy values for their nodes and fuzzy degrees of interrelationships between nodes [1],[9]. He also proposed the differential Hebian rule [9] to estimate the FCM weights expressing the fuzzy interrelationships between nodes based on acquired data. After this pioneering work, fuzzy cognitive maps attracted the attention of scientists in many fields and they have been used in a variety of different scientific problems. Fuzzy cognitive maps have been used for planning and making decisions in the field of international relations and political developments and to model the behavior and reactions of virtual worlds. FCMs have been proposed as a generic system for decision analysis [8], [10] and as coordinator of distributed cooperative agents. Extensions of FCM are the Dynamic Cognitive Networks (DCN) [11], the Fuzzy Causal Networks [12] and recently the Fuzzy Cognitive Networks (FCN) [13]. The latter extension was presented as a complete computational and storage framework to facilitate the use of FCM in cooperation with the physical system they describe.

Regarding FCM weight estimation and updating, recent publications [14], [15], [16], [17] extend the initially proposed differential Hebian rule [9] to achieve better weight estimation. Another group of methods for training FCM structure involves genetic algorithms and other exhaustive search techniques [18], [19], [20], [21], where the training is based on a collection of particular values of input output historical examples and on the definition of appropriate fitness function to incorporate design restrictions.

An issue that needs more theoretical investigation concerns the conditions under which the concept values of FCM reach an equilibrium point and whether this point is unique. According to Kosko [22], starting from an initial state, simple FCMs follow a path, which ends in a fixed point or limit cycle, while more complex ones may end in an aperiodic or "chaotic" attractor. These fixed points and attractors could represent *meta rules* of the form "If input then attractor or fixed point". The relation of the existence of these attractors or fixed points to the weight interconnections of the FCM has not been fully investigated. This is, however, of paramount importance if one wants to use FCMs in reliable adaptive system identification and control schemes.

In this paper, we study the existence of the above fixed points by using an appropriately defined contraction mapping theorem. It is proved that when the weight interconnections fulfill certain conditions, related to the size of the FCM, the concept values will converge to a unique solution regardless of their initial states. Otherwise the existence or the uniqueness of equilibria can not be assured. In view of these

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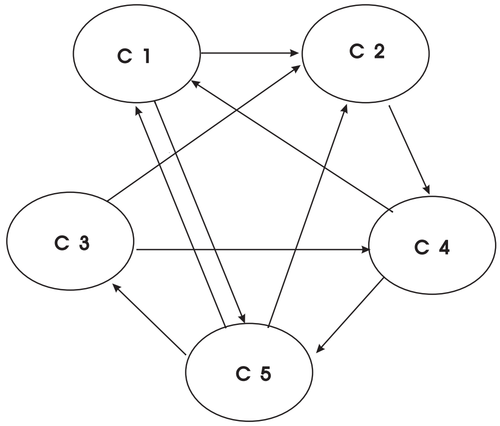

Fig. 1. An FCM with 5 nodes.

results *meta rules* of the form "If weights then fixed point" are more appropriate to represent the behavior of an FCM. Fuzzy Cognitive Networks (FCN) [13], [23]-[25], introduced recently as an extension of FCMs can work on the basis of such *meta rules*. It is demonstrated that when the necessary weight conditions are fulfilled by an FCN during its updating procedure, its information storage mechanism is actually a depository of this kind of *meta rules*.

The paper is organized as follows. Section II describes the representation and mathematical formulation of Fuzzy Cognitive Maps. Section III presents the main results, where the proof of the existence solution of the concept values of a Fuzzy Cognitive Map is given. FCNs are briefly invoked in section IV and some of their important aspects are presented. Finally, Section V concludes the work providing also hints for future extensions.

II. FUZZY COGNITIVE MAPS

Fuzzy Cognitive Maps (FCM) is a modeling methodology for complex systems, originated from the combination of Fuzzy Logic and Neural Networks. The graphical illustration of an FCM is a signed fuzzy graph with feedback, consisting of nodes and weighted interconnections. The nodes of the graph are related to concepts that are used to describe main behavioral characteristics of the system. Nodes are connected by signed and weighted arcs representing the causal relationships that exist among concepts. Graphical representation illustrates which concept influences other concepts, showing the interconnections between them. This simple illustration permits thoughts and suggestions in reconstructing FCM, such as the adding or deleting of an interconnection or a concept. In conclusion, an FCM is a fuzzy-graph structure, which allows systematic causal propagation, in particular forward and backward chaining.

A. *Fuzzy Cognitive Map representation*

A graphical representation of FCMs is depicted in Fig. 1. Each concept represents a characteristic of the system; in general it represents events, actions, goals, values and trends of the system . Each concept is characterized by a number A_i that represents its value and it results from the transformation of the real value of the systems variable, represented by this concept, in the interval [0,1]. All concept values form Vector A are expressed as:

$$A = \begin{bmatrix} A_1 & A_2 & .... & A_n \end{bmatrix}^T$$

with n being the number of the nodes (in Fig. 1 $n = 5$) . Causality between concepts allows degrees of causality and not the usual binary logic, so the weights of the interconnections can range in the interval [-1,1] (see [22]).

The existing knowledge on the behavior of the system is stored in the structure of nodes and interconnections of the map Each node-concept represents one of the key-factors of the system. Relationships between concepts have three possible types; either express positive causality between two concepts ($W_{ij} > 0$) or negative causality ($W_{ij} < 0$) or no relationship ($W_{ij} = 0$). The value of W_{ij} indicates how strongly concept C_i influences concept C_j. The sign of W_{ij} indicates whether the relationship between concepts C_i and C_j is direct or inverse. The direction of causality indicates whether concept C_i causes concept C_j, or vice versa. These parameters have to be considered when a value is assigned to weight W_{ij}. For the FCM of Fig. 1 matrix W is equal to

$$W = \begin{bmatrix} 0 & 0 & 0 & W_{41} & W_{51} \\ W_{12} & 0 & W_{32} & 0 & W_{52} \\ 0 & 0 & 0 & 0 & W_{53} \\ 0 & W_{24} & W_{34} & 0 & 0 \\ W_{15} & 0 & 0 & W_{45} & 0 \end{bmatrix}$$

The equation that calculates the values of concepts of Fuzzy Cognitive Map, according to [4] is equal to:

$$A_i(k) = f(\sum_{\substack{j=1 \\ j\neq i}}^{n} W_{ij}^T A_j(k-1) + A_i(k-1)) \qquad (1)$$

Where $A_i(k)$ is the value of concept C_i at discrete time k, $A_i(k-1)$ the value of concept C_i at discrete time $k-1$, $A_j(k-1)$ the value of concept C_j at discrete time $k-1$, and W_{ij} is the weight of the interconnection from concept C_j to concept C_i. f is a sigmoid function used in the Fuzzy Cognitive Map, which squashes the result in the interval [0,1] and is expressed as,

$$f = \tfrac{1}{1+e^{-x}}.$$

Equation (1) can also be written as:

$$A(k) = f(W^{ext} \cdot A(k-1)) \qquad (2)$$

where W^{ext} is such that: $W_{ij}^{ext} \underset{i\neq j}{=} W_{ji}$, $W_{ij}^{ext} \underset{i=j}{=} d_{ii}$, where d_{ii} is a variable that takes on values in the interval $[0, 1]$, depending upon the existence of "strong" or "weak" self-feedback to node i. Note that the case d_{ii} close to 0 is generic, while the d_{ii} close to 1 is an exception. See among other examples and the virtual undersea world example in Kosko ([22], p.513) where only two out of 24 nodes are using self-feedback. For the FCM of Fig. 1 matrix W^{ext} is equal to

$$W^{ext} = \begin{bmatrix} d_{11} & 0 & 0 & W_{41} & W_{51} \\ W_{12} & d_{22} & W_{32} & 0 & W_{52} \\ 0 & 0 & d_{33} & 0 & W_{53} \\ 0 & W_{24} & W_{34} & d_{44} & 0 \\ W_{15} & 0 & 0 & W_{45} & d_{55} \end{bmatrix}$$

From now on, in this paper the matrix W^{ext} will be just called W. Equation (2) can be rewritten as:

$$A(k) = f(W \cdot A(k-1)) \tag{3}$$

In the next Section we are deriving conditions, which determine the existence of a unique solution of (3).

III. Existence and uniqueness of solutions in Fuzzy Cognitive Maps

In this Section we check the existence of solutions in equation (3). We know that the allowable values of the elements of FCM vectors A lie in the closed interval $[0,1]$. This is a subset of $\Re$ and is a complete metric space with the usual l_2 metric. We will define the regions where the FCM has a unique solution, which does not depend on the initial condition since it is the unique equilibrium point.

A. The Contraction mapping principle

We now introduce the Contraction Mapping Theorem [26].

Definition 1: Let X be a metric space, with metric d. If φ maps X into X and there is a number $c < 1$ such that

$$d(\varphi(x), \varphi(y)) \leq cd(x,y) \tag{4}$$

for all $x, y \in X$, then φ is said to be a contraction of X into X.

Theorem 1: [26] *If X is a complete metric space, and if φ is a contraction of X into X, then there exists one and only one $x \in X$ such that $\varphi(x) = x$.*

In other words, φ has a unique fixed point. The uniqueness is a triviality, for if $\varphi(x) = x$ and $\varphi(y) = y$, then (4) gives $d(x,y) \leq cd(x,y)$, which can only happen when $d(x,y) = 0$.

Equation (3) can be written as:

$$A(k) = G(A(k-1)) \tag{5}$$

where $G(A(k-1))$ is equal to $f(W \cdot A(k-1))$.

In FCM's $A \in [0,1]^n$ and it is also clear according to (3) that $G(A(k-1)) \in [0,1]^n$. If the following inequality is true:

$$d(G(A), G(A')) \leq cd(A, A')$$

then G has a unique fixed point A such that:

$$G(A) = A$$

Before presenting the main theorem we need to explore the role of f as a contraction function.

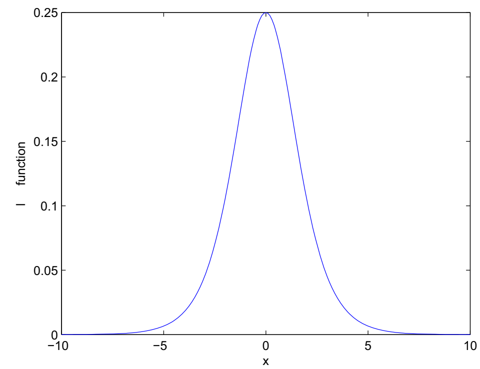

Fig. 2. Inclination of sigmoid function f.

Theorem 2: The scalar sigmoid function f, ($f = \frac{1}{1+e^{-x}}$) is a contraction of metric space X into X, were $X = [a,b], a \leq 0, b \geq 1$ according to Definition 1, where:

$$d(f(x), f(y)) \leq cd(x,y) \tag{6}$$

Proof: Here f is the sigmoid function, x,$y \in X$, X is as defined above and c is a real number such that $0 \leq c < 1$

The inclination l of a sigmoid function f is equal to:

$$l = \frac{\partial f}{\partial x} = \frac{e^{-x}}{(1+e^{-x})^2} = \frac{1}{e^x}\left(\frac{1}{1+e^{-x}}\right)^2 = \frac{1}{e^x} f^2 \tag{7}$$

for $x \in X$. Equation (7) is plotted in Fig. 2. According to Fig. 2 one can see that the inclination l of $f(x)$ in the bounded set X is always smaller than 1/4, as follows:

$$\frac{1}{4} \geq l \tag{8}$$

and l also equals to:

$$\frac{d\left(f(x), f(y)\right)}{d\left(x,y\right)} = l \tag{9}$$

From (8) and (9) we get:

$$\frac{d\left(f(x), f(y)\right)}{d\left(x,y\right)} = l < 1 \tag{10}$$

Thus there is always a number c for which $0 \leq c < 1$, such that (10) is:

$$\frac{d\left(f(x), f(y)\right)}{d\left(x,y\right)} < c < 1 \tag{11}$$

■

Theorem 3: There is one and only one solution for any concept value A_i of any FCM, if:

$$\left(\sum_{i=1}^{n} \|w_i\|^2\right)^{1/2} < 4 \tag{12}$$

where w_i is the i_{th} row of matrix W and $\|w_i\|$ is the l_2 norm of w_i.

Proof:

Let X be the complete metric space $[0,1]^n$ and $G: X \rightarrow X$ be a map such that:

$$d(G(A), G(A')) \leq c d(A, A') \quad (13)$$

for some $0 \leq c < 1$.

Vector G is equal to:

$$G = \begin{bmatrix} f(w_1 \cdot A) \\ f(w_1 \cdot A) \\ f(w_1 \cdot A) \\ \vdots \\ f(w_n \cdot A) \end{bmatrix} \quad (14)$$

where n is the number of concepts of the FCM, f is the sigmoid function defined above, w_i is the i_{th} row for matrix W of the FCM, where $i = 1, 2, ..., n$, and by $\cdot$ we denote the inner product between two equidimensional vectors which both belong in $\Re^n$.

Assume A and A' are two different concept values for the FCM. Then we want to prove the following inequality:

$$\|G(A) - G(A')\| \leq c \|A - A'\| \quad (15)$$

But $\|G(A) - G(A')\|$ according to (14) equals to:

$$\|G(A) - G(A')\| = \left(\sum_{i=1}^{n} (f(w_i \cdot A) - f(w_i \cdot A'))^2 \right)^{1/2}$$

According to Theorem 2 for the scalar argument of $f(.)$ which is $w_i \cdot A$ in the bounded and closed interval $[-a, a]$ with a being a finite number it is true that:

$$|f(w_i \cdot A) - f(w_i \cdot A')| \leq c_i' |(w_i \cdot A) - (w_i \cdot A')|$$

for every $i = 1, 2, ..., n$.

Thus

$$|f(w_i \cdot A) - f(w_i \cdot A')| \leq c' |(w_i \cdot A) - (w_i \cdot A')|$$

where $c' = max(c_1', c_2', ..., c_n')$.

By using the Cauchy-Schwartz inequality we get:

$$c'|(w_i \cdot A) - (w_i \cdot A')| = c'|w_i \cdot (A - A')|$$

$$\leq c'\|w_i\|\|A - A'\|$$

Subsequently, we get:

$$\|G(A) - G(A')\| = \left(\sum_{i=1}^{n} (f(w_i \cdot A) - f(w_i \cdot A'))^2 \right)^{1/2}$$

$$\Rightarrow \|G(A) - G(A')\| \leq \left(\sum_{i=1}^{n} (c' \|w_i\| \|A - A'\|)^2 \right)^{1/2}$$

Finally:

$$\|G(A) - G(A')\| \leq c' \|A - A'\| \left(\sum_{i=1}^{n} \|w_i\|^2 \right)^{1/2}$$

A necessary condition for the above to be a contraction is:

$$c' \left(\sum_{i=1}^{n} \|w_i\|^2 \right)^{1/2} < 1 \quad (16)$$

From eq. (8) we have that:

$$c' \leq 1/4$$

So that condition of eq. (16) now becomes:

$$\left(\sum_{i=1}^{n} \|w_i\|^2 \right)^{1/2} < 4 \quad (17)$$

■

B. Exploring the results

1) An FCM with two concepts: Suppose that we have an FCM with two nodes. The weight matrix W_2 of this FCM is:

$$W_2 = \begin{bmatrix} d_{11} & w_{21} \\ w_{12} & d_{22} \end{bmatrix}$$

According to Theorem 3 in order that an FCM with two nodes has a unique concepts solution inequality (12) must be true. In this case (12) is written as:

$$d_{11} + w_{21}^2 + w_{12}^2 + d_{22} < 16$$

Since $|w_{21}| \leq 1$, $|w_{12}| \leq 1$ and d_{ii} can at most both take the value of 1, one can easily see that the above inequality is **always true** and particularly:

$$1 + w_{21}^2 + w_{12}^2 + 1 \leq 4 < 16$$

2) An FCM with three concepts: Suppose that we have an FCM with three nodes. The weight matrix W_3 of this FCM is:

$$W_3 = \begin{bmatrix} d_{11} & w_{21} & w_{31} \\ w_{12} & d_{22} & w_{32} \\ w_{13} & w_{23} & d_{33} \end{bmatrix}$$

Taking into account that the magnitude of every weight value of W_3 is less than one Eq. (12) is now written:

$$d_{11}+w_{21}^2+w_{31}^2+w_{12}^2+d_{22}+w_{32}^2+w_{13}^2+w_{23}^2+d_{33} \leq 9 < 16$$

where it is obvious that, for an FCM with three concepts, the condition for the uniqueness is **always true**.

3) An FCM with four concepts: Suppose that we have an FCM with four nodes. The weight matrix W_4 of this FCM is:

$$W_4 = \begin{bmatrix} d_{11} & w_{21} & w_{31} & w_{41} \\ w_{12} & d_{22} & w_{32} & w_{42} \\ w_{13} & w_{23} & d_{33} & w_{43} \\ w_{14} & w_{24} & w_{34} & d_{44} \end{bmatrix}$$

The square root of the sum of the square l_2 norm of each row of matrix W_4 is equal to:

$$\sqrt{\sum_{i=1}^{4} \|w_i\|^2} = \sqrt{\|w_1\|^2 + \|w_2\|^2 + \|w_3\|^2 + \|w_4\|^2} \quad (18)$$

The l_2 norm of each row is equal to: $\|w_i\| = \sqrt{\sum_{j=1}^{4} w_{ij}^2}$, where i denotes the i_{th} row of matrix W_4 and j denotes the column index. Equation (18) is now:

$$\sqrt{\sum_{i=1}^{4}\|w_i\|^2} = \sqrt{\|w_1\|^2+\|w_2\|^2+\|w_3\|^2+\|w_4\|^2} =$$

$$\sqrt{\sqrt{\sum_{j=1}^{4}w_{j1}^2}^2+\sqrt{\sum_{j=1}^{4}w_{j2}^2}^2+\sqrt{\sum_{j=1}^{4}w_{j3}^2}^2+\sqrt{\sum_{j=1}^{4}w_{j4}^2}^2} =$$

$$\sqrt{\sum_{j=1}^{4}w_{j1}^2+\sum_{j=1}^{4}w_{j2}^2+\sum_{j=1}^{4}w_{j3}^2+\sum_{j=1}^{4}w_{j4}^2} =$$

$$\sqrt{\sum_{j=1}^{4}\left(\sum_{i=1}^{4}w_{ji}^2\right)} = \sqrt{\sum_{j=1}^{4}(d_{jj})+\sum_{j=1}^{4}\left(\sum_{i=1,i\neq j}^{4}w_{ji}^2\right)}$$

Since for the non diagonal elements $|w_{ji}| < 1$, then : $w_{ji}^2 < 1$.

Finally the above equation concludes to:

$$\sqrt{\sum_{j=1}^{4}(d_{jj})+\sum_{j=1}^{4}\left(\sum_{i=1,i\neq j}^{4}w_{ji}^2\right)} \leq \sqrt{4+12} = 4$$

Subsequently, we get:

$$\sqrt{\sum_{i=1}^{4}\|w_i\|^2} \leq 4$$

According to Theorem 3 in order that only one solution exists for the concepts of an FCM the following inequality must be true: $\sqrt{\sum_{i=1}^{4}\|w_i\|^2} < 4$.

We finally get the next conclusion: Since generically most of the d_{ii} will be close to zero, "an FCM with four concepts has a unique solution generically".

4) An FCM with more than four concepts: Suppose that we have an FCM with more than four nodes. The weight matrix W_n of the FCM is:

$$W_n = \begin{bmatrix} d_{11} & w_{21} & w_{31} & ... & w_{n1} \\ w_{12} & d_{22} & w_{32} & ... & w_{n2} \\ w_{13} & w_{23} & d_{33} & ... & w_{n3} \\ ... & ... & ... & ... & ... \\ w_{1n} & w_{2n} & w_{3n} & ... & d_{nn} \end{bmatrix}$$

where $n > 4$ The square root of the sum of the square l_2 norm of each row of matrix W_n is given by:

$$\sqrt{\sum_{i=1}^{n}\|w_i\|^2} = \sqrt{\|w_1\|^2+\|w_2\|^2+.....+\|w_n\|^2}$$

$$\Rightarrow \sqrt{\sum_{i=1}^{n}\|w_i\|^2} = \sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{n}w_{ji}^2\right)}$$

$$\Rightarrow \sqrt{\sum_{i=1}^{n}\|w_i\|^2} = \sqrt{\sum_{j=1}^{n}(d_{jj})+\sum_{j=1}^{n}\left(\sum_{i=1,i\neq j}^{n}w_{ji}^2\right)}$$

$$\Rightarrow \sqrt{\sum_{i=1}^{n}\|w_i\|^2} \leq \sqrt{\sum_{j=1}^{n}(d_{jj})}+\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1,i\neq j}^{n}w_{ji}^2\right)}$$

Finally we conclude that for an FCM with $n > 4$ concepts Theorem 3 is true when:

$$\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1,i\neq j}^{n}w_{ji}^2\right)} \leq 4-\sqrt{\sum_{j=1}^{n}(d_{jj})} \quad (19)$$

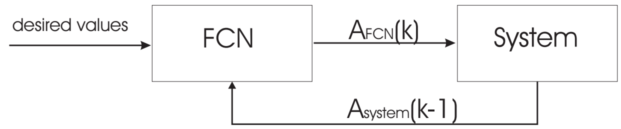

Fig. 3. Interactive operation of the FCN with the physical system.

Therefore, when $n > 4$ the condition for the uniqueness of solution of (3) depends on the number of diagonal d_{ii} elements of the FCM that are nonzero and the size of the FCM. However, Eq. (19) provides us with an upper bound for the weights of the FCM. When the weights are within this bound the solution of (3) is unique and therefore the FCM will converge to one value regardless of its initial concept values. This in turn gives rise to a *meta rules* representation of the FCM having the form "If weights then fixed point". This representation is employed by Fuzzy Cognitive networks (FCN), which are presented in the next Section.

IV. THE FUZZY COGNITIVE NETWORK APPROACH

As shown in Section III the concepts values of the FCM with a specified matrix W have a unique solution as far as (12) and consequently (19) is fulfilled. The perspective of transforming FCMs into a modeling and control alternative requires, first to update its weight matrix W so that the FCM can capture different mappings of the real system and second to store these different kinds of mappings. Fuzzy Cognitive Network (FCN) [13] has been proposed as an operational extension framework of FCM, which updates its weights and reaches new equilibrium points based on the continuous interaction with the system it describes. Moreover, for each equilibrium point a fuzzy rule based storage mechanism of the form "If weights then fixed point" is provided, which facilitates and speeds-up its operation. The components of FCN are briefly presented bellow.

A. Close interaction with the Real System

The operation of the FCN in close cooperation with the real system it describes, might require continuous changes in the weight interconnections, depending on feedback received from the real system. Fig. 3 presents the interactive operation of the FCN with the physical system it describes. The weight updating procedure is described below.

B. Weight Updating Procedure

The updating method takes into account feedback node values from the real system. Using the updated weights the FCN reaches a new equilibrium point. In this approach the updating is made based on the conventional delta rule, which is described by the following equations

$$p_j = A_j^{system}(k) - f(W_{ij} \cdot A_i^{FCN}(k-1)) \quad (20)$$

$$W_{ij}^k = W_{ij}(k-1) + ap_j(1-p_j)A_i^{FCN}(k) \quad (21)$$

where p_j is the error and a is the learning rate, usually set at $a = 0.1$. As it is shown from the argument of the exponential

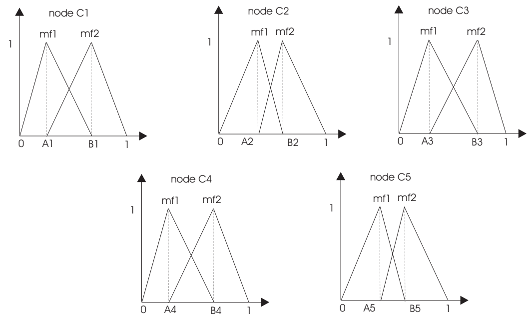

Fig. 4. Left hand side (if-part).

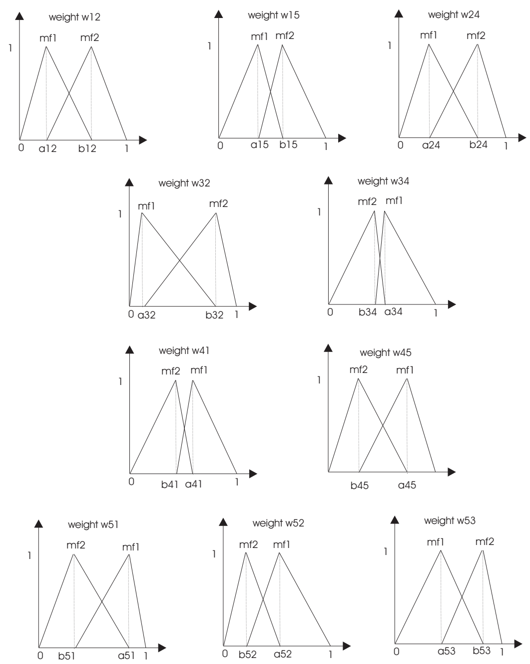

Fig. 5. Right hand side (then-part).

function, the term denoted as A_i^{FCN} refers to the response of the FCN, when the nodes take their values from the feedback of the system. In case the control objective is that one or more nodes reach a desired value then for these nodes Eq. (20) is rewritten as:

$$p_j = A_j^{desired}(k) - f(W_{ij} \cdot A_i^{FCN}(k-1)) \qquad (22)$$

After the weights updating, Eq. (1) will give new equilibrium concept values to the FCN. If the weights are chosen such that they meet the condition derived in Section III, these values will be unique. The calculated node values will be applied to the real system, which in turn provides feedback to the FCN to be used by the new updating cycle according to Fig. 3. Error p_j appearing in Eqs (20) and (21) is actually estimated for each one of the nodes j of the FCN, regardless of its label. Equation (22) is used for calculating the error of desired value nodes, while Eq. (20) is used for the errors of all other node values. When the real node values coming as a feedback from the system are fed to the FCN, this may present non zero error in all of its nodes. The error becomes zero only when the weights are updated so that the node values of the FCN match exactly the values of the corresponding physical quantities.

C. Storing knowledge from previous operating conditions

The procedure described in the previous subsection modifies FCN's knowledge about the system by continuously modifying the weight interconnections and consequently the node values. During the repetitive updating operation the procedure uses feedback from the system variables. This means that in each iteration all the intermediate weight and node values, some of which are control values, are fed to the real system and its response is used to give the new updating direction. It is obvious that this procedure continuously annoys the physical system, something that in many cases is undesirable. In the sequence we propose a methodology that alleviates this annoyance and further speeds up the updating procedure. This is done by storing the previous acquired operational situations in a fuzzy if-then rule database, which associates in a fuzzy manner the various weights with the corresponding equilibrium node values. The procedure is explained as follows. Suppose for example that the FCM of Fig. 1 has a unique equilibrium point

$$A = \begin{bmatrix} A1 & A2 & A3 & A4 & A5 \end{bmatrix}^T$$

which is connected with the weight matrix W:

$$W = \begin{bmatrix} d_{11} & 0 & 0 & a_{41} & a_{51} \\ a_{12} & d_{22} & a_{32} & 0 & a_{52} \\ 0 & 0 & d_{33} & 0 & a_{53} \\ 0 & a_{24} & a_{34} & d_{44} & 0 \\ a_{15} & 0 & 0 & a_{45} & d_{55} \end{bmatrix}$$

in order that A is a unique solution of eq. (1) weight matrix W has to be such that inequality (12) is fulfilled. For weight matrix W inequality (12) takes the form:

$$a_{41}^2 + a_{51}^2 + a_{12}^2 + a_{32}^2 + a_{52}^2 + a_{53}^2 + a_{24}^2 + a_{34}^2 + a_{15}^2 + a_{54}^2 < 16 - \sqrt{\sum_{j=1}^{5} (d_{jj})}$$

where $n = 5$ is the number of concepts of the FCN.

Suppose also that the FCM in another operation point is related to the following weight matrix W, which also fulfills (12):

$$W = \begin{bmatrix} d_{11} & 0 & 0 & b_{41} & b_{51} \\ b_{12} & d_{22} & b_{32} & 0 & b_{52} \\ 0 & 0 & d_{33} & 0 & b_{53} \\ 0 & b_{24} & b_{34} & d_{44} & 0 \\ b_{15} & 0 & 0 & b_{45} & d_{55} \end{bmatrix}$$

with the unique equilibrium point being:

$$A = \begin{bmatrix} B1 & B2 & B3 & B4 & B5 \end{bmatrix}^T$$

Inequality (12) for the weight matrix W has now the form:

$$b_{41}^2 + b_{51}^2 + b_{12}^2 + b_{32}^2 + b_{52}^2 + b_{53}^2 + b_{24}^2 + b_{34}^2 + b_{15}^2 + b_{54}^2 < 16 - \sqrt{\sum_{j=1}^{5} (d_{jj})}$$

The fuzzy rule database, which is obtained using the information from the two previous equilibrium points, is depicted in Fig. 4 and 5 and is resolved as follows:
There are two rules for the description of the above two different equilibrium situations:

Rule 1

if node C1 is mf1 *and* node C2 is mf1 *and* node C3 is mf1 *and* node C4 is mf1 *and* node C5 is mf1
then w_{12} is mf1 *and* w_{15} is mf1 *and* w_{24} is mf1 *and* w_{32} is mf1 *and* w_{34} is mf1 *and* w_{41} is mf1 *and* w_{45} is mf1 *and* w_{51} is mf1 *and* w_{52} is mf1 *and* w_{53} is mf1

Rule 2

if node C1 is mf2 *and* node C2 is mf2 *and* node C3 is mf2 *and* node C4 is mf2 *and* node C5 is mf2
then w_{12} is mf2 *and* w_{15} is mf2 *and* w_{24} is mf2 *and* w_{32} is mf2 *and* w_{34} is mf2 *and* w_{41} is mf2 *and* w_{45} is mf2 *and* w_{51} is mf2 *and* w_{52} is mf2 *and* w_{53} is mf2

The number and shape of the fuzzy membership functions for the variables in both sides of the rules are gradually modified, as new desired equilibria appear in the system during its operation. To add a new triangular membership function in the fuzzy description of a variable, the new value must differ from the previous one more than a specified threshold. The threshold comes usually as a compromise between the maximum number of allowable rules and the detail in fuzzy representation of each variable.

V. CONCLUSIONS

In this paper the uniqueness of the equilibrium values of the concepts of FCMs was studied, using an appropriately defined contraction mapping theorem. It was proved that when the weight interconnections fulfill certain conditions, related to the size of the FCM, the concept values will converge to a unique solution regardless their initial values. The condition is further resolved by exploring its application in FCMs of various representative sizes. The convergence point of the concepts depends exclusively on the values of weight matrix W of the FCM giving rise to an FCM representation using *meta rules* of the form "If weights then equilibrium point". A new alternative scheme called Fuzzy Cognitive Network (FCN) was also presented, which is designed to work in close interaction with the physical system it describes and stores information related to different operational points using fuzzy *meta rules* of this kind. Future work will include the development of reliable system identification and control schemes which will use the concept of FCN and employ the theoretical results presented here.

REFERENCES

[1] B. Kosko, Fuzzy Cognitive Maps, *International Journal of Man-Machine Studies*, 1986, pp 65-75.

[2] C. Stylios and P. Groumpos, "Fuzzy Cognitive Maps in Modelling Supervisory Control Systems", *Journal of Intelligent and Fuzzy Systems*, vol. 8, 2000, pp. 83-98.

[3] C. Stylios and P. Groumpos, "A soft computing approach for modelling the supervisor of manufacturing systems", *Journal of Intelligent and Robotics Systems*, vol. 26, no. 34, 1999, pp. 389-403.

[4] C. Stylios, P. Groumpos and V. Georgopoulos, "A Fuzzy Cognitive Maps approach to process control systems"', *Journal of Intelligent and Robotics Systems*, vol. 26, no. 34, 1999, pp. 389-403.

[5] R. Axelrod, Structure of Decision, *The Cognitive Maps of Political Elites*, Princeton University Press, New Jersey, 1976.

[6] T. Kottas, Y. Boutalis, G. Devedzic and B. Mertzios, "A new method for reaching equilibrium points in Fuzzy Cognitive Maps", *Proceedings of 2nd International IEEE Conference of Intelligent Systems*, Varna Burgaria, 2004, pp. 53-60.

[7] V. Georgopoulos, G. Malandraki and C. Stylios, "A fuzzy cognitive map approach to differential diagnosis of specific language impairment", *Artificial Intelligence in Medicine*, vol. 29, no. 3, 2003, pp. 261-278.

[8] W. Zhang, S. Chen and J. Bezdek, "Pool2: A Generic System for Cognitive Map Development and Decision Analysis", *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 19, no. 1, 1989, pp. 31-39.

[9] B. Kosko, "Differential Hebbian Learning", *Proceedings American Institute of Physics, Neural Networks for Computing*, 1986, pp. 277-282.

[10] W. Zhang, S. Chen, W. Wang and R. King, "A Cognitive Map Based Approach to the Coordination of Distributed Cooperative Agents", *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 22, no. 1, 1992, pp. 103-114.

[11] Y. Miao, Z. Liu, C. Siew, and C. Miao, "Dynamical Cogntive Network-an Extension of Fuzzy Cognitive Map", *IEEE transactions on Fuzzy Systems*, vol. 9, no. 5, 2001, pp. 760-770.

[12] J. Zhang, Zhi-Qiang Liu, and S. Zhou, "Dynamic Domination in Fuzzy Causal Networks", *IEEE Transactions on Fuzzy Systems*, vol. 14, no. 1, 2006, pp. 42 - 57.

[13] T.L. Kottas, Y.S. Boutalis and M.A. Christodoulou, Fuzzy Cognitive Networks: A General Framework", *Inteligent Desicion Technologies*, vol. 1, no. 4, 2007, pp. 183-196.

[14] A. Huerga, "A Balanced Differential Learning algorithm in Fuzzy Cognitive Maps", *Proc. of the Sixteenth Intern.Workshop on Qualitative Reasoning*, poster, 2002.

[15] E. Papageorgiou and P. Groumpos, "A weight adaptation method for Fuzzy Cognitive Maps to a process control problem", *Lecture Notes in Computer Science 3037 (Vol. II),M. Budak et al., (Intern. Conference on Computational Science, ICCS 2004, Krakow, Poland, 69 June)*, Springer Verlag, 2004, pp. 515-522.

[16] E. Papageorgiou, C. Stylios and P. Groumpos, "Active Hebbian Learning Algorithm to train Fuzzy Cognitive Maps", *International Journal of Approximate Reasoning*, vol. 37, no. 3, 2004, pp. 219-247.

[17] J. Aguilar, "Adaptive Random Fuzzy Cognitive Maps", *IBERAMIA 2002, Lecture Notes in Artificial Intelligence 2527, F.J. Garijio, J.C. Riquelme and M. Toro, eds*, Springer-Verlag Berlin Heidelberg, 2002, pp. 402-410.

[18] D.Koulouriotis, I. Diakoulakis and D. Emiris, "Learning Fuzzy Cognitive Maps using evolution strategies: A novel schema for modeling a simulating high-level behavior"', *Proceedings of IEEE Congress on Evolutionary Computation*, vol. 1, 2001, pp. 364-371.

[19] E. Papageorgiou, K. Parsopoulos, C. Stylios, P. Groumpos and M. Vrahatis, "Fuzzy Cognitive Maps Learning Using Particle Swarm Optimization", *International Journal of Intelligent Information Systems*, vol. 25,no. 1, 2005, pp. 95-121.

[20] M. Khan, S. Khor and A. Chong, "Fuzzy cognitive maps with genetic algorithm for goal-oriented decision support", *Int. J. Uncertainty, Fuzziness and Knowledge-based Systems*, vol. 12, 2004, pp. 31-42.

[21] W. Stach, L. Kurgan, W. Pedrycz and M. Reformat, "Genetic learning of fuzzy cognitive maps", *Fuzzy sets and systems*, vol. 153, no. 3, 2005, pp. 371-401.

[22] B. Kosko, *Fuzzy Engineering*, Prentice Hall, 1997.

[23] T. Kottas, Y. Boutalis and M. Christodoulou, "A new method for weight updating in Fuzzy cognitive Maps using system Feedback"', *2nd International Conference on Informatics in Control, Automation and Robotics (ICINCO05)*, Barcelona, Spain 2005, pp. 202-209.

[24] T.L. Kottas, Y.S. Boutalis and A.D. Karlis, "A New Maximum Power Point Tracker for PV Arrays Using Fuzzy Controller in Close Cooperation with Fuzzy Cognitive Networks"', *IEEE Transactions on Energy Conversion*, vol. 21, no. 3, 2006, pp. 793-803.

[25] T. Kottas, Y. Boutalis, V. Diamantis, O. Kosmidou and A. Aivasidis, "A Fuzzy Cognitive Network Based Control Scheme for an Anaerobic Digestion Process"', *14th Mediterranean Conference on Control and Applications*, Session TMS Process Control 1, Ancona, Italy 2006.

[26] Walter Rudin, *Principles of Mathematical Analysis*, McGraw-Hill Inc., 1964, pp. 220-221.