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Topological Geometry and Control for Distributed Port-Hamiltonian Systems with Non-Integrable Structures

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Abstract— **This paper discusses topological geometrical aspects and a control strategy for a distributed port-Hamiltonian system with a non-integrable structure called a distributed energy structure. First, we show a geometrical structure of port variables determined by differential forms. Next, we state the necessary condition for regarding the distributed energy structure as a boundary energy structure which is boundary integrable. From these results, we define the fundamental form that generates the distributed port-Hamiltonian system with distributed energy structures in a variational problem. Finally, we present a new concept of boundary controls for the distributed port-Hamiltonian system with distributed energy structures in space-time coordinates.**

I. Introduction

A *distributed port-Hamiltonian system* is a general framework for boundary control of partial differential equations [2]. The system has two aspects: the physical system defined by a Hamiltonian and the control system based on passivity. If we can formulate a control object as a distributed port-Hamiltonian system, the representations reflects many results in analytical mechanics, and at the same time, we can apply control strategies of port representations (i.e. damping injection, energy shaping and port interconnection) to the system [6], [7], [8].

A boundary integrable structure, called a *Stokes-Dirac structure*, formalizes the distributed port-Hamiltonian system in the sense of Stokes's theorem with differential forms (e.g. Maxwell's equations without current densities). The distributed port-Hamiltonian system consists of dual pairs of variables, called *boundary port variables*, satisfying the Stokes-Dirac structure. The boundary integral of the pairs of boundary port variables is equal to the power flow through the boundary of the system. The formalism can be extended to a boundary non-integrable structure [2] by introducing *distributed port variables*. The domain integral of the pairs of distributed port variables expresses the distributed power flow of the system. We call the boundary integrable (resp. non-integrable) structure, which is defined by boundary (resp. distributed) port variables, *boundary (resp. distributed) energy structures* in the distributed port-Hamiltonian system.

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In this paper, we discuss a geometrical structure determined by differential forms and a boundary control method for the distributed port-Hamiltonian system with the distributed energy structure. First, we show that the spaces of differential forms defining port variables relates to the topological geometry of compact Riemannian manifolds. Next, we state the necessary condition for regarding a distributed energy structure as a boundary energy structure, which we call *boundary completion*. From the results, we define a *fundamental form* and a *partition of the Hamiltonian*. The fundamental form generates the distributed port-Hamiltonian system in terms of variational calculus. The partition of the Hamiltonian gives the information to split a differential form into two port variables and it is introduced from the relation between classical field equations and distributed port-Hamiltonian systems. Finally, after extending the fundamental form to distributed energy structures, we present the new strategy of boundary controls in space-time coordinates. The concept holds for all distributed port-Hamiltonian systems including distributed energy structures.

II. Mathematical Preliminaries

Let M be an m-dimensional orientable C^∞ Riemannian manifold. Let $d : \Omega^k(M) \to \Omega^{k+1}(M)$ denote an *exterior differential operator*, and let $* : \Omega^k(M) \to \Omega^{\bar{k}}(M)$ denote a *Hodge star operator*, where $\bar{k} := m - k$ and $\Omega^k(M)$ is a space of differential k-forms on M. We define $\delta = (-1)^{m(k+1)+1} *d*$ to be the *adjoint operator* of d regarding the inner product $\langle \omega, \eta \rangle = \int_M \omega \wedge *\eta$ for $\omega, \eta \in \Omega^k(M)$.

The form $\omega \in \Omega^k(M)$ is called a *closed form* if $d\omega = 0$, an *exact form* if there exists $\eta \in \Omega^{k-1}(M)$ such that $\omega = d\eta$, a *dual exact form* if there exists $\theta \in \Omega^{k+1}(M)$ such that $\omega = \delta\theta$, and a *harmonic form* if $\triangle\omega = 0$, where $\triangle = d\delta + \delta d$: $\Omega^k(M) \to \Omega^k(M)$ is called the *Laplacian*. We denote exact, dual exact, and harmonic k-forms on M by $\omega_E^k \in \Omega_E^k(M)$, $\omega_D^k \in \Omega_D^k(M)$, and $\omega_H^k \in \Omega_H^k(M)$, respectively.

A. Distributed-port-Hamiltonian systems with distributed energy structures

This section gives the definition of port Hamiltonian systems containing as well boundary as distributed port variables that we shall use. It is a slightly more general definition of the distributed port variables than in the original definition [2] in the sense that the distributed port variables may be defined in a different spatial domain.

Definition 2.1: Let Z be an n-dimensional C^∞ manifold with the boundary ∂Z. Let S be an l-dimensional manifold.

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Let $\mathcal{F}$ and $\mathcal{E}$ be linear spaces satisfying the *complementarity condition* $p+q=n+1$ given by

$$\begin{aligned}\mathcal{F} &= \Omega^p(Z)\times\Omega^q(Z)\times\Omega^{\overline{p}}(\partial Z)\times\Omega^u(S)\,,\\ \mathcal{E} &= \Omega^{\overline{p}}(Z)\times\Omega^{\overline{q}}(Z)\times\Omega^{\overline{q}}(\partial Z)\times\Omega^{\overline{u}}(S)\,,\end{aligned}\tag{1}$$

where $\overline{p}:=n-p$, $\overline{q}:=n-q$, $\overline{u}:=n-u$, $0\le u\le l$, $f:=(f_p,f_q,f_b,f_d)\in\mathcal{F}$ is called a *flow*, $e:=(e_p,e_q,e_b,e_d)\in\mathcal{E}$ is called a *effort*, and the pairs (f,e) are called *power variables*.

Definition 2.2: Let us define the non-degenerate bilinear form on the bond space $\mathcal{F}\times\mathcal{E}$:

$$\begin{aligned}&\langle\!\langle\ (f^1,e^1),(f^2,e^2)\ \rangle\!\rangle\\ &=\int_Z\left(e_p^1\wedge f_p^2+e_q^1\wedge f_q^2+e_p^2\wedge f_p^1+e_q^2\wedge f_q^1\right)\\ &\quad+\int_{\partial Z}\left(e_b^1\wedge f_b^2+e_b^2\wedge f_b^1\right)+\int_S\left(e_d^1\wedge f_d^2+e_d^2\wedge f_d^1\right).\end{aligned}\tag{2}$$

Definition 2.3: Let us define the linear subspace $\mathbb{D}$ of $\mathcal{F}\times\mathcal{E}$ such that

$$\begin{aligned}\mathbb{D}=\{&(f,e)\in\mathcal{F}\times\mathcal{E}\ |\\ &\begin{bmatrix}f_p\\ f_q\end{bmatrix}=\begin{bmatrix}0&(-1)^r d\\ d&0\end{bmatrix}\begin{bmatrix}e_p\\ e_q\end{bmatrix}+Gf_d\,,\ e_d=-\,G^*\begin{bmatrix}e_p\\ e_q\end{bmatrix},\\ &\begin{bmatrix}f_b\\ e_b\end{bmatrix}=\begin{bmatrix}1&0\\ 0&(-1)^p\end{bmatrix}\begin{bmatrix}e_p|_{\partial Z}\\ e_q|_{\partial Z}\end{bmatrix}\Bigg\}\,,\end{aligned}\tag{3}$$

where $r=pq+1$, $|_{\partial Z}$ is the restriction to ∂Z and $G=[G_p,G_q]^\top:\Omega^u(S)\to\Omega^p(Z)\times\Omega^q(Z)$ is a linear map with dual map $G^*=[G_p^*,G_q^*]:\Omega^{\bar{p}}(Z)\times\Omega^{\bar{q}}(Z)\to\Omega^{\bar{u}}(S)$ satisfying

$$\begin{aligned}&\int_Z\left[e_p\wedge G_p(f_d)+e_q\wedge G_q(f_d)\right]\\ &\qquad=\int_S\left[G_p^*(e_p)+G_q^*(e_q)\right]\wedge f_d\end{aligned}\tag{4}$$

for all $(e_p,e_q,f_d)\in\Omega^{\overline{p}}(Z)\times\Omega^{\overline{q}}(Z)\times\Omega^u(S)$.

Definition 2.4: A *Dirac structure* is a linear subspace $\mathbb{D}\subset\mathcal{F}\times\mathcal{E}$ such that $\mathbb{D}=\mathbb{D}^\perp$, where $\perp$ is the orthogonal complement with respect to the bilinear form $\langle\!\langle\,,\ \rangle\!\rangle$.

Theorem 2.1 ([2]): $\mathbb{D}$ is a Dirac structure with respect to the bilinear form $\langle\!\langle\,,\ \rangle\!\rangle$ defined in (2).

Definition 2.5: Let us consider a Hamiltonian functional $\mathscr{H}=\int_Z\mathcal{H}(\bar{\alpha}_q,\bar{\alpha}_p)$ defined with respect to the Hamiltonian density:

$$\mathcal{H}(\bar{\alpha}_q,\bar{\alpha}_p)\colon\Omega^p(Z)\times\Omega^q(Z)\times\mathbb{R}\to\Omega^n(Z)\,,\tag{5}$$

where $\bar{\alpha}_q\in\Omega^q(Z)$ and $\bar{\alpha}_p\in\Omega^p(Z)$. Substituting the following definition of power variables

$$f_p=-\frac{\partial\bar{\alpha}_p}{\partial t}\,,\ f_q=-\frac{\partial\bar{\alpha}_q}{\partial t}\,,\ e_p=\frac{\partial\mathscr{H}}{\partial\bar{\alpha}_p}\,,\ e_q=\frac{\partial\mathscr{H}}{\partial\bar{\alpha}_q}\,,\tag{6}$$

where $\partial\mathscr{H}/\partial\bar{\alpha}$ denotes the variational derivative of $\mathscr{H}$ with respect to $\bar{\alpha}$ [18], into (3), we get a *distributed port-Hamiltonian systems* with a *boundary port variables* (f_b,e_b) and *distributed port variables* (f_d,e_d) .

Proposition 2.2 ([2]): The energy balance of distributed port-Hamiltonian systems is given by

$$\begin{aligned}\frac{d\mathscr{H}}{dt}&=\int_Z(e_p\wedge f_p+e_q\wedge f_q)+\int_S e_d\wedge f_d\\ &=\int_{\partial Z}e_b\wedge f_b+\int_S e_d\wedge f_d\,.\end{aligned}\tag{7}$$

The first term of the right-hand side of (7) corresponds to the flow of energy per time instant through the boundary of the spatial domain. The second term in (7) corresponds to the flow of energy per time distributed in the spatial domain and may arise for instance from a current density in Maxwell's equations, a distributed weight in equations of flexible beams, and a distributed dissipation in telegraph equations, etc. In such cases the domain of the distributed port variables are either $S=Z$ or some connected subset. Then the boundary control with boundary port variables should be completed with some distributed control with distributed port variables to stabilize the systems [5].

B. Differential geometric interpretation of distributed energy structures

In this section we recall that, in the case when, for topological reasons, the space of k-harmonic form is trivial, then the distributed port variables generates a source term in the conservation law in terms of a dual exact form. First, we recall the following important result in differential geometry.

Proposition 2.3 (Hodge decomposition [10], [13], [11]): On an oriented compact Riemannian manifold M, an arbitrary k-form can be uniquely written as the sum of an exact form, a dual exact form, and a harmonic form:

$$\Omega^k(M)=\Omega_E^k(M)\oplus\Omega_D^k(M)\oplus\Omega_H^k(M)\,.\tag{8}$$

If we can assume $\Omega_H^k(M)=0$ for $k>0$ for an appropriate reason (mentioned in the next section), each differential form can be considered as a sum of an exact form and a dual exact form.

Theorem 2.4 ([5]): Consider the Stokes-Dirac structure (3) on a contractible manifold. Then the terms $(-1)^r de_q$ in f_p and de_p in f_q are exact forms and the terms $G_p(f_d)$ in f_p and $G_q(f_d)$ in f_q are dual exact forms.

This theorem says that any port variable belongs to either exact forms or dual exact forms on a contractible manifold. The distributed energy structure consisting of dual exact forms is boundary non-integrable from (7). Therefore, we can say that the distributed energy structure is inappropriate from the viewpoint of boundary controls.

C. Differential forms and de Rham cohomologies

This section explains why we introduce the topological assumption that is contractible. The decomposition in *Prop. 2.3* is significant in the sense of global topology by the following fact.

Definition 2.6: Let $Z^k(M)$ and $B^k(M)$ be the set of all closed k-forms on M and the set of all exact k-forms, respectively:

$$Z^k(M) = \mathrm{Ker}\left(d\colon \Omega^k(M) \to \Omega^{k+1}(M)\right)\,, \tag{9}$$
$$B^k(M) = \mathrm{Im}\left(d\colon \Omega^{k-1}(M) \to \Omega^{k}(M)\right)\,. \tag{10}$$

The quotient space

$$H^k_{DR}(M) = Z^k(M)/B^k(M) \tag{11}$$

is called the *k-dimensional de Rham cohomology* of M.

Theorem 2.5 (Hodge theorem [10], [13], [11]): On an oriented compact Riemannian manifold M, the de Rham cohomology group $H^k_{DR}(M)$ and the space of harmonic forms $\Omega^k_H(M)$ are isomorphic: $\Omega^k_H(M) \cong H^k_{DR}(M)$.

The de Rham cohomology $H^k_{DR}(M)$ is a characterization of topological properties of manifolds [10], [16]. From de Rham's theorem [10], [13], one obtains that $H^k_{DR}(M) \cong H^k(K;\mathbb{R})$, where $H^k(K;\mathbb{R})$ is the cohomology group of a simplical complex K with coefficients in $\mathbb{R}$. As a consequence, $H^k_{DR}(M)$ is equivalent to the topological invariant quantity that is calculated from the triangulation.

In other words, a differential form is defined at each local point of the manifolds, but it essentially relates to global topology through harmonic forms. To understand this, let us consider the simplest situation.

Corollary 2.6 (Poincaré lemma [10], [13], [12]): The de Rham cohomology of $\mathbb{R}^n$ is

$$H^k_{DR}(\mathbb{R}^n) = H^k_{DR}(\mathbb{R}^0) = \begin{cases} \mathbb{R}, & \textit{if } k = 0; \\ 0, & \textit{otherwise}\,, \end{cases} \tag{12}$$

where $\mathbb{R}^0$ is a point.

This lemma says that there exists $\eta \in \Omega^{k-1}(\mathbb{R}^n)$ such that $\omega = d\eta$ if $\omega \in \Omega^k(\mathbb{R}^n)$ for $k > 0$ is an arbitrary closed form. In other words, all closed forms $d\omega = 0$ are exact forms $\omega = d\eta$ in local coordinates on $\mathbb{R}^n$. However, the problem of whether there globally exists such a η depends on the properties of manifolds in general. The de Rham cohomologies of C^∞ manifolds with the same homotopy type are isomorphic. In the case of trivial de Rham cohomologies which are the same as $\mathbb{R}^n$, a manifold is called a *contractible*.

III. Geometrical Structure and Completion of Boundary Energy Structures

From the previous discussion, we shall assume that in *Def. 2.5* of the port Hamiltonian system, the spatial domain is a compact submanifold Z with boundary ∂Z of a contractible manifold N. The algebraic topological properties of the manifolds on which the port Hamiltonian system are defined will play an essential role in the definition of the flow and effort variables and endowing them with a bilinear product. This section presents differential geometrical aspects arising from this setting. First, we show that the topological assumption of manifolds defines a geometrical structure of differential forms. After that, we introduce a necessary condition to regard a distributed energy structure as a boundary energy structure. We call the structure satisfying this condition a completion of distributed energy structures.

A. Geometrical structure of power variables

This section shows that the complementary condition $p + q = n + 1$ determines the main structure of distributed port-Hamiltonian systems defined on a compact submanifold Z with boundary ∂Z of an n-dimensional contractible Riemannian manifold N. The decomposition of the flow variables f_q and f_p given in *Thm. 2.4* may be summarized by the following diagram involving the exterior differential operator d and its adjoint δ:

$$\begin{array}{rcl} f_{p-1} &=& d\alpha_{p-2} + \alpha_{p-1} \in \Omega^{p-1}(Z) = \Omega^{\overline{q}}(Z) \\ & & \quad {\scriptstyle d}\swarrow \\ f_p &=& d\alpha_{p-1} + \delta\beta_{p+1} \in \Omega^{p}(Z) = \Omega^{\overline{q-1}}(Z) \\ & & \quad {\scriptstyle \delta}\nearrow \\ f_{p+1} &=& \beta_{p+1} + \delta\beta_{p+2} \in \Omega^{p+1}(Z) = \Omega^{\overline{q-2}}(Z) \end{array} \tag{13}$$

where there exists $\alpha_{p-1} \in \Omega^{p-1}(Z)$ for $d\alpha_{p-1}$ in f_p, there exists $\beta_{p+1} \in \Omega^{p+1}(Z)$ for $\delta\beta_{p+1}$ in f_p, and $\Omega^{p-1}(Z) = \Omega^{\overline{q}}(Z) = \Omega^{n-q}(Z)$ because $\alpha_{p-1} = e_q$ and $p + q = n + 1$. The same structure exists for f_q:

$$\begin{array}{rcl} f_{q-1} &=& d\alpha_{q-2} + \alpha_{q-1} \in \Omega^{q-1}(Z) = \Omega^{\overline{p+1}}(Z) \\ & & \quad {\scriptstyle d}\swarrow \\ f_q &=& d\alpha_{q-1} + \delta\beta_{q+1} \in \Omega^{q}(Z) = \Omega^{\overline{p}}(Z) \\ & & \quad {\scriptstyle \delta}\nearrow \\ f_{q+1} &=& \beta_{q+1} + \delta\beta_{q+2} \in \Omega^{q+1}(Z) = \Omega^{\overline{p-1}}(Z) \end{array} \tag{14}$$

Proposition 3.1: If Z is contractible, d and δ satisfy

$$d^i\colon \Omega^i_D(Z) \to \Omega^{i+1}_E(Z)\,, \tag{15}$$
$$\delta^{i+1}\colon \Omega^{i+1}_E(Z) \to \Omega^i_D(Z) \tag{16}$$

for $0 \le i \le n-1$.

Proof: This fact is the definition itself; however, we shall check it. $\mathrm{Im}\, d^i = \mathrm{Ker}\, d^{i+1}$ and $\mathrm{Im}\,\delta^{i+2} = \mathrm{Ker}\,\delta^{i+1}$. Because the $(i+1)$-th cohomology vanishes by the contractible topological assumption. The $(i+1)$-th cohomology is isomorphic to the space of harmonic forms. The exactness: $d \circ d = 0$ or $\delta \circ \delta = 0$ holds. ∎

From equations (13) and (14), we obtain the following lemma.

Lemma 3.2: The Hodge star operators

$$*\colon \Omega^q_E(Z) \to \Omega^{p-1}_D(Z);\; d\alpha_{q-1} \mapsto \alpha_p\,, \tag{17}$$
$$*\colon \Omega^p_E(Z) \to \Omega^{q-1}_D(Z);\; d\alpha_{p-1} \mapsto \alpha_q \tag{18}$$

are isomorphisms for each subspace of f_p and f_q.

Proof: First, if ω is an exact form: $\omega = d\eta$, $*\omega$ is a dual exact form: $*\omega = \delta\theta$, because, for $\eta \in \Omega^k(Z)$, $*(\omega) = *(d\eta)$ is equal to $*\omega = (-1)^{k(n-k)} * d * (*\eta) = (-1)^{k(n-k)+n(k+1)+1}\delta(*\eta)$. The converse case also holds. That is, $*(*\omega) = *(\delta\theta)$ yields $(-1)^{(k+1)(n-k-1)}\omega = (-1)^{(n-k+1)(k-1)+n(k+1)+1} d(*\theta)$ for $\theta \in \Omega^k(Z)$. ∎

The mappings can be generalized as follows.

Proposition 3.3: The Hodge star oeprators

$$*\colon \Omega^{q+i}_E(Z) \to \Omega^{p-1-i}_D(Z)\,, \tag{19}$$
$$*\colon \Omega^{p+j}_E(Z) \to \Omega^{q-1-j}_D(Z) \tag{20}$$

are isomorphisms for $-q \le i \le n-q$ and $-p \le j \le n-p$.

The relation can be generalized as follows:

$$\begin{array}{ccccc} f_q \in & \Omega_E^q & \underset{\delta^q}{\overset{d^{q-1}}{\leftrightarrows}} & \Omega_D^{q-1} & \ni e_p \\ & {\scriptstyle *}\downarrow & & \uparrow{\scriptstyle *} & \\ e_q \in & \Omega_D^{p-1} & \underset{\delta^p}{\overset{d^{p-1}}{\rightleftarrows}} & \Omega_E^p & \ni f_p \end{array} \tag{21}$$

where this commutative diagram expresses the structure of power variables of one distributed port-Hamiltonian system.

Next, we clarify the inner product structure. The inner product is defined by the integral of a pair of differential forms $\omega, \eta \in \Omega^k(Z)$ shaping the volume form:

$$\langle f, e\rangle = \int_Z e \wedge f = \int_Z *\eta \wedge \omega\,. \tag{22}$$

From the discussion in the previous section, we arrive at the following propositions immediately.

Proposition 3.4: If we assume that $e_p := *f_p$ and $e_q := *f_q$ on an oriented compact Riemannian manifold Z, then we have the following inner product

$$\langle f, e\rangle = \int_Z e_D^{\overline{k}} \wedge de_D^{k-1} + e_E^{\overline{k}} \wedge \delta e_E^{k+1} + e_H^{\overline{k}} \wedge f_H^k\,, \tag{23}$$

where $f = f_E^k + f_D^k + f_H^k \in \Omega^k(Z)$ and $e = e_E^{\overline{k}} + e_D^{\overline{k}} + e_H^{\overline{k}} \in \Omega^{\overline{k}}(Z)$.

Proof: *Thm. 2.3* gives the following relations: $\left\langle d\alpha^{k-1}, \delta\beta^{k+1}\right\rangle = 0$, $\left\langle d\alpha^{k-1}, \gamma^k\right\rangle = 0$ and $\left\langle \delta\beta^{k+1}, \gamma^k\right\rangle = 0$ for a general representation of k-forms $\omega^k = d\alpha^{k-1} + \delta\beta^{k+1} + \gamma^k \in \Omega^k(Z)$, where $d\alpha^{k-1} \in \Omega_E^k(Z)$, $\delta\beta^{k+1} \in \Omega_D^k(Z)$ and $\gamma^k \in \Omega_H^k(Z)$. The integrand is calculated by

$$\begin{aligned} *f \wedge f &= \left(*f_E^k\right)_D \wedge f_E^k + \left(*f_D^k\right)_E \wedge f_D^k + \left(*f_H^k\right)_H \wedge f_H^k \\ &= e_D^{\overline{k}} \wedge f_E^k + e_E^{\overline{k}} \wedge f_D^k + e_H^{\overline{k}} \wedge f_H^k \\ &= e_D^{\overline{k}} \wedge de_D^{k-1} + e_E^{\overline{k}} \wedge \delta e_E^{k+1} + e_H^{\overline{k}} \wedge f_H^k\,, \end{aligned} \tag{24}$$

where we used $\triangle * f = * \triangle f$ [11], [13] in the third term. ■

Proposition 3.5: If Z is contractible, the integrand of the third term of (23): $e_H^{\overline{k}} \wedge f_H^k$ is equivalent to zero for any k.

Proof: From *Thm. 2.5* and *Cor. 2.6*, the term becomes zero for any k. ■

B. Boundary completion of distributed energy structures

We see that the inner product (23) introduced from the topological assumption itself naturally includes a boundary energy structure as well as a distributed energy structure. The relations in the previous section can be generalized and are illustrated as follows.

$$\begin{tikzcd}
\cdots & & \Omega_E^{q+1} \arrow[ll, dash] \arrow[dd, "*"] & & \Omega_E^q \arrow[ll, "0"'] \arrow[dd, "*"] & & \Omega_E^{q-1} \arrow[ll, "0"'] \arrow[dd, "*"] & & \Omega_E^{q-2} \arrow[ll, "0"'] \arrow[dd, "*"] & \cdots \arrow[l] \\
\cdots & \Omega_D^{q+1} \arrow[l, dash] \arrow[ur, dotted, leftrightarrow] & & \Omega_D^q \arrow[ll, "0"'] \arrow[ul, "d^p"'] \arrow[dl, "\tilde{*}"] \arrow[ur, dotted, leftrightarrow] & & \Omega_D^{q-1} \arrow[ll, "0"'] \arrow[ul, "d^{p-1}"'] \arrow[dl, "\tilde{*}"] \arrow[ur, dotted, leftrightarrow] & & \Omega_D^{q-2} \arrow[ll, "0"'] \arrow[ul, "d^{p-2}"'] \arrow[dl, "\tilde{*}"] \arrow[ur, dotted, leftrightarrow] & & \cdots \arrow[ll] \\
\cdots \arrow[rr] & & \Omega_D^{p-2} \arrow[rr, "0"] \arrow[dr, "d^{p-2}"] \arrow[dl, dotted, leftrightarrow] & & \Omega_D^{p-1} \arrow[rr, "0"] \arrow[dr, "d^{p-1}"] \arrow[dl, dotted, leftrightarrow] & & \Omega_D^p \arrow[rr, "0"] \arrow[dr, "d^p"] \arrow[dl, dotted, leftrightarrow] & & \Omega_D^{p+1} \arrow[r, dash] \arrow[dl, dotted, leftrightarrow] & \cdots \\
\cdots \arrow[r] & \Omega_E^{p-2} \arrow[rr, "0"'] \arrow[uu, "*"] & & \Omega_E^{p-1} \arrow[rr, "0"'] \arrow[uu, "*"] & & \Omega_E^p \arrow[rr, "0"'] \arrow[uu, "*"] & & \Omega_E^{p+1} \arrow[rr, dash] \arrow[uu, "*"] & & \cdots
\end{tikzcd}$$

where $\leftarrow\!\cdots\!\rightarrow$ means the correspondence of the pairs of orthogonal subspaces in a space of differential forms, 0 is a zero map, and $\tilde{*}$ defines the dual pair with respect to the inner product on ∂Z:

$$\tilde{*}\colon \Omega_D^{q-1+i}(Z)|_{\partial Z} \to \Omega_D^{p-1-i}(Z)|_{\partial Z} \tag{25}$$

for $1 - q \le i \le n - q$. In this diagram, the boundary energy structure corresponds to the center parallelogram consisting of $\{\Omega_E^p(Z), \Omega_D^{p-1}(Z), \Omega_E^q(Z), \Omega_D^{q-1}(Z)\}$, which is equivalent to (21), and the distributed energy structure corresponds to the adjacent spaces $\{\Omega_D^p(Z), \Omega_D^q(Z)\}$.

Here, the correspondence indicates that the distributed energy structure is one of spaces of other parallelograms arrayed symmetrically in the diagram. Here, one question arises, that is, whether we can construct a new boundary energy structure for the boundary control of the distributed energy structure. The following answers this question.

Definition 3.1: We define the set of subspaces of differential forms:

$$\mathcal{Z}_i = \{\Omega_E^{p+i}(Z), \Omega_D^{p-1+i}(Z), \Omega_E^{q-i}(Z), \Omega_D^{q-1-i}(Z)\} \tag{26}$$

$$\mathcal{S}_i = \{\Omega_D^{p+i}(Z), \Omega_D^{q-i}(Z)\}\,. \tag{27}$$

A distributed port-Hamiltonian system over $\mathcal{Z}_0$ with distributed energy structures over $\mathcal{S}_0$ is called *boundary complete* if there exist other distributed port-Hamiltonian systems $\mathcal{Z}_i$ for $i = \pm 1$ including the distributed energy structures over $\mathcal{S}_0$ as boundary energy structures.

Theorem 3.6: The following are necessary conditions to be boundary complete for a distributed port-Hamiltonian system over $\mathcal{Z}_0$ with a distributed energy structure over $\mathcal{S}_0$:

1) There exists a pair of distributed port-Hamiltonian systems over $\mathcal{Z}_i$ without a distributed energy structure over $\mathcal{S}_i$ for $i = -1, 1$.
2) $q \ge 2$, $p \ge 2$ and $n \ge 3$.

Proof: The previous diagram clearly proves the theorem; however, it is difficult to see this fact without an illustration. 1) Obviously, we should find two systems defined over $\mathcal{Z}_1$ including $G_p(f_d) \in \Omega_D^p(Z)$ and $\mathcal{Z}_{-1} \in \Omega_D^q(Z)$ including $G_q(f_d)$. 2) If the two systems exist, $q - 2 \ge 0$ and $p - 2 \ge 0$ are necessary, at least. Thus, we have $p - 2 = n - q - 1 \ge 0$. This means $n \ge q + 1$. ■

IV. Boundary Observation of Distributed Energy Structure by Using the Time Coordinate

The completion of the boundary energy structure is theoretically realizable; however, this concept might be ambiguous in a practical situation. In this section, we present a new control strategy to change the obstruction of the distributed energy structure into a useful structure in the sense of boundary controls.

A. Partition of the Hamiltonian

First, we define the fundamental form that yields distributed port-Hamiltonian systems by differentiation.

Definition 4.1: Define the map

$$\begin{aligned} W_n^k\colon &\Omega^{\overline{k}}(Z) \times \Omega^k(Z) \to \Omega^n(Z); \\ &c_{e_i} dx^{\overline{k}} \wedge c_{f_i} dx^k \mapsto \frac{\partial \mathcal{H}}{\partial a_i}\frac{da_i}{dt} dx^n \end{aligned} \tag{28}$$

for $k < n$ and $i = p$ or q, where $c_{e_i}(x), c_{f_i}(x)$ are the coefficients of $\overline{k}$-, k-forms, respectively, $\overline{k} := n - k$ and $\mathcal{H}(a_q(t), a_p(t))$ is the function of t.

Note that the definition of the Hamiltonian is changed from (5) into the function $\mathcal{H}(a_q(t), a_p(t))$.

Proposition 4.1: W_n^k is a surjection.

Proof: The number of basic 1-forms generating k-forms is the combination ${}_nC_k$. An n-form is a monomial. ■

Proposition 4.2: The inverse map $(W_n^k)^{-1} = W_k^n$ such that

$$W_k^n \colon \Omega^n(Z) \to \Omega^{\overline{k}}(Z) \times \Omega^k(Z); \quad \frac{\partial \mathcal{H}}{\partial a_i}\frac{da_i}{dt}dx^n \mapsto c_{e_i}dx^{\overline{k}} \wedge c_{f_i}dx^k \tag{29}$$

is not an injection for $k \neq n$.

Proof: From *Prop. 4.1*, the map is one-to-one if $k = n$. However, in the other cases, the inverse of the projection is not uniquely determined because of the difference in the number of coefficients of differential forms. ■

Definition 4.2: A uniquely defined W_i^n is called a *partition of the Hamiltonian* if information on the coefficients of the differential forms c_e and c_f is given.

These definitions are used in the following proposition.

Theorem 4.3: Consider an m-dimensional manifold M containing an n-dimensional submanifold $N \subset M$ such that $n = m - 1$. We define the fundamental $(m-1)$-form

$$\Theta = (-1)^{n-q}dt \wedge e_q \wedge e_p - \mathcal{H}(a_q, a_p) \wedge ds^n \in \Omega^{m-1}(M), \tag{30}$$

where $e_p \in \Omega^{n-p}(M)$, $e_q \in \Omega^{n-q}(M)$, $ds^n \in \Omega^n(M)$ which consists of basic forms $dx_1, \cdots, dx_n$ generated by spatial variables s of $(t, s) = (x_0, x_1, \cdots, x_n)$ and $\mathcal{H}(a_q(t), a_p(t))$ is the Hamiltonian density. Then $d\Theta \equiv 0$ with a partition of the Hamiltonian W_p^n and W_q^n is a sufficient condition for yielding the distributed port-Hamiltonian system.

Proof: A direct calculation leads to the following:

$$\begin{aligned} d\Theta &= (-1)^{n-q}dt \wedge de_q \wedge e_p + dt \wedge e_q \wedge de_p \\ &\quad - \frac{\partial \mathcal{H}}{\partial a_p}\frac{da_p}{dt}dt \wedge dx^n - \frac{\partial \mathcal{H}}{\partial a_q}\frac{da_q}{dt}dt \wedge dx^n . \end{aligned} \tag{31}$$

By using W_p^n and W_q^n, we get

$$\begin{aligned} d\Theta_W &= (-1)^r dt \wedge e_p \wedge de_q + dt \wedge e_q \wedge de_p \\ &\quad - dt \wedge c_{e_p}dx^{\overline{p}} \wedge c_{f_p}dx^p - dt \wedge c_{e_q}dx^{\overline{q}} \wedge c_{f_q}dx^q \\ &= (-1)^r dt \wedge e_p \wedge de_q + dt \wedge e_q \wedge de_p \\ &\quad + dt \wedge \frac{\partial \mathscr{H}}{\partial \bar{\alpha}_p} \wedge \left(-\frac{d\bar{\alpha}_p}{dt}\right) + dt \wedge \frac{\partial \mathscr{H}}{\partial \bar{\alpha}_q} \wedge \left(-\frac{d\bar{\alpha}_q}{dt}\right), \end{aligned} \tag{32}$$

where $\overline{i} := n - i$. Substituting (3) with $f_d = e_d = 0$ and (6) into the last equation, we get $d\Theta_W = 0$. ■

The fundamental form Θ is related to classical field theory [17].

Corollary 4.4: $*\Theta$ is equivalent to the Poincaré-Cartan fundamental form in the case of a hyper regular Lagrangian.

Proof: Applying the Hodge star operator to the left-hand side of (30), we get

$$\begin{aligned} *\Theta &= *\left(e_q \wedge dt \wedge e_p - \mathcal{H}(a_q, a_p)ds^n\right) \\ &= c_{e_q}c_{e_p}\, ds^1 - \mathcal{H}(a_q, a_p)dt \,, \end{aligned} \tag{33}$$

where ds^1 is the spatial 1-form and c_{e_q}, c_{e_p} are coefficients of e_q, e_p, respectively. $\mathcal{H}$ is uniquely determined by the Legendre transformation of a hyper regular Lagrangian. ■

Thus, the distributed port-Hamiltonian system possessing the partition of the Hamiltonian is a more detailed representation than the classical field equations.

Theorem 4.5: The partition of the Hamiltonian is the necessary condition to formulate a classical field equation as a distributed port-Hamiltonian system.

Proof: From *Cor. 4.4*, distributed port-Hamiltonian systems correspond to classical field equations with the partition of a hyper regular Lagrangian. ■

In some practical cases, the partition of Hamiltonian is given as a problem setting (e.g. Maxwell's equation, etc. [2]).

B. Extension of fundamental forms

In this section, we extend the fundamental form to a wider form that also yields a distributed energy structure. From *Thm. 2.4*, the distributed energy structure is not defined as an exact form, but a dual exact form. However, from the considerations in the previous section, there exists other distributed port-Hamiltonian systems including the distributed energy structure as a boundary energy structure by the boundary completion. Then such a boundary energy structure should be formulated with the fundamental form. To expand the fundamental form, we must first make some preparations.

Definition 4.3: Let us define the following projection:

$$\flat \colon \Omega^m(M) \to \Omega^{m-1}(N);\ d\Theta \mapsto (\partial t \rfloor d\Theta)\,|_N \,, \tag{34}$$

where $N \subset M$ and $\partial t \rfloor := (\partial/\partial t)\rfloor$ is the contraction with respect to dt.

Definition 4.4: Let us define the following injection:

$$\sharp \colon \Omega^m(M) \to \Omega^{m+1}(L);\ d\Theta \mapsto d\tilde{t} \wedge d\Theta \,, \tag{35}$$

which is induced by the inclusion $\iota \colon M \hookrightarrow L$ such that $(t, x) \mapsto (\tilde{t}, t, x)$ in local coordinates, where L is the extended manifold with a new time coordinate $\tilde{t}$.

Proposition 4.6: Consider a distributed port-Hamiltonian system determined by $\Phi := \flat(d\Theta)$. The power balance of the distributed port-Hamiltonian system is expressed by $\Phi|_{t_1} - \Phi|_{t_0} = 0$.

Proof: Φ is integrable with respect to t. Further, Φ is equivalent to $d\mathscr{H}/dt = \int_Z e \wedge f$ if the Hamiltonian density $\mathcal{H}$ is time invariant, where $Z \subset N$. ■

Thus, the system representation $\Phi \equiv 0$ in $d\Theta = dt \wedge \Phi$ can be restricted to $\Omega^{m-1}(M)$ by removing dt. This setting can be illustrated as follows:

$$\begin{tikzcd} \Theta \arrow[d, "d"'] & \Phi & \in \Omega^{m-1}(N) \\ d\Theta \arrow[ur, "\flat"'] & & \in \Omega^m(M) \end{tikzcd}$$

If we attempt to add a distributed energy structure to the fundamental form, from (13), (14) and *Prop. 3.4*, we might select a candidate

$$\begin{aligned}\Delta' = dt \wedge (e_p)_E \wedge \delta\beta_{p+1} \\ + dt \wedge (e_q)_E \wedge \delta\beta_{q+1} \in \Omega^m(M)\end{aligned} \tag{36}$$

such that $d\Theta + \Delta' \equiv 0$ determines the distributed port-Hamiltonian system with distributed energy structures. Note that we cannot differentiate Δ' any longer, because Δ' is already on the space with the maximum degree: $\Omega^m(M)$. However, distributed port-Hamiltonian systems exist to be the completion of the distributed energy structure in fact. Next, we shall check that the candidate is true.

The boundary completion can be considered to be a problem of finding other fundamental forms: Θ_1 and Θ_2 besides Θ which generates the original system. More precisely, the following holds.

Theorem 4.7: The following fundamental forms yield distributed port-Hamiltonian systems which are required for the boundary completion of the distributed energy structures:

$$\Theta_1 = dt \wedge e_D^p \wedge e_D^{q-2}\,, \quad \Theta_2 = dt \wedge e_D^q \wedge e_D^{p-2}\,, \tag{37}$$

where $\Theta_1, \Theta_2 \in \Omega^{m-1}(M)$.

Proof: A direct calculation leads to the following:

$$\begin{aligned}
& d(dt \wedge e_D^p \wedge e_D^{q-2}) \\
&= dt \wedge \left\{ de_D^p \wedge e_D^{q-2} + (-1)^p e_D^p \wedge de_D^{q-2} \right\} \\
&\overset{\flat}{=} d{*}e_E^{q-1} \wedge {*}e_E^{p+1} + (-1)^p {*}e_E^{q-1} \wedge d{*}e_E^{p+1} \\
&= (-1)^{(p+1)(n-p-1)} {*}d{*}e_E^{q-1} \wedge e_E^{p+1} \\
&\quad + (-1)^p (-1)^{(q-1)(n-q+1)} e_E^{q-1} \wedge {*}d{*}e_E^{p+1} \\
&= (-1)^{n-p} \delta e_E^{q-1} \wedge e_E^{p+1} - e_E^{q-1} \wedge \delta e_E^{p+1} \\
&= -\{(-1)^q \delta e_E^{q-1} \wedge e_E^{p+1} + e_E^{q-1} \wedge \delta e_E^{p+1}\}
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
& d(dt \wedge e_D^q \wedge e_D^{p-2}) \\
&= dt \wedge \left\{ de_D^q \wedge e_D^{p-2} + (-1)^q e_D^q \wedge de_D^{p-2} \right\} \\
&\overset{\flat}{=} d{*}e_E^{p-1} \wedge {*}e_E^{q+1} + (-1)^q {*}e_E^{p-1} \wedge d{*}e_E^{q+1} \\
&= (-1)^{(q+1)(n-q-1)} {*}d{*}e_E^{p-1} \wedge e_E^{q+1} \\
&\quad + (-1)^q (-1)^{(p-1)(n-p+1)} e_E^{p-1} \wedge {*}d{*}e_E^{q+1} \\
&= (-1)^{n-q} \delta e_E^{p-1} \wedge e_E^{q+1} - e_E^{p-1} \wedge \delta e_E^{q+1} \\
&= -\{(-1)^p \delta e_E^{p-1} \wedge e_E^{q+1} + e_E^{p-1} \wedge \delta e_E^{q+1}\}\,,
\end{aligned} \tag{39}$$

where $\overset{\flat}{=}$ means the restriction to the n-dimensional submanifold $N \subset M$ and $*$ is the Hodge star operator over N. We can see that the last equations of (38) and (39) include the distributed energy structure (36). ■

C. Utilization of the time coordinates

This section shows another fundamental form that determines the distributed energy structure. The new fundamental form is not an n-form but an $(n+1)$-form on an extended manifold.

In (30), the restrictions of forms dt in the first term and ds^n in the second term are introduced to define the independent equations with respect to dt in $d\Theta$. Now, we consider the time coordinate as a non-special variable, that is, we regard t as one of spatial variables. In this case, we can basically carry out the same procedure as in *Thm. 4.3* to define a distributed port-Hamiltonian system with space-time coordinates. However, we cannot use the time coordinate as an auxiliary variable like dt in (30). By introducing an extra time coordinate $\tilde{t}$, we can define the fundamental form Ξ in the same way as Θ. That is, we can define the injection

$$\begin{aligned}
&\sharp\colon \Omega^m(M) \to \Omega^{m+1}(L); \\
&\quad \Xi = e_E^{q-1} \wedge e_E^{p+1} + e_E^{p-1} \wedge e_E^{q+1} \\
&\quad \mapsto \Psi = d\tilde{t} \wedge e_E^{q-1} \wedge e_E^{p+1} + d\tilde{t} \wedge e_E^{p-1} \wedge e_E^{q+1}\,.
\end{aligned} \tag{40}$$

Using the form Ψ, the following relations hold.

$$\begin{aligned}
&\delta\left(d\tilde{t} \wedge e_E^{q-1} \wedge e_E^{p+1}\right) \\
&\quad = -d\tilde{t} \wedge \left\{(-1)^q \delta e_E^{q-1} \wedge e_E^{p+1} + e_E^{q-1} \wedge \delta e_E^{p+1}\right\}\,,
\end{aligned} \tag{41}$$

$$\begin{aligned}
&\delta\left(d\tilde{t} \wedge e_E^{p-1} \wedge e_E^{q+1}\right) \\
&\quad = -d\tilde{t} \wedge \left\{(-1)^p \delta e_E^{p-1} \wedge e_E^{q+1} + e_E^{p-1} \wedge \delta e_E^{q+1}\right\}\,,
\end{aligned} \tag{42}$$

where δ consists of $*$ on M. Identifying t with $\tilde{t}$, (41) and (42) correspond to $d\Theta_1$ and $d\Theta_2$, respectively. These relations are summarized as follows:

$$\begin{array}{ccccccc}
\Theta_1 & & \Theta_2 & & & & \in \Omega^{m-1}(N) \\
{\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow & & & & \\
d\Theta_1 & + & d\Theta_2 & = & \delta\Psi & \Xi & \in \Omega^m(M) \\
 & & & & {\scriptstyle \delta}\uparrow & \swarrow{\scriptstyle \sharp} & \\
 & & & & \Psi & & \in \Omega^{m+1}(L)
\end{array} \tag{43}$$

In the above extension, it seems that $d\tilde{t}$ is useless because it doesn't have any relation to the system representation. However, the new coordinate $\tilde{t}$ plays an important role in a distributed port-Hamiltonian system with a space-time energy balance. To explain this, we introduce a more general expression defining the extra time coordinate $\tilde{t}$ as follows.

Proposition 4.8 (Cylinder construction [14]): Let $I = [0, 1]$ be the unit interval on the $\tilde{t}$-axis and $(\tilde{t}, x) \in I \times M$ be a product space of I and an m-dimensional manifold. We consider the following maps which identify $(\tilde{t}, x)|_{\tilde{t}=0}$ with $(\tilde{t}, x)|_{\tilde{t}=1}$:

$$j_1\colon M \to I \times M;\; j_i(x) = (1, x)\,, \tag{44}$$

$$j_0\colon M \to I \times M;\; j_0(x) = (0, x)\,. \tag{45}$$

These maps induce the pull-back

$$j_i^*\colon \Omega^k(I \times M) \to \Omega^k(M) \tag{46}$$

for $i = 0, 1$ (e.g. $j_1^*\omega$ is calculated by substituting $\tilde{t} = 1$ and $d\tilde{t} = 0$ to $\omega \in \Omega^k(I \times M)$.). Thus, we can define

$$K\colon \Omega^{k+1}(I \times M) \to \Omega^k(M) \tag{47}$$

such that $K(d\omega) + d(K\omega) = j_1^*\omega - j_0^*\omega$. K is defined on monomials by the formulas

$$K\left(c(\tilde{t},x)\,dx^{k+1}\right) = 0\,, \tag{48}$$

$$K\left(c(\tilde{t},x)\,d\tilde{t}dx^k\right) = \left(\int_0^1 c(\tilde{t},x)d\tilde{t}\right)\,dx^k\,, \tag{49}$$

where $c(\tilde{t},x)$ is the coefficient.

This proposition is used in the *converse of the Poincaré Lemma*, which says that there exists an exact k-form for a closed $(k+1)$-form on a contractible manifold.

By using the above setting, let us consider the system in space-time coordinates. The term $dt \wedge e_q \wedge e_p$ of (30) changes to $e_q \wedge e_p$. Thus, either effort e_q or e_p increases the number of forms by 1 for the form dt.

Theorem 4.9: Consider the distributed port-Hamiltonian system in space-time coordinates by setting $\tilde{p} = p + 1$ and $\tilde{n} = n + 1$. The fundamental form is

$$\Theta' = d\tilde{t} \wedge \tilde{e}_q \wedge \tilde{e}_p - \mathcal{H}(a_q, a_p) \wedge dx^{n+1} \in \Omega^m(L)\,, \tag{50}$$

where $L \simeq I \times M$ is an $(m+1)$-dimensional manifold, $\tilde{e}_p \in \Omega^{n-p+1}(L)$ and $\tilde{e}_q = e_q \in \Omega^{n-q}(L)$.

Proof: We obtain $\tilde{f}_p \in \Omega^{p-1}(L)$, $\tilde{f}_q = f_q \in \Omega^q(L)$ and $\tilde{r} = pq + p - 1$ from the setting. We only have to perform the same procedure as in *Thm. 4.3*, by identifying $da_i/d\tilde{t}$ with da_i/dt for $i = p, q$. ∎

Finally, we can formulate the distributed port-Hamiltonian system in space-time coordinates with Θ' in the same way as $\delta\Psi$. The concept of the given system is illustrated in Fig. 1 [left].

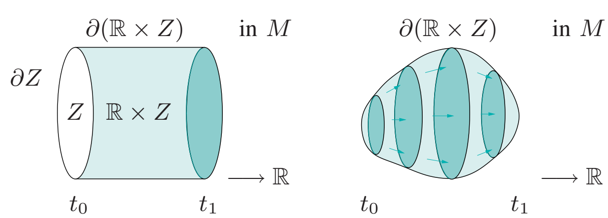

Fig. 1: Energy structures in space-time coordinates

The power balance of the (ordinal) distributed port-Hamiltonian systems on Z from t_0 to t_1 is defined on ∂Z. The power balance of the distributed port-Hamiltonian systems with distributed energy systems on Z from t_0 to t_1 should be defined on Z itself. For the new distributed port-Hamiltonian systems with the time coordinate on $\mathbb{R} \times Z$ from t_0 to t_1, the power balance is defined on $\partial(\mathbb{R} \times Z)$. That is, to calculate the power balance from t_0 to t_1, we only have to observe the distributed energy structure at t_0 and t_1 while the boundary energy structure is being observed on the boundary $\mathbb{R}_{]t_0,t_1[} \times \partial Z$. This result corresponds to a practical situation; for example, if we consider a distributed dissipative element to be time invariant and we know the dissipation structure, we can calculate the power balance on the boundary without observation of the distributed energy structure. Furthermore, this procedure can be extended to observation on a closed boundary $\partial(\mathbb{R} \times Z)$, which is equal to $\mathbb{R}_{[t_0,t_1]} \times \partial Z$ (see Fig. 1 [right]) because both sides in Fig. 1 are homotopic to each other.

V. Conclusions

This paper presented a new concept of boundary controls in space-time coordinates by examining topological geometrical aspects of a distributed port-Hamiltonian system with a distributed energy structure.

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