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# On the Averaging Method for Affine in Control Systems

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*Abstract*— **In this paper we discuss implications of control influence terms on the application of the classical averaging theory to affine in control systems. It is shown that the classical averaging theorem can be extended to the control systems, provided that the controllability and stability properties of the system are preserved in the course of the averaging. Sufficient conditions are given for the equivalence between the original and averaged system in the sense that the stabilizing control law designed for the averaged system also stabilizes the original system and vise versa.**

## I. INTRODUCTION

Any casual observer would note that there has been a proliferation of technical papers that discuss methods for analyzing the stability of *collections* of dynamical systems, in contrast to a single system, in the controls community. Indeed, such diverse topics as robust control theory, hybrid system theory and control formulations in terms of differential inclusions seek at their core to guarantee stability of families of trajectories. Averaging theory may be one of the oldest methods for representing a family of very complex systems in terms of one that is much simpler. Given the popularity of averaging methods over time, and the emergence of interest in stabilization of families of solutions or trajectories, it is surprising that significant open questions yet remain in the application of these techniques to control systems. Evidently, the difficulty is not lack of interest by any means but rather the difficulty in the general problem.

The authors' own interest in this problem has arisen in the study of control formulations of piezoelectric energy harvesting transducers. The foundations of these equations is well-understood, at least for operation in the regimes wherein the equations of linear piezoelectricity apply. It is shown in [5] that the equations governing a linear piezoelectric transducer that is connected to a switched shunt circuit to actively harvest energy can be written in the form

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A_k\boldsymbol{x}(t) + B_k\boldsymbol{u}(t) \qquad \text{for all } t \in [t_{k-1}^+, t_k^-] \\ k(t_k^+) &= S(x(t_k^-), k(t_k^-), t_k^-) \\ \boldsymbol{x}(0) &= x_0 \end{aligned} \tag{1}$$

In these equations the state $\boldsymbol{x}(t) \in \mathbb{R}^n$ includes mechanical and electrical unknowns, the control $\boldsymbol{u}(t) \in \mathbb{R}^m$ may include voltages, currents or duty cycles associated with them, the switching rule is given by $S = S(x(t_k^-), k(t_k^-), t_k^-)$, and the switching times are $\{\ldots t_{k-1}, t_k, t_{k+1}, \ldots\}$. The state matrices $\{A_k\}_{k=1\ldots p}$ and control influence matrices $\{B_k\}_{k=1\ldots p}$ are defined by each topology the electromechanical system takes as the switching state $k = 1\ldots p$ evolves. The precise and rigorous formulation of these equations can be cast in a number of interesting ways. There are several hybrid system formulations within which this system may be studied including the very early work of [15], or the more recent frameworks employed by [13]. In contrast to the hybrid formulations that keep the dependence on the switching process quite explicit, it is also possible to eliminate the explicit dependence on the switching sequence by employing a relaxation of the governing equations cast in terms of Young measures. A representative foundational work in this direction can be found in [14], while more recent relaxation techniques are presented in [11] and [4]. Perhaps most relevant for this paper is the work by numerous authors who have studied techniques for modeling pulse width modulated (PWM) converters in power electronics. The volume [7] gives a good overview of this approach and the reader is referred to the numerous sources therein for a detailed discussion. This technique begins by assuming the the switching times occur on a time scale many orders of magnitude faster than the dynamics of interest. A change of variables is introduced so that the governing equations can be recast in the form

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \varepsilon\left[\boldsymbol{f}(t, \boldsymbol{x}(t)) + G(t, \boldsymbol{x}(t))\boldsymbol{u}(t)\right] \\ \boldsymbol{x}(0) &= \boldsymbol{x}_0\,, \end{aligned} \tag{2}$$

where $\varepsilon$ is a small parameter that is inversely proportional to the switching frequency. At this point a typical averaging analysis involves classical theorems, such as Theorem 1 in [3] (p. 431), which do not include a control term. As we will discuss shortly this control influence term can be a source of difficulty. In effect, the methodology carries out a straightforward averaging of the equations (1) to obtain

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \varepsilon\left[A(D)\boldsymbol{x}(t) + B(D)\boldsymbol{u}(t)\right] \\ \boldsymbol{x}(0) &= \boldsymbol{x}\,. \end{aligned} \tag{3}$$

In these equations the explicit switching time sequence $\ldots t_{k-1}, t_k, t_{k+1}, \ldots$ is eliminated via averaging to obtain a duty cycle $D$ that represents, roughly speaking, the occupancy time that the switching sequence resides in a particular state. The qualitative differences between the explicit representation obtained in (1) and the averaged represented in (3) is clear. The average representation yields a single equation whose stability properties should, in principle, be

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more amenable to analysis. Of course the averaged representation has eliminated some of the information associated with the hybrid system formulation. A critical issue is then to establish when it is possible to carry out control synthesis for the averaged equations and utilize that feedback control for the original (hybrid, in this case) system.

In this paper we address the following question: is it possible to design a stabilizing control law for the averaged system in the form

$$\dot{\boldsymbol{y}}_\varepsilon(t) = \varepsilon\left[\boldsymbol{f}_a(\boldsymbol{y}_\varepsilon(t)) + G_a(\boldsymbol{y}_\varepsilon(t))\boldsymbol{u}(t)\right], \quad \boldsymbol{y}_\varepsilon(0) = \boldsymbol{x}_0\,, \quad (4)$$

where $\boldsymbol{f}_a(\boldsymbol{x})$ and $G_a(\boldsymbol{x})$ are the averages of $\boldsymbol{f}(t,\boldsymbol{x})$ and $G(t,\boldsymbol{x})$ respectively, that will also stabilize the original system given in (2)? Here we retain the notion $\boldsymbol{y}_\varepsilon$ to emphasize the dependence of the averaged system on $\varepsilon$. To simplify notations, the subscript $\varepsilon$ is suppressed in the sequel.

One may think that the general averaging theory introduced in [3] and the subsequent stability analysis (see for example [9], [13], [10] and the references therein) can provide the answer to this question. For, as long as the stabilizing controller $\boldsymbol{u} = \boldsymbol{\varphi}(\boldsymbol{y})$ for the averaged system is constructed, the original and averaged closed loop systems take the form

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \varepsilon\bar{\boldsymbol{f}}(t,\boldsymbol{x}(t)) \\ \dot{\boldsymbol{y}}(t) &= \varepsilon\bar{\boldsymbol{f}}_a(\boldsymbol{y}(t))\,, \end{aligned} \quad (5)$$

where $\bar{\boldsymbol{f}}(t,\boldsymbol{x}) = \boldsymbol{f}(t,\boldsymbol{x}) + G(t,\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x})$ and $\bar{\boldsymbol{f}}_a(\boldsymbol{y}) = \boldsymbol{f}_a(\boldsymbol{y}) + G_a(\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})$, and obviously $\bar{\boldsymbol{f}}_a(\boldsymbol{x})$ is the average of the function $\bar{\boldsymbol{f}}(t,\boldsymbol{x})$. Therefore, one may attempt to apply the existing stability results to conclude the stability of the original system. For instance, if $\bar{\boldsymbol{f}}(t,0) = 0$ uniformly in $t$, then the hypothesis of Theorem 10.5 from [9] (p.417) are satisfied, which states that there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, from the exponential stability of the origin $\boldsymbol{y} = 0$ of the averaged system follows the exponential stability of the origin $\boldsymbol{x} = 0$ of the original system. However, there are two obstacles in this approach. The first obstacle is the possible loss of controllability because of the averaging. That is the original system may well be stabilizible, but the averaged system may be uncontrollable and unstable.

The second obstacle is the dependence of the stability properties of the system on the parameter $\varepsilon^*$, which is in general unknown. Only for linear systems a conservative estimate for $\varepsilon^*$ has been derived [2]. Therefore for a given $\varepsilon$ the stability of the original system cannot be predicted from the stability of the averaged system, and it is imperative to find a controller from the perspective of the stabilization of the averaged system that stabilizes the original system independent of the $\varepsilon$.

The rest of the paper is organized as follows. In Section II we give a motivating example, which shows that the application of straightforward averaging to control systems can be misleading when the parameter $\varepsilon$ is fixed. We show that the controller that asymptotically stabilizes the original system cannot stabilize the averaged system and vise versa. Moreover, we also show that even in the case when the averaged system is exponentially stable, the original system can have a finite escape time, implying that the existing results on the stability of the original system when coming from the averaged system are strongly dependent on the parameter $\varepsilon$. In Section III we show that the conventional averaging method can be applied to the control systems, if the trajectories of both original and averaged system remain bounded. In Section IV we give sufficient conditions for simultaneous stabilization of both the original and the averaged system by a controller designed for the averaged system. The concluding remarks are given in Section V.

Throughout the manuscript bold symbols are used for vectors, capital letters for matrices and small letters for scalars.

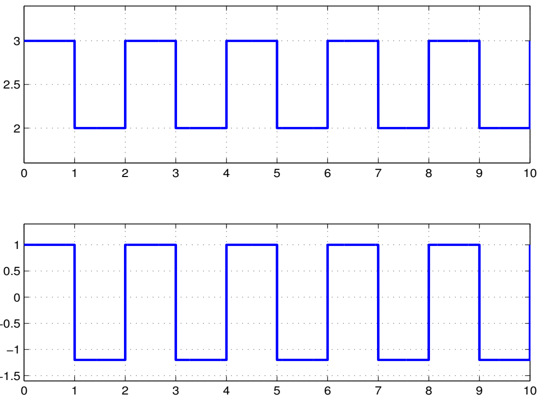

Fig. 1. Functions f(t) and g(t)

## II. Motivating Example

Consider the following system

$$\dot{x}(t) = \varepsilon[f(t)x^3(t) + g(t)u], \quad x(0) = x_0\,, \quad (6)$$

where the periodic functions $f(t)$ and $g(t)$ are square waves presented in Figure 1, where $f(t)$ changes between 2 and 3, and $g(t)$ changes between $-1.2$ and 1. This system is locally Lipschitz in $x$, globally Lipschitz in $u$, and measurable and uniformly bounded in $t$. That is the existence and uniqueness conditions are satisfied for Caratheodory solution of equation (6). Moreover, the system is globally stabilizible. For example, the application of the control input $u(t,x) = \phi(t)(x^3 + kx)$, where $k > 0$ is a constant gain and the function $\phi(t)$, presented in Figure 2, is designed such that $g(t)\phi(t) = -f(t)$, results in the closed-loop system

$$\dot{x}(t) = -k\varepsilon f(t)x(t)\,, \quad (7)$$

which is globally exponentially stable with a rate of convergence that can be adjusted by selection of the parameter $k$. The averaged system has the form

$$\dot{y}(t) = \varepsilon\left[\frac{5}{2}y^3(t) - \frac{1}{10}u\right], \quad y(0) = x_0\,, \quad (8)$$

which is obviously controllable. However, the application

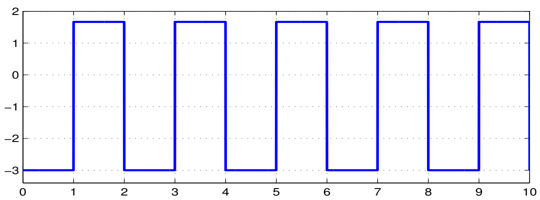


Fig. 2. Functions $\phi(t)$

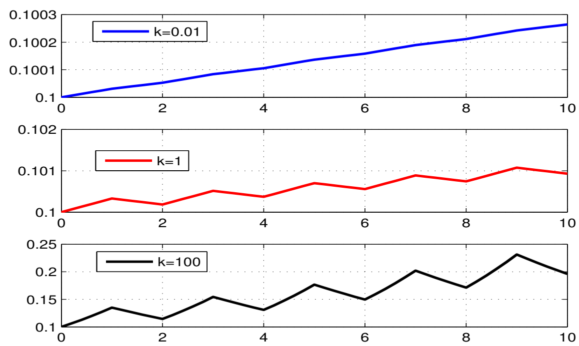


Fig. 3. Instability of the averaged system in Example 2

of the same control input $u(t, y) = k\phi(t)(y^3 + ky)$ results in the closed-loop system

$$\dot{y}(t) = \varepsilon\left[\frac{5}{2}y^3(t) - \frac{1}{10}\phi(t)[y^3(t) + ky(t)]\right], \tag{9}$$

which has a finite escape time on the interval $(0, 1)$, if the initial condition satisfies the inequality

$$x_0 > \sqrt{\frac{3k}{28(\exp 0.6k\varepsilon - 1)}}.$$

It can be shown that the closed loop system is unstable for a larger set of initial conditions. Figure 3 displays the simulation result with $k = 0.01,\ k = 1,\ k = 100,\ \varepsilon = 0.01$ and $y(0) = 0.1$, where $y(t)$ obviously grows unbounded.

On the other hand the averaged system in (8) can be globally exponentially stabilized by the feedback control $u(y) = 25y^3 + 10ky$. The resulting closed-loop system is

$$\dot{y}(t) = -\varepsilon k y(t), \tag{10}$$

the solution to which exponentially converges to zero with the rate of $\varepsilon k$. However, this feedback controller can not stabilize the original system in (6). Indeed, the resulting closed-loop system has the form

$$\dot{x}(t) = \varepsilon[f(t)x^3(t) + 25g(t)x^3(t) + 10kg(t)x(t)]. \tag{11}$$

This system has a finite escape time on the interval $(0, 1)$, if the initial condition satisfies the inequality

$$x_0 > \sqrt{\frac{10k}{28(\exp 2k\varepsilon - 1)}}.$$

Thus, we have constructed an example, in which the controller that stabilizes the original system cannot stabilize the averaged system, and the controller that stabilizes the averaged system cannot stabilize the original one.

## III. Averaging problem for control systems

In this section we present formulation of the averaging problem for general affine in control systems and give the asymptotic analysis.

### A. *Formulation of the averaging problem*

Let the dynamical system be described by the differential equation (2), where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state of the system, $\boldsymbol{u}(t) \in \mathbb{R}^m$ is the control input, and $\varepsilon > 0$ is a small parameter. It is assumed that the functions $\boldsymbol{f}(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $G(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfy the common conditions that ensure existence and uniqueness of solutions to the initial value problem in (2): in the domain $\mathbb{R}^+ \times \mathcal{D}$, where $\mathcal{D} \subset \mathbb{R}^n$ is an open set, the functions $\boldsymbol{f}$ and $G$ are uniformly bounded

$$\|\boldsymbol{f}(t, \boldsymbol{x})\| \le M_f, \quad \|G(t, \boldsymbol{x})\| \le M_G, \tag{12}$$

measurable in $t$ for any $\boldsymbol{x} \in \mathcal{D}$, and satisfy the Lipschitz condition in $x$

$$\begin{aligned} &\|\boldsymbol{f}(t, \boldsymbol{x}) - \boldsymbol{f}(t, \boldsymbol{y})\| \le \lambda_f \|\boldsymbol{x} - \boldsymbol{y}\| \\ &\|G(t, \boldsymbol{x}) - G(t, \boldsymbol{y})\| \le \lambda_G \|\boldsymbol{x} - \boldsymbol{y}\|. \end{aligned} \tag{13}$$

Here $M_f, M_G, \lambda_f, \lambda_G$ are positive constants. Additionally, we assume that the functions $\boldsymbol{f}(t, \boldsymbol{x})$ and $G(t, \boldsymbol{x})$ have well defined averages in the sense of the following definition [9] (p. 414):

*Definition 1:* A continuous, bounded function $g : \mathbb{R}^+ \times \mathcal{D} \to \mathbb{R}^n$ is said to have an average $g_a(x)$ if the limit

$$g_a(\boldsymbol{x}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T g(t, \boldsymbol{x})dt \tag{14}$$

exists and

$$\left\|\frac{1}{T}\int_0^T g(t, \boldsymbol{x})dt - g_a(\boldsymbol{x})\right\| \le k\sigma(T) \tag{15}$$

for any $\boldsymbol{x} \in \mathcal{D}_0$, where $\mathcal{D}_0 \subset \mathcal{D}$ is a compact set, $k$ is a positive constant (possibly dependent on $\mathcal{D}_0$) and $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly decreasing, continuous, bounded function such that $\sigma(T) \to 0$ as $T \to \infty$. The function $\sigma$ is called the convergence function.

It is also assumed that the control input (designed from any control perspective) is admissible, that is, the function $t \to \boldsymbol{u}(t, \boldsymbol{x})$ is measurable, the mapping $\boldsymbol{x} \to \boldsymbol{u}(t, \boldsymbol{x})$ is continuous, and $\boldsymbol{u}(t, \boldsymbol{x})$ uniformly bounded in the domain $\mathbb{R}^+ \times \mathcal{D}$.

*Remark 1:* The condition in (14) is satisfied for time-periodic systems. □

*Remark 2:* From the relationships in (12), (13) and (14) it follows that the function $\boldsymbol{f}_a(\boldsymbol{x})$ and the matrix $G_a(\boldsymbol{x})$ are bounded by the constants $M_f$ and $M_G$ respectively, and satisfy the Lipschitz conditions

$$\begin{aligned} &\|\boldsymbol{f}_a(\boldsymbol{x}) - \boldsymbol{f}_a(\boldsymbol{y})\| \le \lambda_f \|\boldsymbol{x} - \boldsymbol{y}\| \\ &\|G_a(\boldsymbol{x}) - G_a(\boldsymbol{y})\| \le \lambda_G \|\boldsymbol{x} - \boldsymbol{y}\| \end{aligned} \tag{16}$$

in the domain $\mathcal{D}$. □

The objective is to show that the averaging method developed in [3] can be applied to the control system in (2).

### B. Asymptotic Analysis

As noted in the introduction our goal is to establish a correspondence between the solution of the original equation in (2) and of the averaged equation in (4). This correspondence requires that we introduce the notion of $\mathcal{D}_\rho$ subset and $\varepsilon$-approximation.

*Definition 2:* The set $\mathcal{D}_\rho$ is a $\rho$-subset of $\mathcal{D}$, if $\mathcal{D}_\rho \subseteq \mathcal{D}$ and $\inf \|\boldsymbol{x} - \boldsymbol{y}\| \geq \rho$ for all $\boldsymbol{x} \in \partial\mathcal{D},\ \boldsymbol{y} \in \mathcal{D}_\rho$, where $\partial\mathcal{D}$ denotes the boundary of the set $\mathcal{D}$. □

*Definition 3:* The function $\boldsymbol{y}(t) \in \mathcal{D}_\rho$ is called a $\gamma$-approximation of the $\boldsymbol{x}(t) \in \mathcal{D}$ if for any $\gamma > 0$ there exist constants $L(\rho,\gamma) > 0$ and $\varepsilon_0(\rho,\gamma) > 0$ such that for any $0 < \varepsilon < \varepsilon_0(\rho,\gamma)$ the inequality

$$\|\boldsymbol{x}(t) - \boldsymbol{y}(t)\| < \gamma \tag{17}$$

holds on the interval $0 < t < \frac{L(\rho,\gamma)}{\varepsilon}$. □

The asymptotic properties of the averaged system are given by the following theorem that extends the similar theorem from [3] for the systems with no control contribution.

*Theorem 1:* Let $\boldsymbol{y}(t)$ be the solution of the averaged system in (4) and $\boldsymbol{x}(t)$ be the solution of the original system in (2). If there exists a constant $\rho > 0$ such that $\boldsymbol{y}(t)$ lives in $\rho$-subset $\mathcal{D}_\rho$ of the domain $\mathcal{D}$ corresponding to system in (2) for all $t > 0$, then $\boldsymbol{y}(t)$ is the $\gamma$-approximation of the solution $\boldsymbol{x}(t)$.

*Proof:* Consider the following auxiliary function

$$h(\boldsymbol{x}) = \begin{cases} c\left(1 - \frac{\|\boldsymbol{x}\|^2}{a^2}\right), & \|\boldsymbol{x}\| \leq a \\ 0, \quad \|\boldsymbol{x}\| > a\,, \end{cases} \tag{18}$$

with the property

$$\int_{\mathbb{R}^n} h(\boldsymbol{x})d\boldsymbol{x} = 1\,, \tag{19}$$

where $a$ is a positive constant. Since $h(\boldsymbol{x})$ is continuously differentiable with support contained in the ball $\|\boldsymbol{x}\| \leq a$, it follows that the integral

$$I = \int_{\mathbb{R}^n} \left\| \frac{\partial h(\boldsymbol{x})}{\partial \boldsymbol{x}} \right\| d\boldsymbol{x} \tag{20}$$

is bounded. Next we introduce a new variable $\boldsymbol{z} = \boldsymbol{y} + \varepsilon \boldsymbol{e}(t,\boldsymbol{y})$, where the $\boldsymbol{e}(t,\boldsymbol{y})$ is defined as follows

$$\begin{aligned} \boldsymbol{e}(t,\boldsymbol{y}) \;&=\; \int_{\mathcal{D}} h(\boldsymbol{y}-\boldsymbol{s})\Big\{ \int_0^t \big[\boldsymbol{f}(\tau,\boldsymbol{s}) - \boldsymbol{f}_a(\boldsymbol{s}) \\ &+\; [G(\tau,\boldsymbol{s}) - G_a(\boldsymbol{s})]\boldsymbol{u}(\tau,\boldsymbol{s})\big] d\tau \Big\} d\boldsymbol{s} \end{aligned} \tag{21}$$

It should be noted that since the functions $\boldsymbol{f}(\tau,\boldsymbol{s})$, $G(\tau,\boldsymbol{s})$ and $\boldsymbol{u}(\tau,\boldsymbol{s})$ are bounded and measurable in $\tau$, it follows that the function

$$\begin{aligned} \boldsymbol{v}(t,\boldsymbol{s}) \;&=\; \int_0^t \big[\boldsymbol{f}(\tau,\boldsymbol{s}) - \boldsymbol{f}_a(\boldsymbol{s}) \\ &+\; [G(\tau,\boldsymbol{s}) - G_a(\boldsymbol{s})]\boldsymbol{u}(\tau,\boldsymbol{s})\big] d\tau \end{aligned} \tag{22}$$

is well defined and absolutely continuous in $t$. Therefore $\boldsymbol{e}(t,\boldsymbol{y})$ is well defined and differentiable in time. To obtain the dynamic equation for $\boldsymbol{z}(t)$, we differentiate it, taking into account the equations in (4) and (21):

$$\begin{aligned} \dot{\boldsymbol{z}}(t) \;&=\; \dot{\boldsymbol{y}}(t) + \varepsilon\frac{\partial \boldsymbol{e}(t,\boldsymbol{y})}{\partial t} + \varepsilon\frac{\partial \boldsymbol{e}(t,\boldsymbol{y})}{\partial \boldsymbol{y}}\dot{\boldsymbol{y}}(t) \qquad (23) \\ &=\; \varepsilon \boldsymbol{f}_a(\boldsymbol{y}(t)) + \varepsilon G_a(\boldsymbol{y}(t))\boldsymbol{u}(t,\boldsymbol{y}(t)) + \varepsilon\frac{\partial \boldsymbol{e}(t,\boldsymbol{y})}{\partial t} \\ &+\; \varepsilon\frac{\partial \boldsymbol{e}(t,\boldsymbol{y})}{\partial \boldsymbol{y}}\big[\varepsilon \boldsymbol{f}_a(\boldsymbol{y}(t)) + \varepsilon G_a(\boldsymbol{y}(t))\boldsymbol{u}(t,\boldsymbol{y}(t))\big] \\ &=\; \varepsilon \boldsymbol{f}(t,\boldsymbol{z}(t)) + \varepsilon G(t,\boldsymbol{z}(t))\boldsymbol{u}(t,\boldsymbol{z}(t)) + \boldsymbol{F}(t,\boldsymbol{y}(t))\,, \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{F}(t,\boldsymbol{y}) \;&=\; \varepsilon\big[G(t,\boldsymbol{y})\boldsymbol{u}(t,\boldsymbol{y}) - \boldsymbol{f}(t,\boldsymbol{y}+\varepsilon\boldsymbol{e}(t,\boldsymbol{y})) \\ &-\; G(t,\boldsymbol{y}+\varepsilon\boldsymbol{e}(t,\boldsymbol{y}))\boldsymbol{u}(t,\boldsymbol{y}+\varepsilon\boldsymbol{e}(t,\boldsymbol{y})) \\ &+\; \boldsymbol{f}(t,\boldsymbol{y})\big] + \varepsilon\frac{\partial \boldsymbol{e}(t,\boldsymbol{y})}{\partial t} - \varepsilon[\boldsymbol{f}(t,\boldsymbol{y}) - \boldsymbol{f}_a(\boldsymbol{y})] \\ &-\; \varepsilon[G(t,\boldsymbol{y}) - G_a(\boldsymbol{y})]\boldsymbol{u}(t,\boldsymbol{y}) \\ &+\; \varepsilon^2\frac{\partial \boldsymbol{e}(t,\boldsymbol{y})}{\partial \boldsymbol{y}}\big[\boldsymbol{f}_a(\boldsymbol{y}) + G_a(\boldsymbol{y})\boldsymbol{u}(t,\boldsymbol{y})\big] \end{aligned} \tag{24}$$

Denoting the convergence functions for the functions $\boldsymbol{f}(t,\boldsymbol{x})$ and $G(t,\boldsymbol{x})$ by $\delta_f(t)$ and $\delta_G(t)$ respectively, where for simplicity the constant $k$ is absorbed into the convergence functions, the following inequalities can be easily derived

$$\begin{aligned} \|\boldsymbol{e}(t,\boldsymbol{y})\| \;&\leq\; \int_{\mathcal{D}} h(\boldsymbol{y}-\boldsymbol{s})\Big\| \int_0^t \big[\boldsymbol{f}(\tau,\boldsymbol{s}) - \boldsymbol{f}_a(\boldsymbol{s}) \\ &+\; [G(\tau,\boldsymbol{s}) - G_a(\boldsymbol{s})]\boldsymbol{u}(\tau,\boldsymbol{s})\big] d\tau\Big\| d\boldsymbol{s} \\ &\leq\; \int_{\mathcal{D}} h(\boldsymbol{y}-\boldsymbol{s})t\big[\delta_f(t) + u^*\delta_G(t)\big]ds \\ &=\; t\big[\delta_f(t) + u^*\delta_G(t)\big]\int_{\mathcal{D}} h(\boldsymbol{y}-\boldsymbol{s})ds \\ &\leq\; t\big[\delta_f(t) + u^*\delta_G(t)\big] \qquad (25) \\ \left\|\frac{\partial \boldsymbol{e}(t,\boldsymbol{y})}{\partial \boldsymbol{y}}\right\| \;&\leq\; \int_{\mathcal{D}} \left\|\frac{\partial h(\boldsymbol{y}-\boldsymbol{s})}{\partial \boldsymbol{y}}\right\| \Big\| \int_0^t \big[\boldsymbol{f}(\tau,\boldsymbol{s}) - \boldsymbol{f}_a(\boldsymbol{s}) \\ &+\; [G(\tau,\boldsymbol{s}) - G_a(\boldsymbol{s})]\boldsymbol{u}(\tau,\boldsymbol{s})\big] d\tau\Big\| d\boldsymbol{s} \\ &\leq\; \int_{\mathcal{D}} \left\|\frac{\partial h(\boldsymbol{y}-\boldsymbol{s})}{\partial \boldsymbol{y}}\right\| t\big[\delta_f(t) + u^*\delta_G(t)\big]d\boldsymbol{s} \\ &\leq\; It\big[\delta_f(t) + u^*\delta_G(t)\big]\,, \qquad (26) \end{aligned}$$

where $u^* = \max\|\boldsymbol{u}(t,\boldsymbol{x})\|,\ t > 0,\ \boldsymbol{x} \in \mathcal{D}$. Therefore the terms in the first square brackets in (24) can be upper bounded as follows:

$$\begin{aligned} &\big\|\boldsymbol{f}(t,\boldsymbol{y}) + G(t,\boldsymbol{y})\boldsymbol{u}(t,\boldsymbol{y}) - \boldsymbol{f}(t,\boldsymbol{y}+\varepsilon\boldsymbol{e}(t,\boldsymbol{y})) \\ -\;& G(t,\boldsymbol{y}+\varepsilon\boldsymbol{e}(t,\boldsymbol{y}))\boldsymbol{u}(t,\boldsymbol{y}+\varepsilon\boldsymbol{e}(t,\boldsymbol{y}))\big\| \\ \leq\;& \big\|\boldsymbol{f}(t,\boldsymbol{y}) - \boldsymbol{f}(t,\boldsymbol{y}+\varepsilon\boldsymbol{e}(t,\boldsymbol{y}))\big\| + \big\|G(t,\boldsymbol{y})\boldsymbol{u}(t,\boldsymbol{y}) \\ -\;& G(t,\boldsymbol{y}+\varepsilon\boldsymbol{e}(t,\boldsymbol{y}))\boldsymbol{u}(t,\boldsymbol{y}+\varepsilon\boldsymbol{e}(t,\boldsymbol{y}))\big\| \\ \leq\;& \varepsilon\lambda_f\|\boldsymbol{e}(t,\boldsymbol{y})\| + \varepsilon u^*\lambda_G\|\boldsymbol{e}(t,\boldsymbol{y}))\| \\ \leq\;& \varepsilon(\lambda_f + u^*\lambda_G)t\big[\delta_f(t) + u^*\delta_G(t)\big] \end{aligned} \tag{27}$$

The second square bracket in (24) can be upper bounded using the expression for $\frac{\partial \boldsymbol{e}(t,\boldsymbol{y})}{\partial t}$

$$\begin{aligned}\frac{\partial \boldsymbol{e}(t,\boldsymbol{y})}{\partial t} &= \int_{\mathcal{D}} \big[\boldsymbol{f}(t,\boldsymbol{s}) - \boldsymbol{f}_a(\boldsymbol{s}) \\ &+ [G(t,\boldsymbol{s}) - G_a(\boldsymbol{s})]\boldsymbol{u}(t,\boldsymbol{s})\big] h(\boldsymbol{y}-\boldsymbol{s}) d\boldsymbol{s}\end{aligned} \tag{28}$$

and the property in (19). The computations result in

$$\begin{aligned}&\Big\|\frac{\partial \boldsymbol{e}(t,\boldsymbol{y})}{\partial t} - \big[\boldsymbol{f}(t,\boldsymbol{y}) - \boldsymbol{f}_a(\boldsymbol{y}) + [G(t,\boldsymbol{y}) \\ -\ & G_a(\boldsymbol{y})]\boldsymbol{u}(t,\boldsymbol{y})\big] \int_{\mathcal{D}} h(\boldsymbol{y}-\boldsymbol{s})d\boldsymbol{s}\Big\| \le \int_{\mathcal{D}} \big\|\boldsymbol{f}(t,\boldsymbol{s}) \\ -\ & \boldsymbol{f}_a(\boldsymbol{s}) + [G(t,\boldsymbol{s}) - G_a(\boldsymbol{s})]\boldsymbol{u}(t,\boldsymbol{s})\big] - \boldsymbol{f}(t,\boldsymbol{y}) \\ +\ & \boldsymbol{f}_a(\boldsymbol{y}) - [G(t,\boldsymbol{y}) - G_a(\boldsymbol{y})]\boldsymbol{u}(t,\boldsymbol{y})\big\| h(\boldsymbol{y}-\boldsymbol{s})d\boldsymbol{s} \\ \le\ & \int_{\mathcal{D}} [2\lambda_f \|\boldsymbol{s}-\boldsymbol{y}\| + 2u^*\lambda_G \|\boldsymbol{s}-\boldsymbol{y}\| h(\boldsymbol{y}-\boldsymbol{s})]d\boldsymbol{s} \\ \le\ & 2(\lambda_f + u^*\lambda_G)a\end{aligned} \tag{29}$$

The last term in (24) can be upper bounded taking into account the inequalities in (16) and (25):

$$\begin{aligned}&\left\|\frac{\partial \boldsymbol{e}(t,\boldsymbol{y})}{\partial \boldsymbol{y}}\big[\boldsymbol{f}_a(\boldsymbol{y}) + G_a(\boldsymbol{y})\boldsymbol{u}(t,\boldsymbol{y})\big]\right\| \\ \le\ & (M_f + u^* M_G)\, It[\delta_f(t) + u^*\delta_G(t)]\end{aligned} \tag{30}$$

Combining the bounds in (27), (29) and (30) we obtain

$$\|F(t,\boldsymbol{y})\| \le \varepsilon^2 \lambda t \delta(t) + 2\varepsilon\lambda a + \varepsilon^2 MIt\delta(t) \tag{31}$$

where $\lambda = \lambda_f + u^*\lambda_G$, $\delta(t) = \delta_f(t) + u^*\delta_G(t)$ and $M = M_f + u^* M_G$. We notice that if $u^*$ is finite, then $\delta(t)$ is bounded and vanishes as $t \to \infty$.

We introduce functions

$$\varphi(\varepsilon) = \sup_{\tau \le L} \tau\delta\left(\frac{\tau}{\varepsilon}\right), \quad \psi(t) = \frac{1}{t^2}\int_0^t \tau\delta(\tau)d\tau \tag{32}$$

From the definition of $\varphi(\varepsilon)$ it follows that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varphi(\varepsilon) = \lim_{\varepsilon\to 0}\sup_{\tau\le L}\tau\delta\left(\frac{\tau}{\varepsilon}\right) \\ =\ & \sup_{\tau\le L}\tau\lim_{\varepsilon\to 0}\delta\left(\frac{\tau}{\varepsilon}\right) = \sup_{\tau\le L}\tau\lim_{t\to\infty}\delta(t) = 0\end{aligned} \tag{33}$$

With respect to the function $\psi(t)$ we note that it is well defined for any fixed $t$, since $\delta(t)$ is absolutely continuous. Moreover, $\psi(t)$ is a smooth function on the interval $(0,\infty)$. Therefore the Hospital's rule can be applied to conclude that

$$\begin{aligned}&\lim_{t\to\infty}\psi(t) = \lim_{t\to\infty}\frac{\int_0^t \tau\delta(\tau)d\tau}{t^2} \\ =\ & \lim_{t\to\infty}\frac{t\delta(t)}{2t} = \lim_{t\to\infty}\frac{\delta(t)}{2} = 0\end{aligned} \tag{34}$$

Therefore, for any fixed $\rho > 0$ and $\gamma > 0$ there exists $\varepsilon_0 = \varepsilon_0(\rho,\gamma) > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ the following inequalities hold [3]:

$$\varphi(\varepsilon) < \rho, \quad \varphi(\varepsilon) < \frac{\gamma}{2}, \quad \psi\left(\frac{L}{\varepsilon}\right) \le \frac{\gamma}{4L^2 e^{\lambda L}(\lambda + IM)} \tag{35}$$

Then $\varepsilon\|\boldsymbol{e}(t,\boldsymbol{y})\| \le \varepsilon t\delta(t) \le \varphi(\varepsilon) < \rho$ for all $0 < t < \frac{L}{\varepsilon}$. That is $\|\boldsymbol{z}(t)\| \le \|\boldsymbol{y}(t)\| + \varepsilon\|\boldsymbol{e}(t,\boldsymbol{y}(t))\| \le \|\boldsymbol{y}(t)\| + \rho$. Hence, $\boldsymbol{y}(t) \in \mathcal{D}_\rho$ implies that $\boldsymbol{z}(t) \in \mathcal{D}$. Therefore the Lipschitz conditions can be applied. It follows that

$$\begin{aligned}&\|\dot{\boldsymbol{x}}(t) - \dot{\boldsymbol{z}}(t)\| \le \varepsilon(\|\boldsymbol{f}(t,\boldsymbol{x}(t)) - \boldsymbol{f}(t,\boldsymbol{z}(t))\| \\ +\ & \|G(t,\boldsymbol{x}(t))\boldsymbol{u}(t,\boldsymbol{x}(t)) - G(t,\boldsymbol{z}(t))\boldsymbol{u}(t,\boldsymbol{z}(t))\|) \\ +\ & \|\boldsymbol{F}(t,\boldsymbol{y}(t))\| \le \lambda\varepsilon\|\boldsymbol{x}(t) - \boldsymbol{z}(t)\| + \|\boldsymbol{F}(t,\boldsymbol{y}(t))\|,\end{aligned} \tag{36}$$

For the same $\rho$ and $\gamma$ we choose $a$ to satisfy the inequality $a < \frac{\gamma}{8\lambda L e^{\lambda L}}$. Then, taking into account the inequality $\frac{d}{dt}\|\boldsymbol{x}(t) - \boldsymbol{z}(t)\| \le \|\dot{\boldsymbol{x}}(t) - \dot{\boldsymbol{z}}(t)\|$ and the equation $\boldsymbol{x}(0) = \boldsymbol{z}(0)$, we integrate the inequality in (36) and obtain

$$\begin{aligned}\|\boldsymbol{x}(t) - \boldsymbol{z}(t)\| &\le \int_0^t e^{\varepsilon\lambda(t-\tau)}\|\boldsymbol{F}(\tau,\boldsymbol{y}(\tau))\|d\tau \\ &\le e^{\varepsilon\lambda t}\int_0^{\frac{L}{\varepsilon}}\|\boldsymbol{F}(\tau,\boldsymbol{y}(\tau))\|d\tau \\ &\le \left[2\lambda aL + (IM+\lambda)L^2\psi\left(\frac{L}{\varepsilon}\right)\right]e^{\varepsilon\lambda t} \\ &\le \frac{\gamma}{4} + \frac{\gamma}{4} = \frac{\gamma}{2}.\end{aligned} \tag{37}$$

Therefore the difference $\boldsymbol{x}(t) - \boldsymbol{y}(t)$ can be upper bounded as follows

$$\begin{aligned}\|\boldsymbol{x}(t) - \boldsymbol{y}(t)\| &\le \|\boldsymbol{x}(t) - \boldsymbol{z}(t)\| + \|\boldsymbol{y}(t) - \boldsymbol{z}(t)\| \\ &\le \frac{\gamma}{2} + \varphi(\varepsilon) < \gamma.\end{aligned} \tag{38}$$

The proof is complete. ■

*Remark 3:* In this paper we do not treat the control design problem. The only requirement is that the control signal $\boldsymbol{u}(t,\boldsymbol{x})$ satisfy conventional measurability and continuity properties, and be bounded both in $t$ and in $x$. Boundedness in $t$ is common requirement for control systems and is usually met when the right hand side of the dynamic equations are bounded in $t$, which is true in realistic scenarios. Boundedness in $x$ is essentially a stability requirement. Indeed, in most cases the control signal is a continuous function of the state, meaning that if the state is bounded the control signal is bounded. Control signals that are discontinuous in the state usually appear in the form of signum or saturation functions, which are bounded by default.

*Remark 4:* Theorem 1 implies that the averaging method requires preliminary analysis of the original system in two aspects. First, the averaging process must not alter the controllability. Second, one needs to make sure that the states of the original system and of the averaged system can be included in bounded domains, which relate to each other via Definition 2

## IV. STABILIZATION WITH AVERAGING

In this section we give sufficient conditions under which the stabilizing controller for the averaged system will also stabilize the original system and vise versa. To this end we will use the notion of a control Lyapunov functions (CLF) introduced in [1]:

*Definition 4:* A continuously differentiable function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ is a CLF for the system (2) if there exist class $\mathcal{K}_\infty$ functions $\alpha_1(\|\boldsymbol{x}\|)$ and $\alpha_2(\|\boldsymbol{x}\|)$ such that

$$\alpha_1(\|\boldsymbol{x}\|) \le V(t,\boldsymbol{x}) \le \alpha_2(\|\boldsymbol{x}\|)$$
$$\inf_u \left\{ a(t,\boldsymbol{x}) + \boldsymbol{b}^\top(t,\boldsymbol{x})\boldsymbol{u} \right\} < 0\,, \quad (39)$$

where $a(t,\boldsymbol{x}) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \boldsymbol{x}} f(t,\boldsymbol{x})$ and $b(t,\boldsymbol{x}) = G(t,\boldsymbol{x})\frac{\partial V}{\partial \boldsymbol{x}}^\top$.

The property in (39) is equivalent to requiring that in the set $S = \{\boldsymbol{x} : \boldsymbol{b}(t,\boldsymbol{x}) = 0, \boldsymbol{x} \neq 0\}$ the inequality $a(t,\boldsymbol{x}) < 0$ holds.

From Definition 4 and converse Lyapunov theorems the following lemma follows.

*Lemma 1:* The system in (2) is uniformly globally asymptotically stabilizible (UGAS) iff it admits a CLF.

As we mention in the introduction and showed by constructing an example, from the stability of the averaged system in general does not follow the stability of the original system. It strongly depends on the value of the parameter $\varepsilon$, which may not be of the designers disposal. The following lemmas give sufficient conditions independent of $\varepsilon$ under which from the UGAS of the original system follows that the averaged system is globally asymptotically stable (GAS) and vise versa.

*Lemma 2:* If there exists a time invariant CLF $V(\boldsymbol{x})$ for the system (2), then $V(\boldsymbol{x})$ is a CLF for the averaged system provided that it is controllable.

*Proof:* Let $V(\boldsymbol{x})$ be a CLF for the system (2). Then $V(\boldsymbol{x})$ is positive definite and radially unbounded. Moreover, from the equation $\boldsymbol{b}(t,\boldsymbol{x}) = 0$ it follows that $\frac{\partial V}{\partial \boldsymbol{x}} f(t,\boldsymbol{x}) < 0$ or $\boldsymbol{x} = 0$. Since $V(\boldsymbol{x})$ does not depend on $t$, the integration and limiting operation result in

$$\begin{aligned} 0 &= \lim_{t\to\infty} \frac{1}{t}\int_0^t \boldsymbol{b}^\top(t,\boldsymbol{x}) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{\partial V}{\partial \boldsymbol{x}} G(\tau,\boldsymbol{x})d\tau \\ &= \lim_{t\to\infty} \frac{\partial V}{\partial \boldsymbol{x}} \frac{1}{t}\int_0^t G(\tau,\boldsymbol{x})d\tau = \frac{\partial V}{\partial \boldsymbol{x}} G_a(\boldsymbol{x})\,, \quad (40) \end{aligned}$$

implying that the set $S$ is the same for the original and averaged systems. Next, integrating the inequality $\frac{\partial V}{\partial \boldsymbol{x}} f(t,\boldsymbol{x}) < 0$ and taking the limit we obtain

$$\begin{aligned} 0 &> \lim_{t\to\infty} \frac{1}{t}\int_0^t a(t,\boldsymbol{x}) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{\partial V}{\partial \boldsymbol{x}} f(\tau,\boldsymbol{x})d\tau \\ &= \frac{\partial V}{\partial \boldsymbol{x}} \lim_{t\to\infty} \frac{1}{t}\int_0^t f(\tau,\boldsymbol{x})d\tau = \frac{\partial V}{\partial \boldsymbol{x}} f_a(\boldsymbol{x})\,. \quad (41) \end{aligned}$$

Therefore $V(\boldsymbol{x})$ is a CLF for the averaged system in (4). That is there exists control law that stabilizes both the original and the averaged systems. Notice that this control can be given in many different ways. In particular, it can be given by the universal formulas such as Sontag's formula [12], Freeman and Kokotovic's formula [8] or satisficing formula [6]. ■

To go in opposite direction we impose more restrictions on the system in (2).

*Lemma 3:* Let $V(\boldsymbol{x})$ be CLF for the averaged system in (4). If $G(t,\boldsymbol{x})$ is a sign definite matrix, then $V(\boldsymbol{x})$ is a CLF for system in (2), provided that $a(t,\boldsymbol{x}) < 0$ on the set $S_a = \{\boldsymbol{x} : \frac{\partial V}{\partial \boldsymbol{x}} G_a(\boldsymbol{x}) = 0,\ \boldsymbol{x} \neq 0\}$.

*Proof:* Since $V(\boldsymbol{x})$ is a CLF for the system (4), it is positive definite, radially unbounded and on the set $S_a$ the inequality holds: $\frac{\partial V}{\partial \boldsymbol{x}} f_a(\boldsymbol{x}) < 0$. Sign definiteness of the matrix $G(t,\boldsymbol{x})$ implies that of the materix $G_a(\boldsymbol{x})$. Therefore the sets $S_a$ for the averaged system in (4) and $S$ for the original system in (2) are the same. Hence, $V(\boldsymbol{x})$ is a CLF for the original system if $a(t,\boldsymbol{x}) < 0$ on the set $S_a$. ■

*Remark 5:* Theorem 1 along with Lemmas 2 and 3 imply that if the sufficient conditions are satisfied, the control signal that stabilizes the averaged system guarantees $\varepsilon$-closeness of the solutions of the original and averaged systems in transient and in steady-state. The same is true when tracking a reference model.

## V. Conclusion

In this paper we have discussed the question of the extendibility of the classical averaging theory to affine in control systems. It has been shown that for the control systems the straightforward application of averaging can be misleading in the sense that the controller that stabilizes the original system may fail to stabilize the averaged system, and the controller that stabilizes the averaged system may fail to stabilize the original one. This is a consequence of the dependence of the stability properties on the parameter $\varepsilon$. Sufficient conditions are derived to guarantee the validity of the extension of the classical averaging to affine in control systems, provided that the controllability is preserved after averaging.

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