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Instability Conditions for Neutral type time delay systems

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Abstract— **A complete type Lyapunov-Krasovskii functional for neutral type-time delay systems with given cross terms in the time derivative is presented. The facts that the existance of this functional is guaranteed for exponentially stable systems and that it admits a quadratic lower bound allows to propose new instability conditions for this class of systems.**

I. INTRODUCTION

The construction of Lyapunov-Krasovskii functionals with prescribed time derivative have been addressed in several works, starting with the contributions of Repin [14] and Datko [3]. This topic was recently revisited by Kharitonov and coauthors [9], [15], with the so called functionals of complete type whose time derivative includes the whole state of the system. Unlike the functionals of predetermined form proposed in the literature, if the linear system is exponentially stable, the functional exists [9]. Moreover, in contrast with the complete type functional presented by Huang [6] and Repin [14], that were only shown to admit cubic lower bounds [5], the complete type functionals admit a useful quadratic lower bound [9].

Using the negation of the fact that if a system is exponentially stable the existence of the functional is guaranteed and it admits a quadratic lower bound, it was recently possible to provide sufficient instability conditions for the case of retarded linear time delay systems [11].

The instability of functional differential equations has also been studied in the past years by Hale [4] who states sufficient conditions for stability and instability based on Lyapunov functionals, by Barnea [1] who proposes conditions that involve the knowledge of a Lyapunov function and finally by Buslowicz [2] who gives instability sufficient delay-dependent conditions derived from the Theorem of Pontryagin [13].

The aim of our contribution is to extend to the case of neutral type time delay systems the results on complete type functional with derivative including special cross terms [10] and to use them to obtain instability conditions in the spirit of Ochoa and Mondié [11].

The contribution is organized as follows: some basic definitions, concepts and previous results on neutral type-time delay systems are recalled in Section 2. In Section 3 a complete type Lyapunov-Krasovskii functional with given cross terms in its time derivative is constructed and the existence of a special quadratic lower bound is established. Sufficient instability conditions based on the converse idea of the existence of a quadratic lower bound of the functional is presented in Section 4. Illustrative examples are given in Section 5 and some concluding remarks end this contribution.

II. PRELIMINARIES

In this section some useful basic results are given.

We consider neutral type time delay systems of the form

$$\dot{x}(t) - C\dot{x}(t-h) = Ax(t) + Bx(t-h), \tag{1}$$

where A, B and C are given $n \times n$ constant matrices and $h > 0$ is the time delay and the initial condition is

$$x(\theta) = \varphi(\theta), \quad -h \leq \theta \leq 0,\ \varphi \in \mathcal{C}^1[-h, 0]. \tag{2}$$

We denote by $x(t, \varphi)$ or $x(t)$ the solution of system (1) with the initial conditions (2) and x_t or $x_t(\varphi) = \{x(t+\theta, \varphi) \mid \theta \in [-h, 0]\}$ is the state of the system.

Definition 1: The system (1) is said to be exponentially stable if there exist $\gamma \geq 1$ and $\alpha > 0$ such that every solution $x(t, \varphi)$ satisfies the inequality

$$\|x(t, \varphi)\| \leq \gamma e^{-\alpha t} \|\varphi\|_h \quad \forall t \geq 0,$$

where

$$\|\varphi\|_h = \sup_{\theta \in [-h,0]} \|\varphi(\theta)\|.$$

Remark 2: If C is a Schur matrix, then the asymptotic stability of system (1) is equivalent to the exponential stability of the sytem.

Under the assumption that system (1) is exponentially stable, then the matrix

$$U(\tau) = \int_0^\infty K^\intercal(t) W K(t+\tau) dt, \tag{3}$$

is well defined for $\tau \in \mathbb{R}$ and every constant matrix W. Matrix $K(t)$ is the fundamental matrix of system (1) and matrix $U(\tau)$ is known as the Lyapunov matrix of system (1) associated to matrix W [15].

A. Lyapunov matrix

It was shown [15] that the Lyapunov matrix satisfies the following properties

- The dynamic property
$$U'(\tau) - U'(\tau - h)C = U(\tau)A + U(\tau - h)B.$$
- The symmetry property
$$U(-\tau) = U(\tau).$$
- The algebraic property
$$\begin{aligned}-W = {} & A^\intercal U(0) + U(0)A - A^\intercal U^\intercal(h)C - C^\intercal U(h)A + \\ & + B^\intercal U(h) + U^\intercal(h)B - B^\intercal U(0)C - C^\intercal U(0)B.\end{aligned}$$

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B. Lyapunov-Krasovskii functionals of complete type

Given a functional of the form

$$w(x_t) = x^\intercal(t)W_0x(t) + x^\intercal(t-h)W_1x(t-h) + + \int_{-h}^{0} x^\intercal(t-\theta)W_2x(t-\theta)d\theta, \quad (4)$$

where W_0, W_1 and W_2 are positive definite matrices, then the system (1) is exponentially stable if there exists a functional $v(x_t)$ such that

$$\frac{d}{dt}v(x_t) = -w(x_t)$$

along the solutions of the system. The functional $v(x_t)$, called of complete type, is given in the following statement:

Theorem 3: [15] Let system (1) be exponentially stable, then the complete type functional

$$v(x_t) = v_0(x_t, W) + \int_{-h}^{0} x^\intercal(t+\theta)\left[W_1 + (h+\theta)W_2\right]x(t+\theta)d\theta,$$

is such that

$$\frac{d}{dt}v(x_t) = -w(x_t).$$

Here,

$$\begin{aligned} v_0(x_t, W) = {}& x^\intercal(t)[U(0) - C^\intercal U(h) - U^\intercal(h)C + \\ &+C^\intercal U(0)C]x(t) + 2x^\intercal(t)\int_{-h}^{0}[U^\intercal(h+\theta) - C^\intercal U^\intercal(\theta)] \times \\ &\times [Bx(t+\theta) + C\dot{x}(t+\theta)]\,d\theta + \int_{-h}^{0}\int_{-h}^{0}[Bx(t+\theta_1) + \\ &+C\dot{x}(t+\theta_1)]^\intercal U(\theta_1-\theta_2)\,[Bx(t+\theta_2)+C\dot{x}(t+\theta_2)]\,d\theta_1 d\theta_2, \end{aligned} \quad (5)$$

and $W = W_0 + W_1 + hW_2$.

III. FUNCTIONALS OF COMPLETE TYPE WITH A GIVEN CROSS TERM

This section is devoted to the construction of a complete type Lyapunov-Krasovkii functional that includes a cross term in its time derivative. We assume that system (1) is exponentially stable.

We are looking for functionals of complete type that satisfies

$$\frac{dv_c(x_t)}{dt} = -w_c(x_t), \quad (6)$$

where

$$\begin{aligned} w_c(x_t) = {}& \begin{bmatrix} x^\intercal(t) & x^\intercal(t-h) \end{bmatrix} \times \\ & \begin{bmatrix} W_0 & PB - A^\intercal PC \\ B^\intercal P - C^\intercal PA & W_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} + \\ & + \int_{-h}^{0} x^\intercal(t+\theta)W_2x(t+\theta)d\theta, \end{aligned}$$

with P a symmetric matrix such that

$$\begin{bmatrix} W_0 & PB - A^\intercal PC \\ B^\intercal P - C^\intercal PA & W_1 \end{bmatrix} > 0.$$

Notice that $w_c(x_t)$ can be expressed as $w_c(x_t) = w(x_t) + \tilde{w}_c(x_t)$, where $w(x_t)$ is given by (4) and $\tilde{w}_c(x_t)$ corresponds to the cross terms, namely

$$\begin{aligned} \tilde{w}_c(x_t) = {}& x^\intercal(t-h)B^\intercal Px(t) - x^\intercal(t-h)C^\intercal PAx(t) + \\ & x^\intercal(t)PBx(t-h) - x^\intercal(t)A^\intercal PCx(t-h) \end{aligned}$$

Substituting into $\tilde{w}_c(x_t)$ $Ax(t)$ by $\dot{x}(t) - C\dot{x}(t-h) - Bx(t-h)$ and $Bx(t-h)$ by $\dot{x}(t) - C\dot{x}(t-h) - Ax(t)$ gives

$$\begin{aligned} \tilde{w}_c(x_t) = {}& \dot{x}^\intercal Px(t) + x^\intercal(t)P\dot{x}(t) - \dot{x}^\intercal(t-h)C^\intercal Px(t) - \\ & - x^\intercal(t-h)C^\intercal P\dot{x}(t) - \dot{x}^\intercal(t)PCx(t-h) - \\ & - x^\intercal(t)PC\dot{x}(t-h) + \dot{x}^\intercal(t-h)C^\intercal PCx(t-h) + \\ & + x^\intercal(t-h)C^\intercal PC\dot{x}(t-h) - x^\intercal(t)A^\intercal Px(t) - \\ & - x^\intercal(t)PAx(t) + x^\intercal(t-h)C^\intercal PBx(t-h) + \\ & + x^\intercal(t-h)B^\intercal PCx(t-h), \end{aligned}$$

or

$$\begin{aligned} \tilde{w}_c(x_t) = {}& - x^\intercal(t)\left[A^\intercal P + PA\right]x(t) + x^\intercal(t-h) \times \\ & \times \left[C^\intercal PB + B^\intercal PC\right]x(t-h) + \frac{d}{dt}x^\intercal(t)Px(t) - \\ & - \frac{d}{dt}x^\intercal(t)PCx(t-h) - \frac{d}{dt}x^\intercal(t-h)C^\intercal Px(t) + \\ & + \frac{d}{dt}x^\intercal(t-h)C^\intercal PCx(t-h). \end{aligned}$$

Because of the exponential stability assumption for system (1), the integration from 0 to ∞ of expression (6) gives

$$\begin{aligned} v_c(x_t) = {}& - x^\intercal(t)Px(t) - x^\intercal(t-h)C^\intercal PCx(t-h) + \\ & + x^\intercal(t-h)C^\intercal Px(t) + x^\intercal(t)PCx(t-h) + \\ & + \int_0^\infty x^\intercal(t)\left[W_0 - A^\intercal P - PA\right]x(t)dt + \\ & + \int_0^\infty x^\intercal(t-h)\left[W_1 + C^\intercal PB + B^\intercal PC\right] \times \\ & x(t-h)dt + \int_0^\infty \int_{-h}^{0} x^\intercal(t+\theta)W_2x(t+\theta)d\theta dt. \end{aligned}$$

If we define

$$\begin{aligned} W = {}& W_0 + W_1 + hW_2 - A^\intercal P - PA + \\ & + C^\intercal PB + B^\intercal PC, \end{aligned} \quad (7)$$

functional $v_c(x_t)$ appears as

$$\begin{aligned} v_c(x_t) = {}& v_0(x_t, W) - \\ & - x^\intercal(t)Px(t) - x^\intercal(t-h)C^\intercal PCx(t-h) + \\ & + x^\intercal(t-h)C^\intercal Px(t) + x^\intercal(t)PCx(t-h) + \\ & + \int_{-h}^{0} x^\intercal(t+\theta)\left[W_1 + C^\intercal PB + B^\intercal PC + \right. \\ & \left. +(h+\theta)W_2\right]x(t+\theta)d\theta. \end{aligned} \quad (8)$$

Now, we prove that the functional $v_c(x_t)$ admits quadratic upper and lower bounds.

Lemma 4: The functional (8) admits a quadratic lower bound of the form

$$\alpha\left\|x(t) - Cx(t-h)\right\|^2 \leq v_c(x_t).$$

Proof: Let us define the functional

$$\tilde{v}_c(x_t) = v_c(x_t) - \alpha \|x(t) - Cx(t-h)\|^2 .$$

Its time derivative is of the form

$$\frac{d}{dt}\tilde{v}_c(x_t) = - w_c(x_t) - 2\alpha \left[x(t) - Cx(t-h)\right]^{\mathsf{T}} \times \left[Ax(t) + Bx(t-h)\right],$$

equivalently,

$$\frac{d}{dt}\tilde{v}_c(x_t) = - \begin{bmatrix} x(t) & x(t-h) \end{bmatrix} L(\alpha) \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} - - \int_{-h}^{0} x^{\mathsf{T}}(t+\theta) W_2 x(t+\theta) d\theta,$$

with

$$L(\alpha) = \begin{bmatrix} W_0 & PB - A^{\mathsf{T}}PC \\ B^{\mathsf{T}}P - C^{\mathsf{T}}PA & W_1 \end{bmatrix} + + \alpha \begin{bmatrix} A + A^{\mathsf{T}} & B - C^{\mathsf{T}}A \\ B^{\mathsf{T}} - A^{\mathsf{T}}C & -CB - B^{\mathsf{T}}C^{\mathsf{T}} \end{bmatrix}.$$

Let $\alpha_0 > 0$ be the first positive value for which the determinant of the matrix pencil $L(\alpha)$ vanishes for the first time. Since $L(0)$ is positive definite, then for $\alpha^* \in [0, \alpha_0)$, $L(\alpha^*)$ is also positive definite. Under these assumption $\tilde{w}_c(x_t) \geq 0$ for $t \geq 0$ and it follows that

$$\tilde{v}_c(x_t) = \int_0^{\infty} \tilde{w}_c(x_t) \geq 0,$$

and the result follows. ■

Lemma 5: The functional (8) admits a quadratic upper bound of the form

$$v_c(x_t) \leq \rho \left\{ \|x(t)\|^2 + \|x(t-h)\|^2 + \int_{-h}^{0} \|x(t+\theta)\|^2 d\theta \right\}.$$

Proof: Let

$$\mu = \max_{\tau \in [0,h]} \|U(\tau, W)\|,$$

where W is given by (7). Then the majorization of $v_c(x_t)$ appears as

$$v_c(x_t) \leq \rho_1 \|x(t)\|^2 + \rho_2 \|x(t-h)\|^2 + \rho_3 \int_{-h}^{0} \|x(t+\theta)\|^2 d\theta + + \rho_4 \int_{-h}^{0} \|\dot{x}(t+\theta)\|^2 d\theta,$$

where

$$\begin{aligned}
\rho_1 &= \|U(0)\| + 2\|C\|\|U(h)\| + \|C\|^2 \|U(0)\| + \\
&\quad + \mu h \|B\| + \mu h \|C\| + \mu h \|B\| \|C\| + \mu h \|C\|^2 + \\
&\quad + \|P\| + \|C\| \|P\| \\
\rho_2 &= \|C\| \|P\| + \|C\|^2 \|P\| \\
\rho_3 &= \mu \|B\| + \mu \|B\| \|C\| + \mu h \|B\|^2 + \mu h \|B\| \|C\| + \\
&\quad + \left\|\bar{W}_1\right\| + h \|W_2\| \\
\rho_4 &= 2\mu \|C\| + \mu h \|B\| \|C\| + \mu h \|C\|^2 .
\end{aligned}$$

Since

$$\int_{-h}^{0} \|\dot{x}(t+\theta)\|^2 d\theta = \|x(t)\|^2 - \|x(t-h)\|^2 ,$$

then

$$v_c(x_t) \leq (\rho_1 - \rho_4) \|x(t)\|^2 + (\rho_2 - \rho_4) \|x(t-h)\|^2 + + \rho_3 \int_{-h}^{0} \|x(t+\theta)\|^2 d\theta,$$

or

$$v_c(x_t) \leq \rho_1^* \|x(t)\|^2 + \rho_2^* \|x(t-h)\|^2 + \rho_3 \int_{-h}^{0} \|x(t+\theta)\|^2 d\theta.$$

If we select ρ such that $\rho \geq \max\{\rho_1^*, \rho_2^*, \rho_3\}$ then

$$v_c(x_t) \leq \rho \left\{ \|x(t)\|^2 + \|x(t-h)\|^2 + \int_{-h}^{0} \|x(t+\theta)\|^2 d\theta \right\}.$$

which ends the proof of the lemma. ■

Remark 6: The functional $v_c(x_t)$ also satisfies the following upper bound

$$v_c(x_t) \leq \rho^* \|x_t\|_h^2 ,$$

where $\rho^* = \rho(2+h)$.

The following lemma provides a less restrictive quadratic lower bound.

Lemma 7: Let system (1) be exponentially stable and let symmetric matrices W_0, W_1 and W_2 and a symmetric matrix P such that

$$\begin{bmatrix} W_0 & PB - A^{\mathsf{T}}PC \\ B^{\mathsf{T}}P - C^{\mathsf{T}}PA & W_1 \end{bmatrix} > 0$$

be given. Then, the functional $v_c(x_t)$ admits a quadratic lower bound of the form

$$v_c(x_t) \geq \alpha \left\{ \|x(t) - Cx(t-h)\|^2 + \int_{-h}^{0} \|x(t+\theta)\|^2 d\theta \right\},$$

for some $\alpha \in (0, \alpha_0]$, where $\alpha_0 > 0$ is the first positive value for which the determinant of the matrix pencil

$$L(\alpha) = \begin{bmatrix} W_0 & PB - A^{\mathsf{T}}PC \\ B^{\mathsf{T}}P - C^{\mathsf{T}}PA & W_1 \end{bmatrix} + + \alpha \begin{bmatrix} A + A^{\mathsf{T}} + I & B - C^{\mathsf{T}}A \\ B^{\mathsf{T}} - A^{\mathsf{T}}C & -CB - B^{\mathsf{T}}C^{\mathsf{T}} - I \end{bmatrix},$$

vanishes for the first time.

Proof: Let us define the functional

$$\tilde{v}_c(x_t) = v_c(x_t) - \alpha \Big[\|x(t) - Cx(t-h)\|^2 + + \int_{-h}^{0} \|x(t+\theta)\|^2 d\theta \Big]. \quad (9)$$

Its time derivative is of the form

$$\begin{aligned}
\frac{d\tilde{v}_c(x_t)}{dt} &= -w_c(x_t) - 2\alpha \left[x(t) - Cx(t-h)\right]^{\mathsf{T}} \Big[Ax(t) \\
&\quad + Bx(t-h)\Big] - \alpha \|x(t)\|^2 + \alpha \|x(t-h)\|^2 \\
&= -\tilde{w}_c(x_t),
\end{aligned}$$

where $\tilde{w}_c(x_t)$ can be expressed as

$$\tilde{w}_c(x_t) = \begin{bmatrix} x(t) & x(t-h) \end{bmatrix} L(\alpha) \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} + + \int_{-h}^{0} x^{\mathsf{T}}(t+\theta) W_2 x(t+\theta) d\theta.$$

Here

$$L(\alpha)=\begin{bmatrix} W_0 & PB-A^\intercal PC \\ B^\intercal P - C^\intercal PA & W_1\end{bmatrix}+ \\ +\alpha\begin{bmatrix} A+A^\intercal+I & B-C^\intercal A \\ B^\intercal - A^\intercal C & -C^\intercal B - B^\intercal C - I\end{bmatrix}.$$

Since $L(0)$ is positive definite then $L(\alpha)>0$ for $\alpha\in[0,\alpha_0)$ implies that

$$\tilde{w}_c(x_t)\geq 0,$$

then

$$\frac{d\tilde{v}_c\left(x_t\right)}{dt}=-\tilde{w}_c(x_t)\leq 0.$$

Integrating the above inequality gives

$$\int_0^\infty \frac{d\tilde{v}_c\left(x_t\right)}{dt}=-\int_0^\infty \tilde{w}_c(x_t)\leq 0.$$

The results follows from the stability assumption on system (1) and definition (9). ∎

Remark 8: The functional $v_c(x_t)$ can be completely determined by computing the Lyapunov matrix $U(\tau),\tau\in[-h,0]$. Equation (3) cannot be used for the construction of matrix $U(\tau)$. In [12], a semi-analytic procedure and a piece-wise linear approximation are proposed for the computation of Lyapunov matrix $U(\tau)$.

IV. INSTABILITY CONDITION

The main result of this contribution is based on the idea of writting the converse of Lemma 7.

Lemma 9: If there exists an initial condition

$$x(\theta)=\varphi(\theta),\quad -h\leq\theta\leq 0,\ \varphi\in\mathcal{C}^1[-h,0]$$

such that the functional $v_c(x_t)$, defined in (8), satisfies the inequality

$$v_c(x_t)<\alpha\left\{\|\varphi(0)-C\varphi\left(-h\right)\|^2+\int_{-h}^0\|\varphi\left(\theta\right)\|^2 d\theta\right\} \quad (10)$$

where α is the first positive value for which the matrix pencil

$$L(\alpha)=\begin{bmatrix} W_0 & PB-A^\intercal PC \\ B^\intercal P - C^\intercal PA & W_1\end{bmatrix}+ \\ +\alpha\begin{bmatrix} A+A^\intercal+I & B-C^\intercal A \\ B^\intercal - A^\intercal C & -C^\intercal B - B^\intercal C - I\end{bmatrix},$$

vanishes for the first time for matrices W_0, W_1 and W_2 that satisfies $W=W_0+W_1+hW_2-A^\intercal P-PA+C^\intercal PB+B^\intercal PC$ and

$$\begin{bmatrix} W_0 & PB-A^\intercal PC \\ B^\intercal P - C^\intercal PA & W_1\end{bmatrix}>0,$$

then system (1) is unstable.

The above conditions cannot be checked by direct computations because it is written in terms of a general initial conditions. However it is sufficient to find just one special initial condition $\varphi(\theta)$ to prove the instability of system (1). To this aim we consider one of the simplest case: we assume that the initial function is of the form

$$\varphi(\theta)=\varphi(0),\theta\in\left[-h,0\right]. \quad (11)$$

Lemma 10: Let a positive definite matrix W be given and let $U(\tau)$ be the Lyapunov matrix of system (1), then if

$$U(0)-C^\intercal U(h)-U^\intercal(h)C+C^\intercal U(0)C-P-C^\intercal PC+ \\ +C^\intercal P+PC+h\bar{W}_1+\frac{h^2}{2}W_2+\Gamma_1 B+B^\intercal\Gamma_1- \\ -\left[C^\intercal\Gamma_2 B+B^\intercal\Gamma_2 C\right]+B^\intercal\Gamma_3 B-\hat{\alpha}I<0, \quad (12)$$

where

$$\begin{aligned}\hat{\alpha} &= \alpha_0[\|I-C\|^2+h]\\ \Gamma_1 &= \int_{-h}^0 U^\intercal(h+\theta)d\theta\\ \Gamma_2 &= \int_{-h}^0 U^\intercal(\theta)d\theta\\ \Gamma_3 &= \int_{-h}^0\int_{-h}^0 U^\intercal(\theta_1-\theta_2)d\theta_1 d\theta_2,\end{aligned}$$

and α_0 is the first positive value for which the determinant of the matrix pencil $L(\alpha)$ vanishes for some positive definite matrices W_0, W_1 and W_2 satisfying

$$W=W_0+W_1+hW_2-A^\intercal P-PA+ \\ +C^\intercal PB+B^\intercal PC,$$

and

$$\begin{bmatrix} W_0 & PB-A^\intercal PC \\ B^\intercal P - C^\intercal PA & W_1\end{bmatrix}>0,$$

then system (1) is unstable.

Proof: For initial conditions of the form (11) the functional $v_c(x_t)$ given by (8) is

$$\begin{aligned}v_c(\varphi)=&\varphi^\intercal(0)\big[U(0)+C^\intercal U(h)-U^\intercal(h)C+C^\intercal U(0)C-\\ &-P-C^\intercal PC+C^\intercal P+PC\big]\varphi(0)+\\ &+2\varphi^\intercal(0)\int_{-h}^0\left[U^\intercal(h+\theta)-C^\intercal U^\intercal(\theta)\right]d\theta B\varphi(0)+\\ &+\varphi^\intercal(0)B^\intercal\int_{-h}^0\int_{-h}^0 U(\theta_1-\theta 2)d\theta_1 d\theta_2 B\varphi(0)+\\ &+\varphi^\intercal(0)\int_{-h}^0\left[\bar{W}_1+(h+\theta)W_2\right]d\theta\varphi(0).\end{aligned}$$

Then the left hand side of (10) can be written in terms of $\Gamma_{1,2,3}$ as:

$$\begin{aligned}v_c(\varphi)=&\varphi^\intercal(0)\big[U(0)+C^\intercal U(h)-U^\intercal(h)C+C^\intercal U(0)C-\\ &-P-C^\intercal PC+C^\intercal P+PC+\Gamma_1 B+B^\intercal\Gamma_1+\\ &+C^\intercal\Gamma_2+\Gamma_2 C+B^\intercal\Gamma_3 B+hW_1+\frac{h^2}{2}W_2\big]\varphi(0).\end{aligned}$$

The right hand side of inequality (10) appears as

$$\varphi^\intercal(0)\left[\alpha\|I-C\|+h\right]\varphi^\intercal(0)$$

and expression (12) follows from defining

$$\hat{\alpha}=\alpha_0\left[\|I-C\|+h\right].$$

∎

A. Proposed Methodology

Here we propose a methodology to verify the instability conditions of Lemma 10.

- Using the piece-wise linear approximation or the semi-analytic procedure proposed in [12], compute the Lyapunov Matrix $U(\tau), \tau \in [-h, 0]$ for $W > 0$.
- Compute matrices $\Gamma_{1,2,3}$ by using a numerical integration method.
- Solve for $W_0, W_1 > 0$ and $P = P^\intercal$ the maximum eigenvalue problem for the following LMI's

$$-\begin{bmatrix} W_0 & PB - A^\intercal PC \\ B^\intercal P - C^\intercal PA & W_1 \end{bmatrix} < \\ < \alpha \begin{bmatrix} A + A^\intercal + I & B - C^\intercal A \\ B^\intercal - A^\intercal C & -C^\intercal B - B^\intercal C - I \end{bmatrix}, \quad (13)$$

$$W > W_0 + W_1 - A^\intercal P - PA + C^\intercal PB + B^\intercal PC, \quad (14)$$

and compute

$$W_2 = \frac{1}{h}[W - W_0 - W_1 + A^\intercal P + PA - \\ - C^\intercal PB - B^\intercal PC].$$

- If condition (12) is satisfied when the matrices $W_1, W_2, P, U(0), U(h), \Gamma_{1,2,3}$ are substituted then the system is unstable.

Remark 11: If condition (12) is not satisfied for some matrices $W_1, W_2, P, U(0), U(h), \Gamma_{1,2,3}$, we cannot conclude on the instability or stability of the system.

V. EXAMPLES

Example 12: Consider the following scalar neutral type-time delay system

$$\dot{x}(t) = ax(t) + bx(t-h) + c\dot{x}(t-h). \quad (15)$$

It was shown in [2] that system (15) is unstable if the following inequality holds

$$(-a-b)(1-c-ah) \leq 0.$$

Let us select $a = 1, b = -2$ and $c = 0.2$. We first compute the Lyapunov function $u(\tau)$ (see Fig. 1) for $w = 1$ using the piece-wise linear approximation procedure introduced in [12] and we obtain the values $u(0) = -0.098203$ and $u(h) = 0.164779$. Computing the values of $\Gamma_{1,2,3}$ with the trapezoidal rule, we get $\Gamma_1 = \Gamma_2 = -0.304519$ and $\Gamma = -0.858073$.

Solving the maximum eigenvalue problem for LMI's (13) and (14) we obtain $\alpha = 0.55445, w_0 = 0.007478, w_1 = 0.888127, w_2 = 8.17614 \times 10^{-4}$ and $p = -0.03669$. For these values, condition (12) is satisfied and we conclude that system (15) is unstable.

Now let us consider the two dimensional example introduced in [7].

Example 13: Consider the neutral type time delay system

$$\dot{x}(t) - C\dot{x}(t-h) = Ax(t) + Bx(t-h), \quad (16)$$

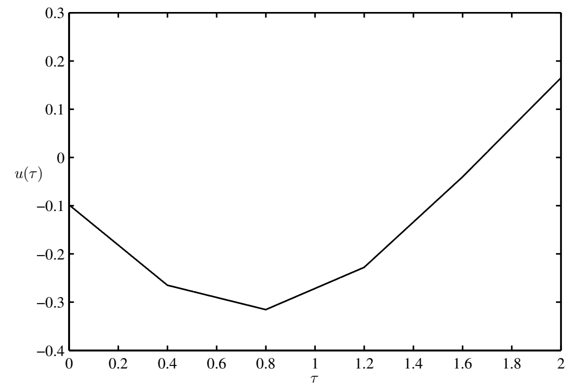

Fig. 1. Piece-wise linear approximation of the Lyapunov function $u(\tau)$

where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

In [8] it was shown that system (16) is unstable for $h > 6.03$. Let us select $h = 7$. Using the semi-analytic procedure introduced in [12] we compute the Lyapunov matrix $U(\tau)$ for $W = I$ (see Fig. (2)) and we obtain the values

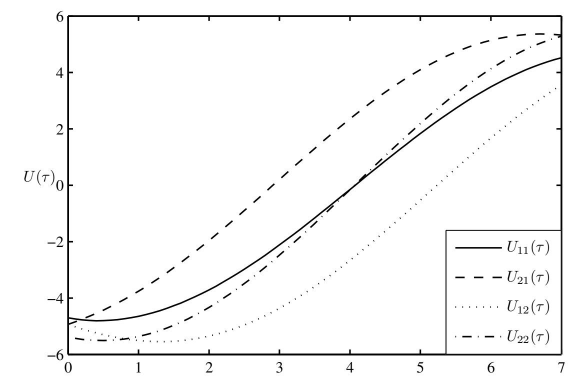

Fig. 2. Lyapunov matrix $U(\tau)$

$$U(0) = \begin{bmatrix} -4.7017 & -4.9355 \\ -4.9355 & -5.3852 \end{bmatrix} U(h) = \begin{bmatrix} 4.5181 & 3.5643 \\ 5.3252 & 5.2837 \end{bmatrix}.$$

Solving the maximum eigenvalue problem for LMI'S (13) and (14) we get $\alpha = 0.33319$ and matrices

$$W_0 = \begin{bmatrix} 0.9997 & -0.000013 \\ -0.000013 & 0.65469 \end{bmatrix} W_1 = \begin{bmatrix} 0.7261 & 0.0111 \\ 0.0111 & 0.6165 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 0.07801 & -0.0067 \\ -0.0067 & 0.0387 \end{bmatrix} P = \begin{bmatrix} -0.3347 & 0.00088 \\ 0.00088 & -0.3391 \end{bmatrix}.$$

Finally $\Gamma_{1,2,3}$ are obtained by direct computations and we can directly check that condition (12) is satisfied hence system (16) is unstable.

It is worth to mention that for the stable case, that is, for $h = 0.5$, one cannot find matrices W_0, W_1, W_2 and P such that inequality (12) holds.

VI. CONCLUSIONS

A complete type Lyapunov-Krasovskii functional with given cross terms in its time derivative is presented. It is shown that, if the system is exponentially stable, the functional exists and it admits a quadratic lower bound. The converse idea leads to sufficient instability conditions stated in terms of LMI's.

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