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# New Results on Control Synthesis for Time-Varying Delay Systems with Actuator Saturation

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*Abstract*— **The control synthesis problem for a class of linear time-delay systems with actuator saturation is investigated in this paper. The time delay is considered to be time-varying and has a lower and upper bounds. A delay-range-dependent approach is adopted and the corresponding existence conditions of the stabilizing state-feedback controller are derived in terms of LMIs. An estimate for the domain of attraction of the origin can be obtained for the underlying systems with different time-delay ranges. Two numerical examples are presented to show the effectiveness and less conservatism of the developed theoretical results.**

## I. INTRODUCTION

During the past decades, time-delay systems have been widely studied and many analysis and synthesis results using delay-dependent approach have also been reported in concern of conservatism, see for example, [1], [4], [9], [12], [14], [15]. Very recently, a new so-called delay-range-dependent concept was proposed [7], [8], in which the delays are considered to vary in a range and thereby more applicable in practice. To further reduce conservatism, more appropriate Lyapunov functional candidates for the underlying systems are constructed such that new stability criterion is proposed depending on the delay variation rate, upper and lower bounds, see for example [5]. However, to the best of authors' knowledge, delay-range-dependent control synthesis problems for linear time-delay systems have not been investigated yet, which will be challenging due to the hard extension of the existing stability results. In fact, how to build a tradeoff between conservatism and extension to control synthesis of adopted approach is still an open problem up to date.

In addition, actuator saturation are often the source of system instability or performance degradation in many physical and industrial systems. Considerable attention has been devoted to the kind of liner system subject to saturating controllers, with or without time-delay in the system, see for example, [6], [11]. Also, delay-dependent approach for such systems with time-delay has been used to estimate the domain of safe admissible initial states (domain of attraction), see, [2], [3], [10]. To be more practical and significant, the advanced and less conservative delay-range-dependent idea is worth considering and attempting to solve the control problems for time-delay systems with actuator saturation. Note that the expected control synthesis results can not be obtained by simple fusion of the available results on time-delay systems and constrained control.

In this paper, we are interested in designing a state feedback controller for a class of linear time-delay systems with actuator saturation. The time-delay is considered to be time-varying and has a lower and upper bounds. The delay-range-dependent approach is adopted and the corresponding existence criterion of the stabilizing controller is derived via LMI formulation. Furthermore, the domain of attraction of the origin can be estimated for the underlying systems with different time-delay ranges. Two numerical examples are given to show the effectiveness and potential of the developed theoretical results.

*Notation:* The notation used in this paper is fairly standard. The superscript "T" stands for matrix transposition, and $\mathbb{R}^n$ denotes the $n$ dimensional Euclidean space. The space of the continuously differentiable vector functions $\phi$ over $[-d_2, 0]$ is denoted by $C^1[-d_2, 0]$. In symmetric block matrices or long matrix expressions, we use * as an ellipsis for the terms that are introduced by symmetry and $diag\{\cdot\}$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. The notation $P > 0$ $(\geq 0)$ means $P$ is real symmetric and positive (semi-positive) definite. $I$ and $0$ represent respectively, identity and zero matrices.

## II. PROBLEM FORMULATION AND PRELIMINARIES

Consider a class of time-varying state-delayed systems with the following dynamics:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t - d(t)) + B\sigma(Fx), t > 0 \\ x(s) &= \phi(s),\ s \in \begin{bmatrix} -d_2 & 0 \end{bmatrix} \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A$ and $A_d$ are constant matrices with appropriate dimensions, $F \in \mathbb{R}^{m\times n}$ is the memoryless state feedback controller gain that has to be designed. The actuator is described by the following non-linearity

$$\sigma(Fx(t)) = [\sigma(f_1x(t)), ..., \sigma(f_mx(t))]^T \tag{2}$$

$$\sigma(f_ix(t)) \triangleq \begin{cases} u_i, & \text{if} \quad f_ix(t) > u_i \\ f_ix(t), & \text{if} \quad -u_i \leq f_ix(t) \leq u_i \\ -u_i, & \text{if} \quad f_ix(t) < u_i \end{cases} \tag{3}$$

where $f_i$ is the $i$th row of $F$. In addition, for feedback matrix $F$, we define

$$\mathcal{L}(F) \triangleq \{x \in \mathcal{R}^n : |f_ix| \leq u_i, i = 1, 2, ..., m\}$$

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then $\mathcal{L}(F)$ is the region in the state space where the control input is linear in $x$.

The time-delay, $d(t)$, is a time-varying continuous function that satisfies

$$0 < d_1 \le d(t) \le d_2,\ 0 \le \dot{d}(t) \le \mu \quad (4)$$

where $d_1$ and $d_2$ are the delay lower and upper bounds, respectively, and $\mu$ is the delay variation rate.

Now, for later development, we revisit the definition on domain of attraction for system (1).

*Definition 1:* Denoting the solution of system (1) with the initial condition $x_0 = \phi \in C^1[-d_2, 0]$ by $\psi(t, x_0)$, then the domain of attraction of the origin of system by (1) is

$$\mathcal{T} \triangleq \left\{\phi \in C^1[-d_2, 0] : \lim_{t\to\infty} \psi(t, x_0) = 0\right\},$$

Our main purpose in this paper is to design a state feedback controller for system (1) such that the closed-loop system is asymptotically stable for all time-varying delays satisfying (4). Also, we are interested by obtaining an estimate of the domain of attraction $X_\delta \subset \mathcal{T}$, where

$$X_\delta = \left\{\phi \in C^1[-d_2, 0] : \max|\phi| \le \delta_1, \max|\dot{\phi}| \le \delta_2\right\}$$

with scalars $\delta_i > 0, i = 1, 2$ that will be maximized in the sequel.

## III. Main results:

The stability problem based on the delay-range-dependent idea has been studied for systems with time-varying delays, however, the obtained conditions are generally hard to extend to solve the stabilization problem, even for the absence of actuator saturation. To overcome the difficulty, an appropriate transformation will be proposed in this paper while dealing with the constructed Lyapunov functional differential.

In addition, as shown in [6], to reduce the conservatism of handling the actuator saturation, the technique of adding auxiliary feedback matrix will be used here, namely, for two matrices $F, H \in \mathbb{R}^{m\times n}$, a matrix set is introduced as:

$$\mathcal{W}(\alpha, F, H) \triangleq \left\{W \in \mathbb{R}^{m\times n} : W = \begin{bmatrix} \alpha_1 f_1 + (1-\alpha_1 h_1) \\ \vdots \\ \alpha_m f_m + (1-\alpha_m h_m) \end{bmatrix}\right\} \quad (5)$$

where $h_i$ is the $i$th row of $H$ and $\alpha_i = 0$ or $1$ (then we define $\psi(\alpha) \triangleq \{\alpha \in \mathcal{R}^m : \alpha_i = 1, 0\}$ for later use). To satisfy the actuator saturation, the technique requires that the auxiliary matrix $H$ also satisfies $|h_i x| \le u_i, i = 1, ..., m$. To this end, a subset of the set $\mathcal{L}(H)$ will be found and chosen to be an ellipsoid of the form:

$$\mathcal{E}(P, 1) \triangleq \{x : x^T P x \le 1\}$$

where $P > 0$ will be determined. Combined $\mathcal{E}(P, 1)$ with

$$\begin{bmatrix} u_i & h_i \\ * & u_i P \end{bmatrix} \ge 0, i = 1, ..., m$$

we can obtain that $\mathcal{E}(P, 1) \subset \mathcal{L}(H)$ (see [6] for details).

Based on the above ideas, the following theorem gives the existence conditions of a stabilizing state-feedback controller for system (1) and the corresponding estimation of domain of attraction.

*Theorem 1:* Consider system (1) and let $0 < d_1 \le d_2$ and $\mu > 0$ be given constants. If there exist matrices $X > 0, \bar{P} > 0,\ \bar{Q}_i > 0, i = 1, 2, 3,\ \bar{Z}_i > 0, i = 1, 2,\ \bar{M}_i,\ \bar{N}_i,\ \bar{S}_i,\ i = 1, ..., 5, Y,\ L,\ W(v, Y, L) \in \mathcal{W}(\alpha, Y, L)$ and $W(s, Y, L) \in \mathcal{W}(\alpha, Y, L)$, $\forall v, s \in \psi(\alpha)$, such that the following hold

$$\begin{bmatrix} \bar{\Pi} & d_2\bar{\mathcal{N}} & d_{12}\bar{\mathcal{M}} & d_{12}\bar{\mathcal{S}} \\ * & -d_2\bar{Z}_1 & 0 & 0 \\ * & * & -d_{12}(\bar{Z}_1 + \bar{Z}_2) & 0 \\ * & * & * & -d_{12}\bar{Z}_2 \end{bmatrix} < 0 \quad (6)$$

$$\begin{bmatrix} u_i & l_i \\ * & u_i\bar{P} \end{bmatrix} \ge 0, i = 1, ..., m \quad (7)$$

where $l_i$ denotes the $i$th row of $L$, $d_{12} \triangleq d_2 - d_1$ and

$$\bar{\Pi} \triangleq \begin{bmatrix} \bar{\Pi}_{11} & \bar{\Pi}_{12} & \bar{\Pi}_{13} & \bar{\Pi}_{14} & \bar{\Pi}_{15} \\ * & \bar{\Pi}_{22} & \bar{\Pi}_{23} & \bar{\Pi}_{24} & \bar{\Pi}_{25} \\ * & * & \bar{\Pi}_{33} & \bar{\Pi}_{34} & \bar{\Pi}_{35} \\ * & * & * & \bar{\Pi}_{44} & \bar{\Pi}_{45} \\ * & * & * & * & \bar{\Pi}_{55} \end{bmatrix}, \bar{\mathcal{N}} \triangleq \begin{bmatrix} \bar{N}_1 \\ \bar{N}_2 \\ \bar{N}_3 \\ \bar{N}_4 \\ \bar{N}_5 \end{bmatrix}$$

$$\bar{\mathcal{M}} \triangleq \begin{bmatrix} \bar{M}_1 \\ \bar{M}_2 \\ \bar{M}_3 \\ \bar{M}_4 \\ \bar{M}_5 \end{bmatrix}, \bar{\mathcal{S}} \triangleq \begin{bmatrix} \bar{S}_1 \\ \bar{S}_2 \\ \bar{S}_3 \\ \bar{S}_4 \\ \bar{S}_5 \end{bmatrix}$$

with

$$\begin{aligned}
\bar{\Pi}_{11} &\triangleq \sum\nolimits_{i=1}^{3} \bar{Q}_i + \bar{N}_1 + \bar{N}_1^T + AX + BW(v, Y, L) \\
&\quad + (AX + BW(v, Y, L))^T, \\
\bar{\Pi}_{12} &\triangleq \bar{N}_2^T - \bar{N}_1 + \bar{S}_1 - \bar{M}_1 + A_d X, \\
\bar{\Pi}_{13} &\triangleq \bar{M}_1 + \bar{N}_3^T, \quad \bar{\Pi}_{14} \triangleq -\bar{S}_1 + \bar{N}_4^T, \\
\bar{\Pi}_{15} &\triangleq \bar{N}_5^T - X + \bar{P} + (AX + BW(s, Y, L))^T, \\
\bar{\Pi}_{22} &\triangleq (\mu - 1)\bar{Q}_3 + \bar{S}_2 + \bar{S}_2^T - \bar{N}_2^T - \bar{N}_2 - \bar{M}_2 - \bar{M}_2^T, \\
\bar{\Pi}_{23} &\triangleq \bar{M}_2 - \bar{N}_3^T + \bar{S}_3^T - \bar{M}_3^T, \\
\bar{\Pi}_{24} &\triangleq -\bar{S}_2 - \bar{N}_4^T + \bar{S}_4^T - \bar{M}_4^T, \\
\bar{\Pi}_{25} &\triangleq -X + \bar{S}_5^T - \bar{N}_5^T - \bar{M}_5^T + XA_d^T, \\
\bar{\Pi}_{33} &\triangleq -\bar{Q}_1 + \bar{M}_3 + \bar{M}_3^T, \quad \bar{\Pi}_{34} \triangleq -\bar{S}_3 + \bar{M}_4^T, \\
\bar{\Pi}_{35} &\triangleq -X + \bar{M}_5^T, \quad \bar{\Pi}_{44} \triangleq -\bar{Q}_2 - \bar{S}_4 - \bar{S}_4^T, \\
\bar{\Pi}_{45} &\triangleq -\bar{S}_5^T - X, \quad \bar{\Pi}_{55} \triangleq d_2\bar{Z}_1 + d_{12}\bar{Z}_2 - 2X,
\end{aligned}$$

then the underlying closed-loop system is asymptotically stable and an estimate of the domain of attraction is given by $\Gamma_\delta \le 1$, where

$$\begin{aligned}
\Gamma_\delta &= \delta_1^2[\lambda_{\max}(X^{-1}\bar{P}X^{-1}) + d_1\lambda_{\max}(X^{-1}\bar{Q}_1X^{-1}) \\
&\quad + d_2\lambda_{\max}(X^{-1}\bar{Q}_2X^{-1}) + d_2\lambda_{\max}(X^{-1}\bar{Q}_3X^{-1})] \\
&\quad + \delta_2^2[\frac{1}{2}d_2^2\lambda_{\max}(X^{-1}\bar{Z}_1X^{-1}) \\
&\quad + \frac{(d_1 + d_2)d_{12}}{2}\lambda_{\max}(X^{-1}\bar{Z}_2X^{-1})]
\end{aligned} \quad (8)$$

**Proof.** See the Appendix for a sketch of proof and [13] for the full one.

*Remark 1:* Note that condition (6) contains $2^{2m}$ LMIs since that there are $2^m$ elements in the matrix $W(v,F,H)$ and $W(s,F,H)$, respectively, due to the special construction of $\mathcal{W}(\alpha,F,H)$ in (5).

*Remark 2:* From Theorem 1, it is seen that an optimization procedure can be proposed to maximize the initial conditions, i.e., to obtain a maximized estimate of domain of attraction. As the method commonly adopted in the literature, we also select $\delta_1 = \delta_2$ in (8), and an approximating optimization problem can be obtained as:

*P1:* $\min\ r$

s.t. (6), (7) and $\begin{bmatrix} w_1 I & I \\ * & X \end{bmatrix} \geq 0,$

$$w_2 I - \bar{P} \geq 0, w_3 I - \bar{Q}_1 \geq 0, w_4 I - \bar{Q}_2 \geq 0,$$
$$w_5 I - \bar{Q}_3 \geq 0, w_6 I - \bar{Z}_1 \geq 0, w_7 I - \bar{Z}_2 \geq 0,$$

where $r = \varepsilon * w_1 + w_2 + d_1 w_3 + d_2 w_4 + d_2 w_5 + \frac{(d_1+d_2)d_{12}}{2} w_7$.

In $P1$, $w_i$, $i = 1,...,7$, are the introduced variables for optimizing and $\varepsilon$ (determined iteratively) represents the relevant weighting in the optimization procedure. The reader can refer to [11] for more details on how the approximating optimization is realized. Then, a maximized estimate of domain of attraction can be obtained by $\delta_{\max} = 1/\sqrt{\Lambda}$, where

$$\begin{aligned}\Lambda &= \lambda_{\max}(X^{-1}\bar{P}X^{-1}) + d_1\lambda_{\max}(X^{-1}\bar{Q}_1X^{-1}) \\ &\quad + d_2\lambda_{\max}(X^{-1}\bar{Q}_2X^{-1}) + d_2\lambda_{\max}(X^{-1}\bar{Q}_3X^{-1}) \\ &\quad + \frac{1}{2}d_2^2\lambda_{\max}(X^{-1}\bar{Z}_1X^{-1}) \\ &\quad + \frac{(d_1+d_2)d_{12}}{2}\lambda_{\max}(X^{-1}\bar{Z}_2X^{-1})\end{aligned}$$

with $X > 0, \bar{P} > 0$, $\bar{Q}_i > 0, i = 1,2,3$, $\bar{Z}_i > 0, i = 1,2$ are the solution of *P1*.

*Remark 3:* In addition, the delay-range-dependent approach proposed here has been shown to be less conservative in stability analysis and more applicable in practice [5]. Therefore, in this paper, the developed results on control synthesis for time-delay systems with actuator saturation will present less conservatism compared with the existing results (using delay-independent [11] or delay-dependent approach [2], [3]).

To compare our derived results with the cases of systems with constant delay, we can select $d_1 = d_2 = d$ in (1) and readily obtain the following corollary.

*Corollary 1:* Consider system (1) with $d_1 = d_2 = d$ and $\mu = 0$. If there exist matrices $X > 0, \bar{P} > 0$, $\bar{Q} > 0$, $\bar{Z} > 0$, $\bar{N}_i$, $i = 1,...,3, Y$, $L$, $W(v,Y,L) \in \mathcal{W}(\alpha,Y,L)$ and $W(s,Y,L) \in \mathcal{W}(\alpha,Y,L)$, $\forall v,s \in \phi(\alpha)$, such that the following hold

$$\begin{bmatrix} \bar{\Pi} & d\bar{\mathcal{N}} \\ * & -d\bar{Z} \end{bmatrix} < 0 \tag{9}$$

$$\begin{bmatrix} u_i & l_i \\ * & u_i\bar{P} \end{bmatrix} \geq 0, i = 1,...,m \tag{10}$$

where $l_i$ denote the $i$th row of $L$ and

$$\bar{\Pi} \triangleq \begin{bmatrix} \bar{\Pi}_{11} & \bar{\Pi}_{12} & \bar{\Pi}_{13} \\ * & \bar{\Pi}_{22} & \bar{\Pi}_{23} \\ * & * & \bar{\Pi}_{33} \end{bmatrix}, \bar{\mathcal{N}} \triangleq \begin{bmatrix} \bar{N}_1 \\ \bar{N}_2 \\ \bar{N}_3 \end{bmatrix}$$

with

$$\begin{aligned}\bar{\Pi}_{11} &\triangleq \bar{Q} + \bar{N}_1 + \bar{N}_1^T + (AX + BW(v,Y,L)) \\ &\quad + (AX + BW(v,Y,L))^T \\ \bar{\Pi}_{12} &\triangleq \bar{N}_2^T - \bar{N}_1 + A_dX,\ \bar{\Pi}_{22} \triangleq -\bar{Q} - \bar{N}_2^T - \bar{N}_2, \\ \bar{\Pi}_{13} &\triangleq \bar{N}_3^T - X + (AX + BW(s,Y,L))^T + \bar{P} \\ \bar{\Pi}_{23} &\triangleq XA_d^T - \bar{N}_3^T,\ \bar{\Pi}_{33} \triangleq d\bar{Z} - 2X\end{aligned}$$

then the underlying closed-loop system is asymptotically stable and an estimate of the domain of attraction is given by

$$\begin{aligned}&\delta_1^2\left[\lambda_{\max}(X^{-1}\bar{P}X^{-1}) + d_1\lambda_{\max}(X^{-1}\bar{Q}X^{-1})\right] \\ &\quad + \delta_2^2\left[\frac{1}{2}d_2^2\lambda_{\max}(X^{-1}\bar{Z}X^{-1})\right] \leq 1.\end{aligned} \tag{11}$$

Moreover, the stabilizing feedback controller gain is given by $F = YX^{-1}$.

The proof of this corollary can be completed following the similar lines as for Theorem 1. We omit it here for brevity. Also, selecting $\delta_1 = \delta_2$ in (11), and the corresponding approximating optimization problem becomes:

*P2:*

$$\min\ r$$

s.t. (9), (10) and $\begin{bmatrix} w_1 I & I \\ * & X \end{bmatrix} \geq 0$, $w_2I - \bar{P} \geq 0$, $w_3 I - \bar{Q} \geq 0$, $w_4 I - \bar{Z} \geq 0$, where $r = \varepsilon * w_1 + w_2 + dw_3 + \frac{1}{2}d_2^2 w_4$. Then, a maximized estimate of domain of attraction can be obtained by $\delta_{\max} = 1/\sqrt{\Lambda}$, where

$$\begin{aligned}\Lambda &= \lambda_{\max}(X^{-1}\bar{P}X^{-1}) + d\lambda_{\max}(X^{-1}\bar{Q}X^{-1}) \\ &\quad + \frac{1}{2}d^2\lambda_{\max}(X^{-1}\bar{Z}X^{-1})\end{aligned}$$

with $X > 0, \bar{P} > 0$, $\bar{Q} > 0, \bar{Z} > 0$ are the solution of *P2*.

## IV. Numerical Example

Now let us consider the following two illustrative examples to show the importance of our results. The first one is provided to check the validness of the results dealing with time-varying delays with ranges, while the second one is borrowed from [2] and [3] to show the less conservatism of the stabilization results.

*Example 1:* Consider a linear state-delayed system (1) with the following matrices:

$$A = \begin{bmatrix} -4.6 & -2.5 \\ 1 & 0.9 \end{bmatrix}, A_d = \begin{bmatrix} 0.3 & 1 \\ 0.1 & -0.4 \end{bmatrix}, B = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

Our purpose here is to design a stabilizing controller for different time-delay range and estimate the domain of attraction for the above system. Now given a delay variation rate $\mu = 1.5$, $u_i = 1$ and by solving *P1 (with* $\varepsilon = 10^4$*)*, we first obtain the upper bound $d_2$ of the time-varying delays when a lower bound is given $(e.g.\ d_1 = 1)$, the different controller gains and the corresponding estimations of the domain of attraction are also obtained. The detailed results are listed in Table 1.

TABLE I

CONTROLLER GAINS CORRESPONDING TO DIFFERENT DELAY RANGES

| $d(t)$ | $\delta_{\max}$ | *Controller gains* |
|---|---|---|
| $0.10 \leq d(t) \leq 0.50$ | 7.52 | $[-0.11\ -0.07]$ |
| $1.0 \leq d(t) \leq 2.0$ | 3.81 | $[-0.11\ -0.17]$ |
| $1.0 \leq d(t) \leq 3.0$ | 2.17 | $[-0.13\ -0.26]$ |
| $1.0 \leq d(t) \leq 3.906$ (*upper bound*) | 0.31 | $[-0.24\ -1.77]$ |

Furthermore, given delay ranges $1 \leq d(t) \leq 3$ and $1 \leq d(t) \leq 3.906$, Fig. 1 illustrates the estimate of the corresponding domain of attraction and trajectories of system states starting from the initial conditions on the margin of the circles. It is clearly observed from Fig. 1 that the state of the examined system converges to origin within the estimated domain of attraction despite actuator saturation and the time-varying delays within different ranges.

*Example 2:* Consider the following linear delay system (1) with:

$$A = \begin{bmatrix} 0.5 & -1 \\ 0.5 & -0.5 \end{bmatrix},\ A_d = \begin{bmatrix} 0.6 & 0.4 \\ 0 & -0.5 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and $u_i = 5$.

Given $d_1 = d_2 = d$ as a constant delay, and solving $P2$ (with $\varepsilon = 10^3$), we obtain a maximal admissible delay bound equal to $d = 2.248$, $\delta_{\max} = 0.3272$ and the corresponding stabilizing controller gain $F = [-2.82\ \ 0.21]$. Table 2 gives the detailed comparison of $\delta_{\max}$with the results in [2] and [3]:

TABLE II

COMPUTATION RESULTS OF EXAMPLE 2

| Methods | $d = 0.35$ | $d = 1.0$ | $d = 1.854$ | $d = 2.248$ |
|---|---|---|---|---|
| Theorem 5 in [2] | 0.968 | infeasible | infeasible | infeasible |
| Theorem 1 in [3] | 2.852 | 1.7442 | 0.091 | infeasible |
| Corollary 1 | 6.0044 | 2.4571 | 0.4521 | 0.3272 |

From the above two examples, one can see that our derived results can not only solve the stabilization problems for the systems involving time-varying delays with ranges and actuator saturation based on the advanced delay-range-dependent stability ideas, but also present much less conservatism for upper bound of delay and for the estimate of domain of attraction.

## V. CONCLUSIONS

The control synthesis problem for a class of linear time-delay systems with actuator saturation is investigated in this paper. The time-delay is considered to be time-varying and has a lower and upper bounds. A delay-range-dependent approach is used and the corresponding LMI-based stabilizing state-feedback controller is derived. The domain of attraction of the origin can be further estimated for the underlying systems with different time-delay ranges.

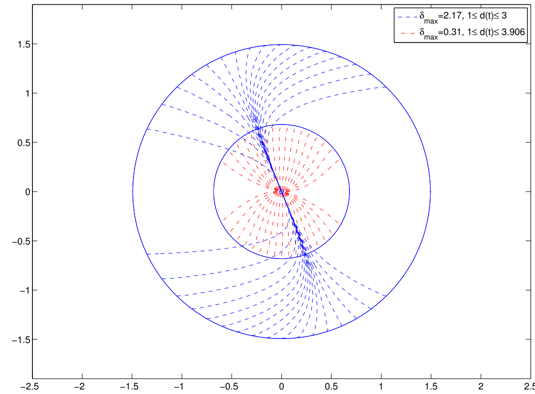


Fig. 1. Estimates of the domain of attraction for different delay ranges

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## VI. APPENDIX

**Proof sketch of Theorem 1**. Constructing a Lyapunov functional for the system as follows

$$\begin{aligned}\mathbf{V}(x_t) &\triangleq x^T(t)Px(t) + \sum_{i=1}^{2}\int_{t-d_i}^{t} x^T(s)Q_ix(s)ds \\ &+ \int_{t-d(t)}^{t} x^T(s)Q_3x(s)ds + \int_{-d_2}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z_1\dot{x}(s)dsd\theta \\ &+ \int_{-d_2}^{-d_1}\int_{t+\theta}^{t} \dot{x}^T(s)Z_2\dot{x}(s)dsd\theta \qquad (12)\end{aligned}$$

where $P = X^{-1}\bar{P}X^{-1}, Q_i = X^{-1}\bar{Q}_iX^{-1}, i = 1,2,3, Z_i = X^{-1}\bar{Z}_iX^{-1}, i = 1,2$. Then we have

$$\begin{aligned}\dot{\mathbf{V}}(x_t) &= 2x^T(t)P\dot{x}(t) \\ &+ \sum_{i=1}^{2}\{x(t)Q_ix(t) - x^T(t-d_i)Q_ix(t-d_i)\} \\ &+x^T(t)Q_3x(t) + d_2\dot{x}(t)Z_1\dot{x}(t) - \int_{t-d_2}^{t-d_1} \dot{x}(s)Z_2\dot{x}(s)ds \\ &-(1-\dot{d}(t))x^T(t-d(t))Q_3x(t-d(t)) \\ &- \int_{t-d_2}^{t} \dot{x}(s)Z_1\dot{x}(s)ds + (d_2-d_1)\dot{x}(t)Z_2\dot{x}(t)\end{aligned}$$

Note that the following equations are true for matrix $T > 0$ and any matrices $N_i$, $S_i$ and $M_i$, $i = 1, ..., 5$ with appropriate dimensions:

$$\begin{aligned}2\mathcal{N}_{\mathcal{X}}[x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s)ds] &= 0 \\ 2\mathcal{S}_{\mathcal{X}}[x(t-d(t)) - x(t-d_2) - \int_{t-d_2}^{t-d(t)} \dot{x}(s)ds] &= 0 \\ 2\mathcal{M}_{\mathcal{X}}[x(t-d_1) - x(t-d(t)) - \int_{t-d(t)}^{t-d_1} \dot{x}(s)ds] &= 0 \\ 2\mathcal{T}_{\mathcal{X}}[-\dot{x}(t) + Ax(t) + A_dx(t-d(t)) + B\sigma(Fx)] &= 0\end{aligned}$$

where $\mathcal{T}_{\mathcal{X}} = [x^T(t)T + \dot{x}^T(t)T]$ and

$$\begin{aligned}\mathcal{N}_{\mathcal{X}} &= [x^T(t)N_1 + x^T(t-d(t))N_2 + x^T(t-d_1)N_3 \\ &\quad +x^T(t-d_2)N_4 + \dot{x}^T(t)N_5] \\ \mathcal{S}_{\mathcal{X}} &= [x^T(t)S_1 + x^T(t-d(t))S_2 + x^T(t-d_1)S_3 \\ &\quad +x^T(t-d_2)S_4 + \dot{x}^T(t)S_5] \\ \mathcal{M}_{\mathcal{X}} &= [x^T(t)M_1 + x^T(t-d(t))M_2 + x^T(t-d_1)M_3 \\ &\quad +x^T(t-d_2)M_4 + \dot{x}^T(t)M_5]\end{aligned}$$

Then using these relations and similar techniques dealing with the constrained control in [6], we can obtain:

$$\begin{aligned}&\dot{\mathbf{V}}(x_t) \le \zeta^T(t)[\Pi + d_2\mathcal{N}Z_1^{-1}\mathcal{N}^T \\ &+d_{12}\mathcal{S}(Z_1+Z_2)^{-1}\mathcal{S}^T + d_{12}\mathcal{M}Z_2^{-1}\mathcal{M}^T]\zeta(t) \\ &- \int_{t-d(t)}^{t}[\zeta^T(t)\mathcal{N} + \dot{x}(s)Z_1]Z_1^{-1} \times \\ &[\zeta^T(t)\mathcal{N} + \dot{x}(s)Z_1]^T ds \\ &- \int_{t-d_2}^{t-d(t)}[\zeta^T(t)\mathcal{S} + \dot{x}^T(s)(Z_1+Z_2)] \times \\ &(Z_1+Z_2)^{-1}[\zeta^T(t)\mathcal{S} + \dot{x}^T(s)(Z_1+Z_2)]^T ds \\ &- \int_{t-d(t)}^{t-d_1}[\zeta^T(t)\mathcal{M} + \dot{x}(s)Z_2]Z_2^{-1} \times \\ &[\zeta^T(t)\mathcal{M} + \dot{x}(s)Z_2]^T ds \\ \le\ &\zeta^T(t)[\Pi + d_2\mathcal{N}Z_1^{-1}\mathcal{N}^T + d_{12}\mathcal{S}(Z_1+Z_2)^{-1}\mathcal{S}^T \\ &+d_{12}\mathcal{M}Z_2^{-1}\mathcal{M}^T]\zeta(t)\end{aligned}$$

By Schur complement, $\Pi + \mathcal{A}^T(d_2Z_1 + d_{12}Z_2)\mathcal{A} + d_2\mathcal{N}Z_1^{-1}\mathcal{N}^T + d_{12}\mathcal{S}(Z_1+Z_2)^{-1}\mathcal{S}^T + d_{12}\mathcal{M}Z_2^{-1}\mathcal{M}^T < 0$ is equivalent to

$$\begin{bmatrix} \Pi & d_2\mathcal{N} & d_{12}\mathcal{M} & d_{12}\mathcal{S} \\ * & -d_2Z_1 & 0 & 0 \\ * & * & -d_{12}(Z_1+Z_2) & 0 \\ * & * & * & -d_{12}Z_2 \end{bmatrix} < 0 \qquad (13)$$

Note that (6) is equivalent to (13) by setting $X = T^{-1}$ , $\bar{\mathcal{N}} = X\mathcal{N}X$, $\bar{\mathcal{M}} = X\mathcal{M}X$, $\bar{\mathcal{S}} = X\mathcal{S}X$, $\bar{Z}_1 = XZ_1X$, $\bar{Z}_2 = XZ_2X$, $Y = FX$ and performing a congruence transformation to (13) via $diag\{X, X, X, X, X, X, X, X\}$, which means that (13) holds if (6) is satisfied. Then we have $\dot{\mathbf{V}}(x_t) < -e\|x_t\|^2$ for a sufficiently small $e > 0$, and accordingly,

$$\begin{aligned}x^TPx &\le \mathbf{V}(x_t) < \mathbf{V}(x_0) \\ &\le \max_{\theta\in[-d_2,0]}|\phi(\theta)|^2[\lambda_{\max}(P) + d_1\lambda_{\max}(Q_1) \\ &\quad +d_2\lambda_{\max}(Q_2) + d_2\lambda_{\max}(Q_3)] \\ &\quad + \max_{\theta\in[-d_2,0]}|\dot{\phi}(\theta)|^2[\frac{1}{2}d_2^2\lambda_{\max}(Z_1) \\ &\quad +\frac{(d_1+d_2)d_{12}}{2}\lambda_{\max}(Z_2)] \\ &= \Gamma(\phi,\dot{\phi})\end{aligned}$$

thus if the set $\Gamma(\phi,\dot{\phi}) \le 1$, we readily know that $x^TPx \le 1$ and all the trajectories of $x(t)$ that start from $\Gamma(\phi,\dot{\phi}) \le 1$ remain within $x^TPx \le 1$ and thereby the control constraints $|h_ix| \le u_i$ are also satisfied due to (7). Meantime, the controller gain is given by $F = YX^{-1}$, and the estimate of the domain of attraction can be obtained from $\Gamma(\phi,\dot{\phi}) \le 1$, namely $\Gamma_\delta \le 1$ and this completes the proof. □