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Decentralized Control Using Reduced-Order Unknown Input Observers

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***Abstract*— The output feedback stabilization problem of a class of nonlinear interconnected systems is considered. A novel decentralized dynamic output feedback controller is proposed, where local, projection operator based, reduced-order observers are employed to estimate the subsystems' states. The proposed design algorithm is characterized by the separation property of the observer-controller design. The closed-loop system driven by the proposed decentralized dynamic output feedback controller is asymptotically stable. The effectiveness of the proposed control strategy is illustrated with simulation examples.**

I. INTRODUCTION

Decentralized control uses local information available at each subsystem level for the controller implementation. This feature overcomes the limitations of centralized control, which requires sufficiently large communication bandwidth. Moreover, decentralized controllers are simpler and more practical than centralized controllers. Most of the proposed decentralized control strategies assume the availability of the subsystems' states; see, for example, [1], [2]. However, the availability of the states of each subsystem cannot be guaranteed in practice, which restricts the applications of decentralized state feedback controllers. This motivated the development of decentralized output feedback controllers that incorporate local observers to estimate the states of the subsystems; see, for example, [3]–[5].

In this paper, we consider the stabilization problem of a class of large-scale interconnected systems modeled by

$$\dot{\boldsymbol{x}}_i = \boldsymbol{A}_i\boldsymbol{x}_i + \boldsymbol{B}_{i1}\boldsymbol{u}_{i1} + \boldsymbol{B}_{i2}\boldsymbol{u}_{i2}(\boldsymbol{x}), \tag{1}$$

$$\boldsymbol{y}_i = \boldsymbol{C}_i\boldsymbol{x}_i, \qquad i = 1, \ldots, N \tag{2}$$

where $\boldsymbol{x}_i \in \mathbb{R}^{n_i}$, $\boldsymbol{u}_{i1} \in \mathbb{R}^{m_{i1}}$, $\boldsymbol{y}_i \in \mathbb{R}^{p_i}$ are the state, input and output vectors, respectively, of the i-th subsystem, $\boldsymbol{x} = [\boldsymbol{x}_1^\top \ \cdots \ \boldsymbol{x}_N^\top]^\top \in \mathbb{R}^n$ is the state vector of the whole system with $n = \sum_{i=1}^N n_i$, and $\boldsymbol{u}_{i2}(\boldsymbol{x})$ models the unknown interconnection of the i-th subsystem with other subsystems. Several decentralized dynamic output feedback control strategies have been recently proposed for the above nonlinear interconnected system in [4]–[7], where the controller design has certain degree of interdependence with the observer design. In [5], [7], decentralized dynamic output feedback controllers were developed based on the distance to uncontrollable (unobservable) pair of matrices. In [4], [6], the controller and the observer designs were formulated together in the linear matrix inequalities framework, and then the resulting convex optimization problem was solved to obtain the parameters of the controller and the observer. However, the local observers used in [4]–[7] are full-order observers. In [4]–[6], local full-order Luenberger-type observers were used, while in [7], local full-order sliding mode observers were employed. The application of local reduced-order Luenberger-type observers has been investigated in [8]. However, the controller and the observer designs in [8] are still interdependent.

In this paper, we propose a decentralized dynamic output feedback controller that incorporates local projection operator based reduced-order observers to estimate the subsystems' states, for which the observer design is independent of the controller design. The closed-loop system driven by the proposed decentralized compensator is guaranteed to be asymptotically stable.

II. PRELIMINARIES

We consider the nonlinear interconnected system described by (1) and (2), where $\boldsymbol{B}_{i1} \in \mathbb{R}^{n_i \times m_{i1}}$ ($m_{i1} \leq n_i$) is of full rank, the pair $(\boldsymbol{A}_i, \boldsymbol{B}_{i1})$ is controllable, and the pair $(\boldsymbol{A}_i, \boldsymbol{C}_i)$ is observable. The interconnection of each subsystem satisfies the following quadratic constraint as, for example, in [4], [5],

$$(\boldsymbol{B}_{i2}\boldsymbol{u}_{i2})^\top (\boldsymbol{B}_{i2}\boldsymbol{u}_{i2}) \leq \alpha_i^2 \boldsymbol{x}^\top \boldsymbol{\Gamma}_i^\top \boldsymbol{\Gamma}_i \boldsymbol{x}, \tag{3}$$

where α_i is a known positive constant and $\boldsymbol{\Gamma}_i \in \mathbb{R}^{n_i \times n}$ is a known interconnection matrix. We assume that $\operatorname{rank}(\boldsymbol{B}_{i2}) = \operatorname{rank}(\boldsymbol{C}_i\boldsymbol{B}_{i2}) = m_{i2}$ ($m_{i2} \leq p_i$) and the system zeros of the system model given by the triple $(\boldsymbol{A}_i, \boldsymbol{B}_{i2}, \boldsymbol{C}_i)$ are in the open left-hand complex plane, that is,

$$\operatorname{rank} \begin{bmatrix} s\boldsymbol{I}_{n_i} - \boldsymbol{A}_i & \boldsymbol{B}_{i2} \\ \boldsymbol{C}_i & \boldsymbol{O} \end{bmatrix} = n_i + m_{i2}, \tag{4}$$

for all s such that $\Re(s) \geq 0$.

The control objective is to design controllers $\boldsymbol{u}_{i1}$ that stabilize the closed-loop system under the assumption that only the system outputs $\boldsymbol{y}_i$ are available. In the following, we first design local, projection based reduced-order observers to obtain asymptotic estimates $\hat{\boldsymbol{x}}_i$ of each subsystem's state vector $\boldsymbol{x}_i$. Then we propose and analyze a decentralized dynamic output feedback controller of the form,

$$\boldsymbol{u}_{i1} = \boldsymbol{K}_i\hat{\boldsymbol{x}}_i = (\boldsymbol{K}_{i1} + \boldsymbol{K}_{i2})\,\hat{\boldsymbol{x}}_i, \tag{5}$$

where $\hat{\boldsymbol{x}}_i$ is an asymptotic estimate of the i-th subsystem's state vector, $\boldsymbol{K}_{i1}$ is a chosen pre-feedback gain matrix so that $\boldsymbol{A}_{ci} = \boldsymbol{A}_i + \boldsymbol{B}_{i1}\boldsymbol{K}_{i1}$ is Hurwitz, and $\boldsymbol{K}_{i2}$ is a feedback gain matrix to be defined later. In the next section, we first design a local projection operator based reduced-order observer to obtain an asymptotic estimate $\hat{\boldsymbol{x}}_i$.

III. Local Reduced-Order Observer Design

In this section, we use the projection operator based reduced-order observer, first introduced in [9], to design local observers for the subsystems. For the i-th subsystem, we can decompose the state vector $\boldsymbol{x}_i$ as

$$\boldsymbol{x}_i = (\boldsymbol{I}_{n_i} - \boldsymbol{M}_i\boldsymbol{C}_i)\boldsymbol{x}_i + \boldsymbol{M}_i\boldsymbol{C}_i\boldsymbol{x}_i = \boldsymbol{q}_i + \boldsymbol{M}_i\boldsymbol{y}_i,$$

where $\boldsymbol{M}_i \in \mathbb{R}^{n_i\times p_i}$ and $\boldsymbol{q}_i = (\boldsymbol{I}_{n_i} - \boldsymbol{M}_i\boldsymbol{C}_i)\boldsymbol{x}_i$. If $\boldsymbol{M}_i$ is chosen so that

$$(\boldsymbol{I}_{n_i} - \boldsymbol{M}_i\boldsymbol{C}_i)\boldsymbol{B}_{i2} = \boldsymbol{O}, \tag{6}$$

then we have

$$\begin{aligned}\dot{\boldsymbol{q}}_i &= (\boldsymbol{I}_{n_i} - \boldsymbol{M}_i\boldsymbol{C}_i)\dot{\boldsymbol{x}}_i \\ &= (\boldsymbol{I}_{n_i} - \boldsymbol{M}_i\boldsymbol{C}_i)(\boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{B}_{i1}\boldsymbol{u}_{i1} + \boldsymbol{B}_{i2}\boldsymbol{u}_{i2}) \\ &= (\boldsymbol{I}_{n_i} - \boldsymbol{M}_i\boldsymbol{C}_i)(\boldsymbol{A}_i\boldsymbol{q}_i + \boldsymbol{A}_i\boldsymbol{M}_i\boldsymbol{y}_i + \boldsymbol{B}_{i1}\boldsymbol{u}_{i1}). \end{aligned}\tag{7}$$

If $\boldsymbol{q}_i(t_0) = (\boldsymbol{I}_{n_i} - \boldsymbol{M}_i\boldsymbol{C}_i)\boldsymbol{x}_i(t_0)$, then we have $\boldsymbol{x}_i(t) = \boldsymbol{q}_i(t) + \boldsymbol{M}_i\boldsymbol{y}_i(t)$ for $t \geq t_0$, where $\boldsymbol{q}_i(t)$ is obtained by solving (7). However, because $\boldsymbol{x}_i(t_0)$ is assumed to be unknown, $\hat{\boldsymbol{x}}_i = \boldsymbol{q}_i + \boldsymbol{M}_i\boldsymbol{y}_i$ is only an estimate of $\boldsymbol{x}_i$. Thus, in order to ensure convergence or improve the convergence rate, we add an extra term to the right-hand side of (7) to obtain

$$\begin{aligned}\dot{\boldsymbol{q}}_i &= (\boldsymbol{I}_{n_i} - \boldsymbol{M}_i\boldsymbol{C}_i)\left(\boldsymbol{A}_i\boldsymbol{q}_i + \boldsymbol{A}_i\boldsymbol{M}_i\boldsymbol{y}_i + \boldsymbol{B}_{i1}\boldsymbol{u}_{i1}\right. \\ &\qquad \left. + \ \boldsymbol{L}_i(\boldsymbol{C}_i\boldsymbol{q}_i + \boldsymbol{C}_i\boldsymbol{M}_i\boldsymbol{y}_i - \boldsymbol{y}_i)\right) \\ &= (\boldsymbol{I}_{n_i} - \boldsymbol{M}_i\boldsymbol{C}_i)\left(\boldsymbol{A}_i\boldsymbol{q}_i + \boldsymbol{A}_i\boldsymbol{M}_i\boldsymbol{y}_i + \boldsymbol{B}_{i1}\boldsymbol{u}_{i1}\right. \\ &\qquad \left. + \ \boldsymbol{L}_i\boldsymbol{C}_i(\boldsymbol{q}_i + \boldsymbol{M}_i\boldsymbol{y}_i - \boldsymbol{x}_i)\right). \end{aligned}\tag{8}$$

Let $\boldsymbol{e}_i = \hat{\boldsymbol{x}}_i - \boldsymbol{x}_i$ be the estimation error of the i-th subsystem's state vector. Taking into account (1), (6) and (8), we obtain the equation governing the estimation error dynamics,

$$\begin{aligned}\dot{\boldsymbol{e}}_i &= \dot{\hat{\boldsymbol{x}}}_i - \dot{\boldsymbol{x}}_i \\ &= \dot{\boldsymbol{q}}_i - (\boldsymbol{I}_{n_i} - \boldsymbol{M}_i\boldsymbol{C}_i)\dot{\boldsymbol{x}}_i \\ &= (\boldsymbol{I}_{n_i} - \boldsymbol{M}_i\boldsymbol{C}_i)(\boldsymbol{A}_i + \boldsymbol{L}_i\boldsymbol{C}_i)(\boldsymbol{q}_i + \boldsymbol{M}_i\boldsymbol{C}_i\boldsymbol{x}_i - \boldsymbol{x}_i) \\ &= (\boldsymbol{I}_{n_i} - \boldsymbol{M}_i\boldsymbol{C}_i)\left(\boldsymbol{A}_i + \boldsymbol{L}_i\boldsymbol{C}_i\right)\boldsymbol{e}_i. \end{aligned}\tag{9}$$

It follows that $(\boldsymbol{C}_i\boldsymbol{B}_{i2})^\dagger(\boldsymbol{C}_i\boldsymbol{B}_{i2}) = \boldsymbol{I}_{m_{i2}}$, where the superscript $\dagger$ denotes the Moore-Penrose pseudo inverse. A general solution to (6) is given by

$$\boldsymbol{M}_i = \boldsymbol{B}_{i2}\left((\boldsymbol{C}_i\boldsymbol{B}_{i2})^\dagger + \boldsymbol{Z}_i\left(\boldsymbol{I}_{p_i} - (\boldsymbol{C}_i\boldsymbol{B}_{i2})(\boldsymbol{C}_i\boldsymbol{B}_{i2})^\dagger\right)\right), \tag{10}$$

where $\boldsymbol{Z}_i \in \mathbb{R}^{m_{i2}\times p_i}$ is a design parameter matrix. Let

$$\boldsymbol{\Pi}_i = \boldsymbol{I}_{n_i} - \boldsymbol{M}_i\boldsymbol{C}_i. \tag{11}$$

It follows from (10) that $\boldsymbol{\Pi}_i = \boldsymbol{\Pi}_i^2$, so $\boldsymbol{\Pi}_i$ is a projection matrix. Thus, there exist an invertible matrix $\boldsymbol{T}_i$ whose columns are eigenvectors of $\boldsymbol{\Pi}_i$ such that

$$\boldsymbol{T}_i^{-1}\boldsymbol{\Pi}_i\boldsymbol{T}_i = \begin{bmatrix} \boldsymbol{I}_{n_i-m_{i2}} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}. \tag{12}$$

A proof of the above fact can be found in [10, pp. 156–158 and pp. 194–195].

Remark 1: Usually, in order to find the invertible matrix $\boldsymbol{T}_i$, a common way is to compute the eigenvectors of $\boldsymbol{\Pi}_i$ directly. However, this approach is numerically unstable. An efficient and numerically stable way of constructing $\boldsymbol{T}_i$ can be found in the proof of Theorem 2 of [9]. We summarize the method here. Let

$$\boldsymbol{F}_i = (\boldsymbol{C}_i\boldsymbol{B}_{i2})^\dagger + \boldsymbol{Z}_i\left(\boldsymbol{I}_{p_i} - (\boldsymbol{C}_i\boldsymbol{B}_{i2})(\boldsymbol{C}_i\boldsymbol{B}_{i2})^\dagger\right),$$

and $\boldsymbol{S}_i = \boldsymbol{F}_i\boldsymbol{C}_i$. It is easy to verify that $\boldsymbol{S}_i\boldsymbol{B}_{i2} = \boldsymbol{I}_{m_{i2}}$, so $\text{rank}\boldsymbol{S} = m_{i2}$. Thus we can find a full rank matrix $\boldsymbol{W}_i \in \mathbb{R}^{n_i\times(n_i-m_{i2})}$ such that $\boldsymbol{S}_i\boldsymbol{W}_i = \boldsymbol{O}$. Then, we can choose the invertible matrix $\boldsymbol{T}_i$ to be $\boldsymbol{T}_i = [\boldsymbol{W}_i \ \ \boldsymbol{B}_{i2}]$, which is shown to satisfy (12) in [9].

Consider now the following coordinate transformation,

$$\tilde{\boldsymbol{e}}_i = \boldsymbol{T}_i^{-1}\boldsymbol{e}_i. \tag{13}$$

It follows from (9) that

$$\begin{aligned}\dot{\tilde{\boldsymbol{e}}}_i &= \boldsymbol{T}_i^{-1}\boldsymbol{\Pi}_i\boldsymbol{T}_i\boldsymbol{T}_i^{-1}(\boldsymbol{A}_i + \boldsymbol{L}_i\boldsymbol{C}_i)\boldsymbol{T}_i\tilde{\boldsymbol{e}}_i \\ &= \boldsymbol{T}_i^{-1}\boldsymbol{\Pi}_i\boldsymbol{T}_i\left(\tilde{\boldsymbol{A}}_i + \tilde{\boldsymbol{L}}_i\tilde{\boldsymbol{C}}_i\right)\tilde{\boldsymbol{e}}_i, \end{aligned}\tag{14}$$

where

$$\tilde{\boldsymbol{A}}_i = \boldsymbol{T}_i^{-1}\boldsymbol{A}_i\boldsymbol{T}_i = \begin{bmatrix} \tilde{\boldsymbol{A}}_{i_{11}} & \tilde{\boldsymbol{A}}_{i_{12}} \\ \tilde{\boldsymbol{A}}_{i_{21}} & \tilde{\boldsymbol{A}}_{i_{22}} \end{bmatrix}, \tag{15}$$

$$\tilde{\boldsymbol{L}}_i = \boldsymbol{T}_i^{-1}\boldsymbol{L}_i = \begin{bmatrix} \tilde{\boldsymbol{L}}_{i_1} \\ \tilde{\boldsymbol{L}}_{i_2} \end{bmatrix}, \tag{16}$$

$$\tilde{\boldsymbol{C}}_i = \boldsymbol{C}_i\boldsymbol{T}_i = \begin{bmatrix} \tilde{\boldsymbol{C}}_{i_1} & \tilde{\boldsymbol{C}}_{i_2} \end{bmatrix}, \tag{17}$$

with $\tilde{\boldsymbol{A}}_{i_{11}} \in \mathbb{R}^{(n_i-m_{i2})\times(n_i-m_{i2})}$, $\tilde{\boldsymbol{L}}_{i_1} \in \mathbb{R}^{(n_i-m_{i2})\times p_i}$, $\tilde{\boldsymbol{C}}_{i_1} \in \mathbb{R}^{p_i\times(n_i-m_{i2})}$. Let

$$\tilde{\boldsymbol{e}}_i = \begin{bmatrix} \tilde{\boldsymbol{e}}_{i_1}^\top & \tilde{\boldsymbol{e}}_{i_2}^\top \end{bmatrix}^\top, \tag{18}$$

with $\tilde{\boldsymbol{e}}_{i_1} \in \mathbb{R}^{n_i-m_{i2}}$. Using the above notation (15)–(18) and (12), we can represent (14) as

$$\begin{bmatrix} \dot{\tilde{\boldsymbol{e}}}_{i_1} \\ \dot{\tilde{\boldsymbol{e}}}_{i_2} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{A}}_{i_{11}} + \tilde{\boldsymbol{L}}_{i_1}\tilde{\boldsymbol{C}}_{i_1} & \tilde{\boldsymbol{A}}_{i_{12}} + \tilde{\boldsymbol{L}}_{i_1}\tilde{\boldsymbol{C}}_{i_2} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{e}}_{i_1} \\ \tilde{\boldsymbol{e}}_{i_2} \end{bmatrix}. \tag{19}$$

It is shown in [9] that if we choose

$$\boldsymbol{q}_i(t_0) = (\boldsymbol{I}_{n_i} - \boldsymbol{M}_i\boldsymbol{C}_i)\boldsymbol{v}_i, \tag{20}$$

for arbitrary $\boldsymbol{v}_i \in \mathbb{R}^{n_i}$, then $\tilde{\boldsymbol{e}}_{i_2}(t) = \boldsymbol{0}$ for all $t \geq t_0$. Then, the dynamics of the estimation error $\tilde{\boldsymbol{e}}_i$ are completely determined by the dynamics of $\tilde{\boldsymbol{e}}_{i_1}$, that is,

$$\dot{\tilde{\boldsymbol{e}}}_{i_1} = \left(\tilde{\boldsymbol{A}}_{i_{11}} + \tilde{\boldsymbol{L}}_{i_1}\tilde{\boldsymbol{C}}_{i_1}\right)\tilde{\boldsymbol{e}}_{i_1}. \tag{21}$$

Thus, if $\tilde{\boldsymbol{L}}_{i_1}$ is chosen such that the matrix $\tilde{\boldsymbol{A}}_{i_{11}} + \tilde{\boldsymbol{L}}_{i_1}\tilde{\boldsymbol{C}}_{i_1}$ is Hurwitz, then we have $\tilde{\boldsymbol{e}}_{i1}(t) \to \boldsymbol{0}$ as $t \to \infty$. It is also shown in [9] that the detectability of the pair $(\boldsymbol{A}_i, \boldsymbol{C}_i)$ guarantees the detectability of the pair $(\tilde{\boldsymbol{A}}_{i_{11}}, \tilde{\boldsymbol{C}}_{i_1})$. Thus, we can choose $\boldsymbol{q}_i(t_0)$ satisfying (20) and

$$\boldsymbol{L}_i = \boldsymbol{T}_i\begin{bmatrix} \tilde{\boldsymbol{L}}_{i_1} \\ \boldsymbol{O} \end{bmatrix} \tag{22}$$

such that the estimation error $\boldsymbol{e}_i = \hat{\boldsymbol{x}}_i - \boldsymbol{x}_i$, where $\hat{\boldsymbol{x}}_i = \boldsymbol{q}_i + \boldsymbol{M}_i\boldsymbol{y}_i$, of the full-order observer described by (8) will asymptotically converge to zero as $t \to \infty$.

Note that the dynamics of the estimation error $\tilde{\boldsymbol{e}}_i$ are completely determined by the $(n_i - m_{i2})$-dimensional dynamics given by (21). Therefore, we can construct a reduced-order observer with only $(n_i - m_{i2})$ observer states. Applying the transformation from $\boldsymbol{e}_i$ to $\tilde{\boldsymbol{e}}_i$ to $\boldsymbol{q}_i$, we obtain

$$\tilde{\boldsymbol{q}}_i = \boldsymbol{T}_i^{-1}\boldsymbol{q}_i. \tag{23}$$

It follows from (23) and (8) that

$$\begin{aligned}\dot{\tilde{\boldsymbol{q}}}_i &= \boldsymbol{T}_i^{-1}\boldsymbol{\Pi}_i\boldsymbol{T}_i\boldsymbol{T}_i^{-1}\left(\boldsymbol{A}_i\boldsymbol{T}_i\tilde{\boldsymbol{q}}_i + \boldsymbol{A}_i\boldsymbol{M}_i\boldsymbol{y}_i + \boldsymbol{B}_{i1}\boldsymbol{u}_{i1}\right.\\ &\qquad \left. +\ \boldsymbol{L}_i(\boldsymbol{C}_i\boldsymbol{T}_i\tilde{\boldsymbol{q}}_i + \boldsymbol{C}_i\boldsymbol{M}_i\boldsymbol{y}_i - \boldsymbol{y}_i)\right)\\ &= \boldsymbol{T}_i^{-1}\boldsymbol{\Pi}_i\boldsymbol{T}_i\left(\boldsymbol{T}_i^{-1}\boldsymbol{A}_i\boldsymbol{T}_i + \boldsymbol{T}_i^{-1}\boldsymbol{L}_i\boldsymbol{C}_i\boldsymbol{T}_i\right)\tilde{\boldsymbol{q}}_i\\ &\quad +\ \boldsymbol{T}_i^{-1}\boldsymbol{\Pi}_i\boldsymbol{T}_i\boldsymbol{T}_i^{-1}\left(\boldsymbol{A}_i\boldsymbol{M}_i - \boldsymbol{L}_i + \boldsymbol{L}_i\boldsymbol{C}_i\boldsymbol{M}_i\right)\boldsymbol{y}_i\\ &\quad +\ \boldsymbol{T}_i^{-1}\boldsymbol{\Pi}_i\boldsymbol{T}_i\boldsymbol{T}_i^{-1}\boldsymbol{B}_{i1}\boldsymbol{u}_{i1}.\end{aligned}$$

Using the notation introduced in (15)–(17), we have

$$\begin{aligned}\dot{\tilde{\boldsymbol{q}}}_i &= \boldsymbol{T}_i^{-1}\boldsymbol{\Pi}_i\boldsymbol{T}_i\left(\tilde{\boldsymbol{A}}_i + \tilde{\boldsymbol{L}}_i\tilde{\boldsymbol{C}}_i\right)\tilde{\boldsymbol{q}}_i\\ &\quad +\ \boldsymbol{T}_i^{-1}\boldsymbol{\Pi}_i\boldsymbol{T}_i\boldsymbol{T}_i^{-1}\Big[\boldsymbol{B}_{i1}\boldsymbol{u}_{i1}\\ &\qquad +\ \Big(\boldsymbol{A}_i\boldsymbol{M}_i - \boldsymbol{T}_i\tilde{\boldsymbol{L}}_i\left(\boldsymbol{I}_{p_i} - \boldsymbol{C}_i\boldsymbol{M}_i\right)\Big)\boldsymbol{y}_i\Big].\end{aligned} \tag{24}$$

Let $\tilde{\boldsymbol{q}}_i = [\tilde{\boldsymbol{q}}_{i_1}^\top\ \tilde{\boldsymbol{q}}_{i_2}^\top]^\top$, where $\tilde{\boldsymbol{q}}_{i_1} \in \mathbb{R}^{n_i - m_{i2}}$ and $\tilde{\boldsymbol{q}}_{i_2} \in \mathbb{R}^{m_{i2}}$. It follows from (12) and (24) that $\dot{\tilde{\boldsymbol{q}}}_{i_2}(t) = \boldsymbol{0}$. Therefore, choosing $\tilde{\boldsymbol{q}}_{i_2}(t_0) = \boldsymbol{0}$ guarantees that $\tilde{\boldsymbol{q}}_{i_2}(t) = \boldsymbol{0}$ for $t \geq t_0$. We can thus remove m_{i2} observer states from the observer dynamics described by (24). Let

$$\tilde{\boldsymbol{G}}_i = \boldsymbol{A}_i\boldsymbol{M}_i - \boldsymbol{T}_i\tilde{\boldsymbol{L}}_i\left(\boldsymbol{I}_{p_i} - \boldsymbol{C}_i\boldsymbol{M}_i\right). \tag{25}$$

The resulting reduced-order observer is given by

$$\begin{aligned}\dot{\tilde{\boldsymbol{q}}}_{i_1} &= \left(\tilde{\boldsymbol{A}}_{i_{11}} + \tilde{\boldsymbol{L}}_{i_1}\tilde{\boldsymbol{C}}_{i_1}\right)\tilde{\boldsymbol{q}}_{i_1} + \left[\,\boldsymbol{I}_{n_i - m_{i2}}\ \boldsymbol{O}\,\right]\\ &\qquad \times\ \boldsymbol{T}_i^{-1}\left(\tilde{\boldsymbol{G}}_i\boldsymbol{y}_i + \boldsymbol{B}_{i1}\boldsymbol{u}_{i1}\right),\end{aligned} \tag{26}$$

$$\hat{\boldsymbol{x}}_i = \boldsymbol{T}_i\begin{bmatrix}\boldsymbol{I}_{n_i - m_{i2}}\\ \boldsymbol{O}\end{bmatrix}\tilde{\boldsymbol{q}}_{i_1} + \boldsymbol{M}_i\boldsymbol{y}_i, \tag{27}$$

with $\tilde{\boldsymbol{q}}_{i1}(t_0) = \boldsymbol{0}$.

IV. Decentralized Dynamic Output Feedback Controller Construction

We now present a method to determine $\boldsymbol{K}_{i2}$. This method is based on the results of [5]. To proceed, we define the distance, $\delta(\boldsymbol{A}, \boldsymbol{B})$, between the pair $(\boldsymbol{A}, \boldsymbol{B})$ and the set of pairs with an uncontrollable purely imaginary mode as

$$\delta(\boldsymbol{A}, \boldsymbol{B}) = \min_{\omega \in \mathbb{R}} \sigma_{\min}\left(\left[\ j\omega\boldsymbol{I} - \boldsymbol{A}\quad \boldsymbol{B}\ \right]\right),$$

where $\sigma_{\min}(\cdot)$ denotes the smallest singular value. The above definition is an adaptation of the distance between the pair $(\boldsymbol{A}, \boldsymbol{B})$ and the set of uncontrollable pairs introduced by [11]. Moreover, $\delta(\boldsymbol{A}, \epsilon\boldsymbol{B})$ is a continuous function of ϵ in ([5], [12], [13]). An efficient bisection algorithm for computing $\delta(\boldsymbol{A}, \boldsymbol{B})$ can be obtained by substituting $(\boldsymbol{A}^\top, \boldsymbol{B}^\top)$ into the algorithm presented in [12], where the distance between the pair $(\boldsymbol{A}, \boldsymbol{C})$ and the set of pairs with an unobservable purely imaginary mode is considered.

To proceed, we need the following lemma which is a modification of a result found in [12].

Lemma: For the quadratic matrix equation,

$$\boldsymbol{A}^\top\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{P}\boldsymbol{R}\boldsymbol{P} + \boldsymbol{Q} = \boldsymbol{O}, \tag{28}$$

if $\boldsymbol{R} = \boldsymbol{R}^\top \geq 0$, $\boldsymbol{Q} = \boldsymbol{Q}^\top > 0$, $\boldsymbol{A}$ is Hurwitz and the associated Hamiltonian matrix,

$$\boldsymbol{H} = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{R}\\ -\boldsymbol{Q} & -\boldsymbol{A}^\top\end{bmatrix},$$

has no eigenvalues on the imaginary axis (i.e. $\boldsymbol{H}$ is hyperbolic), then there exist symmetric positive definite solutions $\boldsymbol{P}$ to the quadratic matrix equation (28).

Proof: If $\boldsymbol{R} = \boldsymbol{R}^\top \geq 0$, $\boldsymbol{A}$ is Hurwitz, and the Hamiltonian matrix $\boldsymbol{H}$ has no eigenvalues on the imaginary axis, there exist symmetric matrices $\boldsymbol{P} = \boldsymbol{P}^\top$ to the quadratic matrix equation (28). A proof of this fact is given in [14]. On the other hand, we can rearrange (28) to obtain $\boldsymbol{A}^\top\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{P}\boldsymbol{R}\boldsymbol{P} - \boldsymbol{Q}$. Because $\boldsymbol{R} = \boldsymbol{R}^\top \geq 0$, $\boldsymbol{P} = \boldsymbol{P}^\top$ and $\boldsymbol{Q} = \boldsymbol{Q}^\top > 0$, therefore $-\boldsymbol{P}\boldsymbol{R}\boldsymbol{P} - \boldsymbol{Q}$ is negative definite. In addition, $\boldsymbol{A}$ is Hurwitz, so $\boldsymbol{P}$ is positive definite, which concludes the proof of the lemma. ■

Proposition: For the controllable pair $(\boldsymbol{A}_{ci}, \boldsymbol{B}_{i1})$, if

$$\delta\left(\boldsymbol{A}_{ci}, \sqrt{2\beta}\boldsymbol{B}_{i1}\left(\boldsymbol{B}_{i1}^\top\boldsymbol{B}_{i1}\right)^{-\frac{1}{2}}\right) > \sqrt{2\beta}, \tag{29}$$

where $\beta = \sum_{i=1}^N \alpha_i^2\lambda_{\max}(\boldsymbol{\Gamma}_i^\top\boldsymbol{\Gamma}_i)$, there exists a $\gamma_i^* > 0$ such that there exist symmetric positive definite solutions $\boldsymbol{P}_i^c$ to the quadratic matrix equation,

$$\boldsymbol{A}_{ci}^\top\boldsymbol{P}_i^c + \boldsymbol{P}_i^c\boldsymbol{A}_{ci} + \boldsymbol{P}_i^c\boldsymbol{R}_i\boldsymbol{P}_i^c + \boldsymbol{Q}_i = \boldsymbol{O}, \tag{30}$$

where $\boldsymbol{R}_i = 2(\boldsymbol{I}_{n_i} - \boldsymbol{B}_{i1}(\boldsymbol{B}_{i1}^\top\boldsymbol{B}_{i1})^{-1}\boldsymbol{B}_{i1}^\top)$ and $\boldsymbol{Q}_i^c = (\beta + \gamma_i)\boldsymbol{I}_{n_i}$ for $\gamma_i \in [0,\ \gamma_i^*)$.

Proof: The following proof is based on the results of [7]. Let $f(\epsilon) = \delta(\boldsymbol{A}_{ci}, \sqrt{2\epsilon}\boldsymbol{B}_{i1}(\boldsymbol{B}_{i1}^\top\boldsymbol{B}_{i1})^{-\frac{1}{2}}) - \sqrt{2\epsilon}$. The composite function, $\delta(\boldsymbol{A}, \sqrt{2\epsilon}\boldsymbol{B}_{i1}(\boldsymbol{B}_{i1}^\top\boldsymbol{B}_{i1})^{-\frac{1}{2}})$, is a continuous function of ϵ, because $\delta(\boldsymbol{A}, \epsilon\boldsymbol{B}_{i1}(\boldsymbol{B}_{i1}^\top\boldsymbol{B}_{i1})^{-\frac{1}{2}})$ and $\sqrt{2\epsilon}$ are continuous functions of ϵ. Hence, $f(\epsilon)$ is a continuous function of ϵ. It follows from (29) and the continuity of $f(\epsilon)$ that there exists a $\gamma_i^* > 0$ such that

$$\delta\left(\boldsymbol{A}_{ci}, \sqrt{2(\beta+\gamma_i)}\boldsymbol{B}_{i1}\left(\boldsymbol{B}_{i1}^\top\boldsymbol{B}_{i1}\right)^{-\frac{1}{2}}\right) > \sqrt{2(\beta+\gamma_i)}, \tag{31}$$

for $\gamma_i \in [0,\ \gamma_i^*)$. For the quadratic matrix equation (30), the associated Hamiltonian matrix is,

$$\boldsymbol{H}_i = \begin{bmatrix}\boldsymbol{A}_{ci} & \boldsymbol{R}_i\\ -\boldsymbol{Q}_i^c & -\boldsymbol{A}_{ci}^\top\end{bmatrix}.$$

It can be shown that the above Hamiltonian matrix $\boldsymbol{H}_i$ has no eigenvalues on the imaginary axis if (31) is satisfied. A proof of this result is given in [5], [7]. Then it follows from the lemma that there exist symmetric positive definite solutions $\boldsymbol{P}_i^c$ to the quadratic matrix equation (30) for $\gamma_i \in [0,\ \gamma_i^*)$, which concludes the proof of the proposition. ■

Theorem: For the interconnected system with the i-th subsystem, $i = 1, \ldots, N$, modeled by (1), if condition (29) is satisfied, there exists a $\gamma_i^* > 0$ such that the closed-loop system driven by the decentralized dynamic output feedback controller (5) is asymptotically stable, where

$$\boldsymbol{K}_{i2} = -\left(\boldsymbol{B}_{i1}^\top \boldsymbol{B}_{i1}\right)^{-1} \boldsymbol{B}_{i1}^\top \boldsymbol{P}_i^c, \tag{32}$$

and $\boldsymbol{P}_i^c$ is a positive definite solution to the quadratic matrix equation (30) for $\gamma_i \in [0,\ \gamma_i^*)$.

Proof: Substituting (5) into (1), we obtain

$$\begin{aligned}\dot{\boldsymbol{x}}_i &= \boldsymbol{A}_i \boldsymbol{x}_i + \boldsymbol{B}_{i1}\boldsymbol{K}_i \hat{\boldsymbol{x}}_i + \boldsymbol{B}_{i2}\boldsymbol{u}_{i2}(\boldsymbol{x}) \\ &= \left(\boldsymbol{A}_{ci} + \boldsymbol{B}_{i1}\boldsymbol{K}_{i2}\right)\boldsymbol{x}_i + \boldsymbol{B}_{i1}\boldsymbol{K}_i \boldsymbol{e}_i + \boldsymbol{B}_{i2}\boldsymbol{u}_{i2}(\boldsymbol{x}).\end{aligned} \tag{33}$$

Then it follows from (9) and (11) that

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\Pi}_i \left(\boldsymbol{A}_i + \boldsymbol{L}_i \boldsymbol{C}_i\right) \boldsymbol{e}_i. \tag{34}$$

Recall that $\boldsymbol{L}_i$ in (22) is chosen such that $\tilde{\boldsymbol{A}}_{i_{11}} + \tilde{\boldsymbol{L}}_{i_1}\tilde{\boldsymbol{C}}_{i_1}$ is Hurwitz. Thus, there exists a symmetric positive definite matrix $\tilde{\boldsymbol{P}}_{i_{11}}^o$ such that

$$\left(\tilde{\boldsymbol{A}}_{i_{11}} + \tilde{\boldsymbol{L}}_{i_1}\tilde{\boldsymbol{C}}_{i_1}\right)^\top \tilde{\boldsymbol{P}}_{i_{11}}^o + \tilde{\boldsymbol{P}}_{i_{11}}^o \left(\tilde{\boldsymbol{A}}_{i_{11}} + \tilde{\boldsymbol{L}}_{i_1}\tilde{\boldsymbol{C}}_{i_1}\right) = -\tilde{\boldsymbol{Q}}_{i_{11}}^o, \tag{35}$$

where we select $\tilde{\boldsymbol{Q}}_{i_{11}}^o$ to be a symmetric positive definite matrix such that

$$\lambda_{\min}\left(\tilde{\boldsymbol{Q}}_{i_{11}}^o\right) > \sigma_{\max}^2 \left(\boldsymbol{B}_{i1}\boldsymbol{K}_i \boldsymbol{T}_i \boldsymbol{\Upsilon}_i\right), \tag{36}$$

where $\boldsymbol{T}_i$ satisfies (12) and $\boldsymbol{\Upsilon}_i = [\boldsymbol{I}_{n_i - m_{i2}}\ \boldsymbol{O}^\top]^\top$.

It follows from the proposition that if condition (29) is satisfied, there exists a $\gamma_i^* > 0$ such that there exist symmetric positive definite solutions $\boldsymbol{P}_i^c$ to the following quadratic matrix equation (30) for $\gamma_i \in [0,\ \gamma_i^*)$. Let $\boldsymbol{P}^c = \text{diag}[\boldsymbol{P}_1^c\ \cdots\ \boldsymbol{P}_N^c]$ and $\boldsymbol{P}^o = \text{diag}[\boldsymbol{P}_1^o\ \cdots\ \boldsymbol{P}_N^o]$, where $\boldsymbol{P}_i^o$ is defined to be

$$\boldsymbol{P}_i^o = \boldsymbol{T}_i^{-\top} \begin{bmatrix} \tilde{\boldsymbol{P}}_{i_{11}}^o & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_{m_{i2}} \end{bmatrix} \boldsymbol{T}_i^{-1} \tag{37}$$

with $\boldsymbol{T}_i$ satisfying (12) and $\tilde{\boldsymbol{P}}_{i_{11}}^o$ satisfying (35). Now we consider the following Lyapunov function candidate,

$$V = \boldsymbol{x}^\top \boldsymbol{P}^c \boldsymbol{x} + \boldsymbol{e}^\top \boldsymbol{P}^o \boldsymbol{e} = \sum_{i=1}^N \left(\boldsymbol{x}_i^\top \boldsymbol{P}_i^c \boldsymbol{x}_i + \boldsymbol{e}_i^\top \boldsymbol{P}_i^o \boldsymbol{e}_i\right).$$

Evaluating the time derivative of $V(\boldsymbol{x}, \boldsymbol{e})$ on the solutions of (33) and (34), we obtain

$$\begin{aligned}\dot{V} &= \sum_{i=1}^N \left(2\boldsymbol{x}_i^\top \boldsymbol{P}_i^c \dot{\boldsymbol{x}}_i + 2\boldsymbol{e}_i^\top \boldsymbol{P}_i^o \dot{\boldsymbol{e}}_i\right) \\ &= \sum_{i=1}^N \left[2\boldsymbol{x}_i^\top \boldsymbol{P}_i^c \left(\boldsymbol{A}_{ci} + \boldsymbol{B}_{i1}\boldsymbol{K}_{i2}\right)\boldsymbol{x}_i \right. \\ &\qquad \left. +\ 2\boldsymbol{x}_i^\top \boldsymbol{P}_i^c \left(\boldsymbol{B}_{i1}\boldsymbol{K}_i\right)\boldsymbol{e}_i + 2\boldsymbol{x}_i^\top \boldsymbol{P}_i^c \left(\boldsymbol{B}_{i2}\boldsymbol{u}_{i2}\right)\right] \\ &\quad + \sum_{i=1}^N 2\boldsymbol{e}_i^\top \boldsymbol{P}_i^o \boldsymbol{\Pi}_i \left(\boldsymbol{A}_i + \boldsymbol{L}_i \boldsymbol{C}_i\right)\boldsymbol{e}_i.\end{aligned} \tag{38}$$

Using the inequality, $2\boldsymbol{a}^\top \boldsymbol{b} \leq \boldsymbol{a}^\top \boldsymbol{a} + \boldsymbol{b}^\top \boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are arbitrary vectors, we obtain

$$2\boldsymbol{x}_i^\top \boldsymbol{P}_i^c \left(\boldsymbol{B}_{i1}\boldsymbol{K}_i\right)\boldsymbol{e}_i \leq \boldsymbol{x}_i^\top \boldsymbol{P}_i^c \boldsymbol{P}_i^c \boldsymbol{x}_i + \boldsymbol{e}_i^\top \boldsymbol{Q}_i \boldsymbol{e}_i, \tag{39}$$

where $\boldsymbol{Q}_i = (\boldsymbol{B}_{i1}\boldsymbol{K}_i)^\top(\boldsymbol{B}_{i1}\boldsymbol{K}_i)$, and

$$2\boldsymbol{x}_i^\top \boldsymbol{P}_i^c \left(\boldsymbol{B}_{i2}\boldsymbol{u}_{i2}\right) \leq \boldsymbol{x}_i^\top \boldsymbol{P}_i^c \boldsymbol{P}_i^c \boldsymbol{x}_i + \left(\boldsymbol{B}_{i2}\boldsymbol{u}_{i2}\right)^\top \left(\boldsymbol{B}_{i2}\boldsymbol{u}_{i2}\right). \tag{40}$$

Let $\beta_i = \alpha_i^2 \lambda_{\max}(\boldsymbol{\Gamma}_i^\top \boldsymbol{\Gamma}_i)$. It follows from (3) and (40) that

$$2\boldsymbol{x}_i^\top \boldsymbol{P}_i^c \left(\boldsymbol{B}_{i2}\boldsymbol{u}_{i2}\right) \leq \beta_i \sum_{j=1}^N \boldsymbol{x}_j^\top \boldsymbol{x}_j + \boldsymbol{x}_i^\top \boldsymbol{P}_i^c \boldsymbol{P}_i^c \boldsymbol{x}_i. \tag{41}$$

It follows from (38), (39) and (41) that

$$\begin{aligned}\dot{V} &\leq \sum_{i=1}^N \left[2\boldsymbol{x}_i^\top \boldsymbol{P}_i^c \left(\boldsymbol{A}_{ci} + \boldsymbol{B}_{i1}\boldsymbol{K}_{i2}\right)\boldsymbol{x}_i \right. \\ &\qquad \left. +\ 2\boldsymbol{x}_i^\top \boldsymbol{P}_i^c \boldsymbol{P}_i^c \boldsymbol{x}_i + \beta_i \sum_{j=1}^N \boldsymbol{x}_j^\top \boldsymbol{x}_j\right] \\ &\quad + \sum_{i=1}^N \left[2\boldsymbol{e}_i^\top \boldsymbol{P}_i^o \boldsymbol{\Pi}_i \left(\boldsymbol{A}_i + \boldsymbol{L}_i \boldsymbol{C}_i\right)\boldsymbol{e}_i + \boldsymbol{e}_i^\top \boldsymbol{Q}_i \boldsymbol{e}_i\right] \\ &= \sum_{i=1}^N \left(\dot{V}_{ci} + \dot{V}_{oi}\right),\end{aligned}$$

where

$$\dot{V}_{ci} = \boldsymbol{x}_i^\top \left(2\boldsymbol{P}_i^c \left(\boldsymbol{A}_{ci} + \boldsymbol{B}_{i1}\boldsymbol{K}_{i2}\right) + 2\boldsymbol{P}_i^c \boldsymbol{P}_i^c + \beta \boldsymbol{I}_{n_i}\right)\boldsymbol{x}_i, \tag{42}$$

and $\dot{V}_{oi} = 2\boldsymbol{e}_i^\top \boldsymbol{P}_i^o \boldsymbol{\Pi}_i (\boldsymbol{A}_i + \boldsymbol{L}_i \boldsymbol{C}_i)\boldsymbol{e}_i + \boldsymbol{e}_i^\top \boldsymbol{Q}_i \boldsymbol{e}_i$. Substituting the gain matrix $\boldsymbol{K}_{i2}$ into (42), we obtain

$$\begin{aligned}\dot{V}_{ci} = \boldsymbol{x}_i^\top \Bigg(&\boldsymbol{A}_{c_i}^\top \boldsymbol{P}_i^c + \boldsymbol{P}_i^c \boldsymbol{A}_{c_i} + \beta \boldsymbol{I}_{n_i} \\ &+ 2\boldsymbol{P}_i^c(\boldsymbol{I} - \boldsymbol{B}_{i1}\left(\boldsymbol{B}_{i1}^\top \boldsymbol{B}_{i1}\right)^{-1} \boldsymbol{B}_{i1}^\top)\boldsymbol{P}_i^c\Bigg)\boldsymbol{x}_i.\end{aligned}$$

Because $\boldsymbol{P}_i^c$ satisfies (30), we have $\dot{V}_{ci} = -\gamma_i \boldsymbol{x}_i^\top \boldsymbol{x}_i < 0$. It follows from (13) and (37) that $\dot{V}_{oi}$ can be rewritten as

$$\begin{aligned}\dot{V}_{oi} = &\ 2\tilde{\boldsymbol{e}}_i^\top \boldsymbol{T}_i^\top \boldsymbol{T}_i^{-\top} \begin{bmatrix} \tilde{\boldsymbol{P}}_{i_{11}}^o & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_{m_{i2}} \end{bmatrix} \boldsymbol{T}_i^{-1} \boldsymbol{\Pi}_i \\ &\times \boldsymbol{T}_i \boldsymbol{T}_i^{-1}\left(\boldsymbol{A}_i + \boldsymbol{L}_i \boldsymbol{C}_i\right)\boldsymbol{T}_i \tilde{\boldsymbol{e}}_i \\ &+\ \tilde{\boldsymbol{e}}_i^\top \left(\boldsymbol{B}_{i1}\boldsymbol{K}_i \boldsymbol{T}_i\right)^\top \left(\boldsymbol{B}_{i1}\boldsymbol{K}_i \boldsymbol{T}_i\right)\tilde{\boldsymbol{e}}_i.\end{aligned} \tag{43}$$

Recall that $\tilde{\boldsymbol{e}}_i = [\tilde{\boldsymbol{e}}_{i_1}^\top\ \boldsymbol{0}^\top]^\top$. It follows from (12), (15)–(17), (35) and (43) that

$$\begin{aligned}\dot{V}_{oi} &= 2\begin{bmatrix} \tilde{\boldsymbol{e}}_{i_1} \\ \boldsymbol{0} \end{bmatrix}^\top \begin{bmatrix} \tilde{\boldsymbol{P}}_{i_{11}}^o & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_{m_{i2}} \end{bmatrix} \\ &\quad \times \begin{bmatrix} \tilde{\boldsymbol{A}}_{i_{11}} + \tilde{\boldsymbol{L}}_{i_1}\tilde{\boldsymbol{C}}_{i_1} & \tilde{\boldsymbol{A}}_{i_{12}} + \tilde{\boldsymbol{L}}_{i_1}\tilde{\boldsymbol{C}}_{i_2} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{e}}_{i_1} \\ \boldsymbol{0} \end{bmatrix} \\ &\quad + \begin{bmatrix} \tilde{\boldsymbol{e}}_{i_1} \\ \boldsymbol{0} \end{bmatrix}^\top \left(\boldsymbol{B}_{i1}\boldsymbol{K}_i \boldsymbol{T}_i\right)^\top \left(\boldsymbol{B}_{i1}\boldsymbol{K}_i \boldsymbol{T}_i\right) \begin{bmatrix} \tilde{\boldsymbol{e}}_{i_1} \\ \boldsymbol{0} \end{bmatrix} \\ &= 2\tilde{\boldsymbol{e}}_{i_1}^\top \tilde{\boldsymbol{P}}_{i_{11}}^o \left(\tilde{\boldsymbol{A}}_{i_{11}} + \tilde{\boldsymbol{L}}_{i_1}\tilde{\boldsymbol{C}}_{i_1}\right)\tilde{\boldsymbol{e}}_{i_1} + \tilde{\boldsymbol{e}}_{i_1}^\top \tilde{\boldsymbol{Q}}_{i_{11}} \tilde{\boldsymbol{e}}_{i_1} \\ &= -\tilde{\boldsymbol{e}}_{i_1}^\top \tilde{\boldsymbol{Q}}_{i_{11}}^o \tilde{\boldsymbol{e}}_{i_1} + \tilde{\boldsymbol{e}}_{i_1}^\top \tilde{\boldsymbol{Q}}_{i_{11}} \tilde{\boldsymbol{e}}_{i_1} \\ &= -\left(\lambda_{\min}\left(\tilde{\boldsymbol{Q}}_{i_{11}}^o\right) - \lambda_{\max}\left(\tilde{\boldsymbol{Q}}_{i_{11}}\right)\right)\|\tilde{\boldsymbol{e}}_{i_1}\|_2^2,\end{aligned} \tag{44}$$

where

$$\tilde{\boldsymbol{Q}}_{i_{11}} = \begin{bmatrix} \boldsymbol{I}_{n_i-m_{i2}} \\ \boldsymbol{O} \end{bmatrix}^\top (\boldsymbol{B}_{i1}\boldsymbol{K}_{i1}\boldsymbol{T}_i)^\top \times (\boldsymbol{B}_{i1}\boldsymbol{K}_{i1}\boldsymbol{T}_i) \begin{bmatrix} \boldsymbol{I}_{n_i-m_{i2}} \\ \boldsymbol{O} \end{bmatrix} = (\boldsymbol{B}_{i1}\boldsymbol{K}_{i1}\boldsymbol{T}_i\boldsymbol{\Upsilon}_i)^\top (\boldsymbol{B}_{i1}\boldsymbol{K}_{i1}\boldsymbol{T}_i\boldsymbol{\Upsilon}_i)$$

and

$$\lambda_{\max}\left(\tilde{\boldsymbol{Q}}_{i_{11}}\right) = \sigma^2_{\max}(\boldsymbol{B}_{i1}\boldsymbol{K}_{i1}\boldsymbol{T}_i\boldsymbol{\Upsilon}_i). \quad (45)$$

Using (36) and (45), we obtain, $\dot{V}_{oi} < 0$. Thus,

$$\dot{V} \le \sum_{i=1}^N \left(\dot{V}_{ci} + \dot{V}_{oi}\right) < 0,$$

which implies that the closed-loop system is asymptotically stable. The proof of the theorem is complete. ■

Remark 2: In the proof, the selection of $\tilde{\boldsymbol{Q}}^o_{i_{11}}$ for the calculation of $\tilde{\boldsymbol{P}}^o_{i_{11}}$ in (35) is not arbitrary. The matrix $\tilde{\boldsymbol{Q}}^o_{i_{11}}$ must satisfy (36), because the resulting $\tilde{\boldsymbol{P}}^o_{i_{11}}$ is essential in the subsequent closed-loop system stability analysis. This seems to couple the observer design with the controller design. However, $\tilde{\boldsymbol{Q}}^o_{i_{11}}$ does not affect the observer design. Thus, as in the linear case, the so-called separation property of the observer-controller design holds.

V. EXAMPLE

In this section, we illustrate the performance of our proposed decentralized dynamic output feedback controller on a nonlinear interconnected system adapted from [5]. The original system in [5] is stable even without control, so we modify the original system to make it more challenging to control.

The system consists of two subsystems. The first subsystem's dynamics are

$$\dot{\boldsymbol{x}}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{x}_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\boldsymbol{u}_{11} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\boldsymbol{u}_{12}(\boldsymbol{x}).$$

where $\boldsymbol{u}_{12}(\boldsymbol{x}) = 0.2\cos(x_4)\sum_{i=1}^5 x_i/\sqrt{10}$. The second subsystem's dynamics are

$$\dot{\boldsymbol{x}}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -40.8 & -41.5 & -9.35 \end{bmatrix}\boldsymbol{x}_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\boldsymbol{u}_{21} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\boldsymbol{u}_{22}(\boldsymbol{x}).$$

where $\boldsymbol{u}_{22}(\boldsymbol{x}) = 0.2\cos(x_1)\sum_{i=1}^5 x_i/\sqrt{15}$. We have $\beta = 0.08$. The initial conditions for the plant are chosen to be $\boldsymbol{x}_1(0) = [1\ 5]^\top$ and $\boldsymbol{x}_2(0) = [1\ 5\ 5]^\top$, respectively and for the observer are set to zero. It is easy to check that the uncontrolled system is unstable. We choose $\boldsymbol{K}_{11} = [-1.5\ -1.25]$ and $\boldsymbol{K}_{21} = \boldsymbol{0}$ such that

$$\delta\left(\boldsymbol{A}_{c1}, \sqrt{2\beta}\boldsymbol{B}_{11}\left(\boldsymbol{B}_{11}^\top\boldsymbol{B}_{11}\right)^{-\frac{1}{2}}\right) = 0.5660 > \sqrt{2\beta},$$

and

$$\delta\left(\boldsymbol{A}_{c2}, \sqrt{2\beta}\boldsymbol{B}_{21}\left(\boldsymbol{B}_{21}^\top\boldsymbol{B}_{21}\right)^{-\frac{1}{2}}\right) = 0.5578 > \sqrt{2\beta}.$$

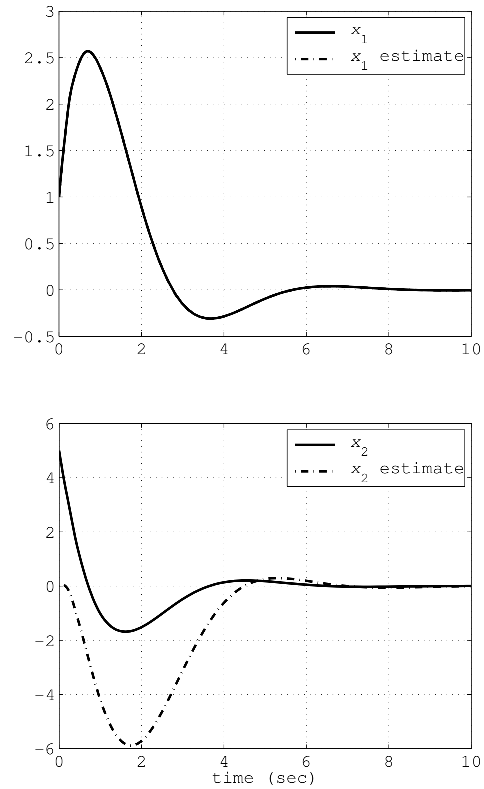


Fig. 1. Decentralized controller performance for the first subsystem.

Then we select $\gamma_1 = 0.1$ and $\gamma_2 = 0.01$. Solving the quadratic matrix equations, we obtain two different $\boldsymbol{P}^c_1$ and four different $\boldsymbol{P}^c_2$, and we select

$$\boldsymbol{P}^c_1 = \begin{bmatrix} 0.4872 & 0.2182 \\ 0.2182 & 0.2847 \end{bmatrix},$$

$$\boldsymbol{P}^c_2 = \begin{bmatrix} 4.7139 & 3.9301 & 0.9243 \\ 3.9301 & 3.8380 & 0.8229 \\ 0.9243 & 0.8229 & 0.2566 \end{bmatrix}.$$

It follows from (32) that $\boldsymbol{K}_{12} = [-0.2182\ -0.2847]$ and $\boldsymbol{K}_{22} = [-0.9243\ -0.8229\ -0.2566]$. We select $\boldsymbol{H}_1 = \boldsymbol{H}_2 = 0$ and we obtain $\boldsymbol{M}_1 = [1\ 1]^\top$, $\boldsymbol{M}_2 = [1\ 1\ 1]^\top$,

$$\boldsymbol{T}_1 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{T}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 5 & 0 & 1 \\ 0 & 5 & 1 \end{bmatrix}.$$

Then it follows from (11) that

$$\boldsymbol{\Pi}_1 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Pi}_2 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

which result in $\tilde{\boldsymbol{G}}_1 = [-0.5\ 0]^\top$ and $\tilde{\boldsymbol{G}}_2 = [0\ -18.53\ 0]^\top$. Simulation results for the first subsystem are shown in Fig. 1, while for the second subsystem in Fig. 2. In Fig. 3, plots of the control inputs u_{11} and u_{21} versus time, are shown. As can be seen from the above figures, the proposed decentralized control strategy performs well.

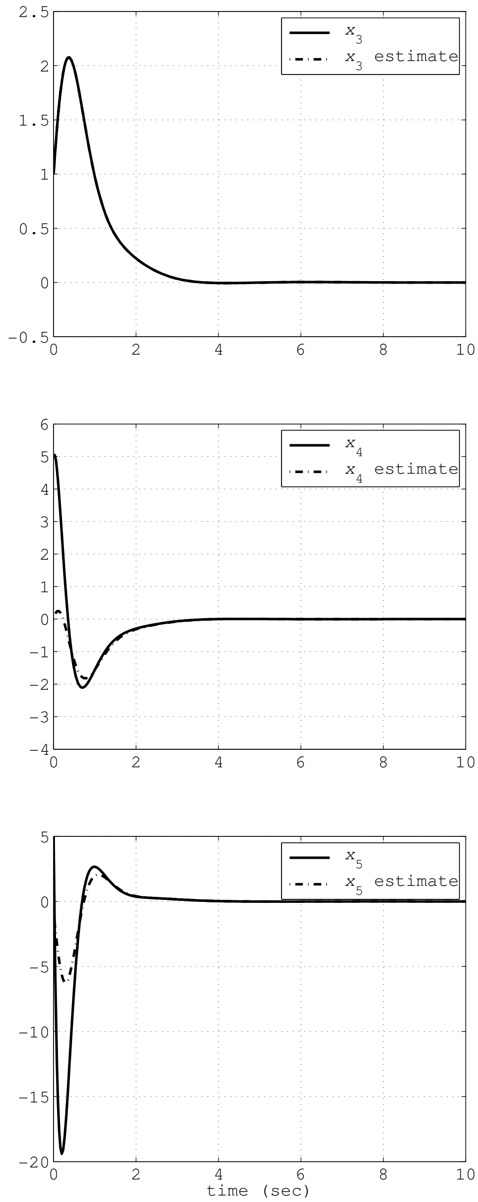
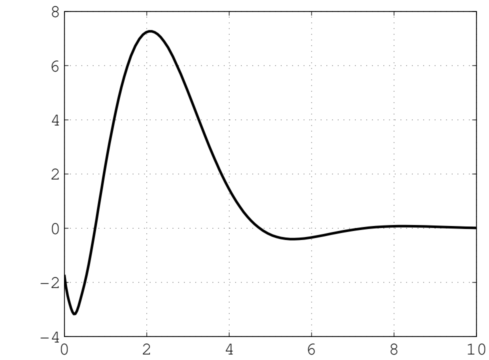
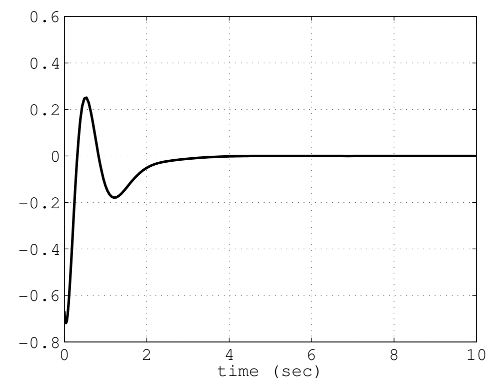


Fig. 2. Decentralized controller performance for the second subsystem.



(a) u_{11}



(b) u_{21}

Fig. 3. Control inputs u_{11} and u_{12}.

VI. Conclusions

An effective decentralized dynamic output feedback controller has been proposed and analyzed for the stabilization problem of a class of nonlinear interconnected systems. The proposed control strategy uses local projection operator based reduced-order observers to estimate the subsystems' states for feedback implementation. The controller and observer can be designed independently of each other, i.e., the separation property of the observer-controller design procedure holds.

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