Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Optimal Control of Uncertain Nonlinear Systems using RISE Feedback

K. Dupree, P. M. Patre, Z. D. Wilcox, and W. E. Dixon
email: {kdupree, pmpatre, zibrus, wdixon}@ufl.edu
Department of Mechanical and Aerospace Engineering
University of Florida, Gainesville, FL 32611

Abstract— **A Hamilton-Jacobi-Bellman optimization scheme is used along with a RISE feedback structure to minimize a quadratic performance index while the generalized coordinates of a nonlinear Euler-Lagrange system asymptotically track a desired time-varying trajectory despite general uncertainty in the dynamics, such as additive bounded disturbances and parametric uncertainty. Motivated by recent previous results that use a neural network structure to approximate the dynamics (i.e., the state space model is approximated with a residual function reconstruction error), the result in this paper uses the implicit learning capabilities of the RISE control structure to learn the dynamics asymptotically. Specifically, a Lyapunov stability analysis is performed to show that the RISE feedback term asymptotically identifies the unknown dynamics, yielding semi-global asymptotic tracking. In addition, it is shown that the system converges to a state space system that has a quadratic performance index which has been optimized by an additional control element. Simulation results are included to demonstrate the performance of the developed controller.**

I. Introduction[1]

Optimal control theory involves the design of controllers that can satisfy some tracking or regulation control objective while simultaneously minimizing some performance metric. A sufficient condition to solve an optimal control problem is to solve the Hamilton-Jacobi-Bellman (HJB) equation. For the special case of linear time-invariant systems, the solution to the HJB equation reduces to solving the algebraic Riccati equation. However, for general systems, the challenge is to find a value function that satisfies the HJB equation. Finding this value function has remained problematic because it requires the solution of a partial differential equation that can not be solved explicitly.

One common technique in developing an optimal controller for a nonlinear system is to assume the nonlinear dynamics are exactly known, feedback linearize the system, and then apply optimal control techniques to the resulting system as in [1]–[3], and others. For example, dynamic feedback linearization was used in [1] to develop a control Lyapunov function to obtain a class of optimal controllers. A review of the optimality of nonlinear design techniques and general results involving feedback linearization as well as Jacobian linearization and other nonlinear design techniques are provided in [4], [5]. A review of inverse optimal control is provided in [6], where the cost function is not a priori provided.

Motivated by the desire to eliminate the requirement for exact knowledge of the dynamics, [7] developed one of the first results to illustrate the interaction of adaptive control with an optimal controller. Specifically, [7] first used exact feedback linearization to cancel the nonlinear dynamics and produce an optimal controller. Then, a self-optimizing adaptive controller was developed to yield global asymptotic tracking despite linear-in-the parameters uncertainty. The analysis in [7] indicated that if the parameter estimation error could somehow converge to zero, then the controller would converge to the optimal solution.

Another method to compensate for system uncertainties is to employ neural networks (NN) to approximate the unknown dynamics. The universal approximation property states that a NN can identify a function up to some function reconstruction error. The use of NN versus feedback linearization allows for general uncertain systems to be examined. However this added robustness comes at the expense of reduced steady-state error (i.e., generally resulting in a uniformly ultimately bounded (UUB) result). NN controllers were developed in results such as [8]–[13] to accommodate for the uncertainty in the system and to solve the HJB equation. Specifically the tracking errors are proven to be uniformly ultimately bounded (UUB) and the resulting state space system, for which the HJB optimal controller is developed, is only approximated.

Our contribution arises from incorporating an optimal control elements with an implicit learning feedback control strategy developed in [14] with modifications in [15] that was later coined the Robust Integral of the Sign of the Error (RISE) method in [16], [17]. The RISE method is used to identify the system and reject disturbances, while achieving asymptotic tracking and the convergence of a control term to the optimal controller. Inspired by the previous work in [7]–[13], [18], [19], a system in which all terms are assumed known (temporarily) is feedback linearized and a control law is developed based on the HJB optimization method for a given quadratic performance index. Under the assumption that parametric uncertainty and unknown bounded disturbances are present in the dynamics, the control law is modified to contain

[1]This research is supported in part by the NSF CAREER award CMS-0547448, AFOSR contract numbers F49620-03-1-0381 and F49620-03-1-0170, AFRL contract number FA4819-05-D-0011, and by research grant No. US-3715-05 from BARD, the United States - Israel Binational Agricultural Research and Development Fund at the University of Florida. The authors would like to gratefully acknowledge the support of the Department of Energy, grant number DE-FG04-86NE37967. This work was accomplished as part of the DOE University Research Program in Robotics (URPR).

the RISE feedback term which is used to identify the uncertainty. Specifically, a Lyapunov stability analysis is included to show that the RISE feedback term asymptotically identifies the unknown dynamics (yielding semi-global asymptotic tracking) provided upper bounds on the disturbances are known and the control gains are selected appropriately. As in previous literature the control law converges to the optimal law, however because our result is asymptotic rather than UUB the control law converges exactly to the optimal law

II. Dynamic Model and Properties

The class of nonlinear dynamic systems considered in this paper is assumed to be modeled by the following Euler-Lagrange [20] formulation:

$$M(q)\ddot{q} + V_m(q,\dot{q})\dot{q} + G(q) + F(\dot{q}) + \tau_d(t) = \tau(t). \quad (1)$$

In (1), $M(q) \in \mathbb{R}^{n\times n}$ denotes the inertia matrix, $V_m(q,\dot{q}) \in \mathbb{R}^{n\times n}$ denotes the centripetal-Coriolis matrix, $G(q) \in \mathbb{R}^n$ denotes the gravity vector, $F(\dot{q}) \in \mathbb{R}^n$ denotes friction, $\tau_d(t) \in \mathbb{R}^n$ denotes a general nonlinear disturbance (e.g., unmodeled effects), $\tau(t) \in \mathbb{R}^n$ represents the input control vector, and $q(t)$, $\dot{q}(t)$, $\ddot{q}(t) \in \mathbb{R}^n$ denote the position, velocity, and acceleration vectors, respectively. The subsequent development is based on the assumption that $q(t)$ and $\dot{q}(t)$ are measurable and that $M(q)$, $V_m(q,\dot{q})$, $G(q)$, $F(\dot{q})$ and $\tau_d(t)$ are unknown. Moreover, the following properties and assumptions will be exploited in the subsequent development.

Property 1: The inertia matrix $M(q)$ is symmetric, positive definite, and satisfies the following inequality $\forall y(t) \in \mathbb{R}^n$:

$$m_1 \|\xi\|^2 \leq \xi^T M(q)\xi \leq \bar{m}(q)\|\xi\|^2, \quad (2)$$

where $m_1 \in \mathbb{R}$ is a known positive constant, $\bar{m}(q) \in \mathbb{R}$ is a known positive function, and $\|\cdot\|$ denotes the standard Euclidean norm.

Property 2: The following skew-symmetric relationship is satisfied:

$$\xi^T\left(\dot{M}(q) - 2V_m(q,\dot{q})\right)\xi = 0 \qquad \forall\xi \in \mathbb{R}^n. \quad (3)$$

Property 3: If $q(t)$, $\dot{q}(t) \in \mathcal{L}_\infty$, then $V_m(q,\dot{q})$, $F(\dot{q})$ and $G(q)$ are bounded. Moreover, if $q(t)$, $\dot{q}(t) \in \mathcal{L}_\infty$, then the first and second partial derivatives of the elements of $M(q)$, $V_m(q,\dot{q})$, $G(q)$ with respect to $q(t)$ exist and are bounded, and the first and second partial derivatives of the elements of $V_m(q,\dot{q})$, $F(\dot{q})$ with respect to $\dot{q}(t)$ exist and are bounded.

Property 4: The desired trajectory is assumed to be designed such that $q_d(t)$, $\dot{q}_d(t)$, $\ddot{q}_d(t)$, $\dddot{q}_d(t)$, $\ddddot{q}_d(t) \in \mathbb{R}^n$ exist, and are bounded.

III. Control Objective

The control objective is to ensure that the system tracks a desired time-varying trajectory, denoted by $q_d(t) \in \mathbb{R}^n$, despite uncertainties in the dynamic model, while minimizing a given performance index. To quantify the tracking objective, a position tracking error, denoted by $e_1(t) \in \mathbb{R}^n$, is defined as

$$e_1 \triangleq q_d - q. \quad (4)$$

To facilitate the subsequent analysis, filtered tracking errors, denoted by $e_2(t)$, $r(t) \in \mathbb{R}^n$, are also defined as

$$e_2 \triangleq \dot{e}_1 + \alpha_1 e_1 \quad (5)$$

$$r \triangleq \dot{e}_2 + \alpha_2 e_2, \quad (6)$$

where $\alpha_1 \in \mathbb{R}^{n\times n}$, denotes a subsequently defined positive definite, constant, gain matrix, and $\alpha_2 \in \mathbb{R}$ is a positive constant. The filtered tracking error $r(t)$ is not measurable since the expression in (6) depends on $\ddot{q}(t)$.

IV. Optimal Control Design

In this section, a state-space model is developed based on the tracking errors in (4) and (5). Based on this model, a controller is developed that minimizes a quadratic performance index under the (temporary) assumption that the dynamics in (1), including the additive disturbance, are known. The development in this section motivates the control design in Section V, where a robust controller is developed to identify the unknown dynamics and additive disturbance.

To develop a state-space model for the tracking errors in (4) and (5), the inertia matrix is premultiplied by the time derivative of (5), and substitutions are made from (1) and (4) to obtain

$$M\dot{e}_2 = -V_m e_2 - \tau + h + \tau_d, \quad (7)$$

where the nonlinear function $h(q,\dot{q},q_d,\dot{q}_d,\ddot{q}_d) \in \mathbb{R}^n$ is defined as

$$\begin{aligned} h \;\triangleq\;& M(\ddot{q}_d + \alpha_1\dot{e}_1) + V_m(\dot{q}_d + \alpha_1 e_1) \\ &+G + F \end{aligned} \quad (8)$$

Under the (temporary) assumption that the dynamics in (1) are known, the control input can be designed as

$$\tau \triangleq h + \tau_d - u, \quad (9)$$

where $u(t) \in \mathbb{R}^n$ is an auxiliary control input that will be designed to minimize a subsequent performance index. By substituting (9) into (7) the closed-loop error system for $e_2(t)$ can be obtained as

$$M\dot{e}_2 = -V_m e_2 + u. \quad (10)$$

A state-space model for (5) and (10) can now be developed as

$$\dot{z} = A(q,\dot{q})z + B(q)u, \quad (11)$$

where $A(q,\dot{q}) \in \mathbb{R}^{2n\times 2n}$, $B(q) \in \mathbb{R}^{2n\times n}$, and $z(t) \in \mathbb{R}^{2n}$ are defined as

$$\begin{aligned} A(q,\dot{q}) &\triangleq \begin{bmatrix} -\alpha_1 & I_{n\times n} \\ 0_{n\times n} & -M^{-1}V_m \end{bmatrix}, \\ B(q) &\triangleq \begin{bmatrix} 0_{n\times n} & M^{-1} \end{bmatrix}^T, \\ z(t) &\triangleq \begin{bmatrix} e_1^T & e_2^T \end{bmatrix}^T, \end{aligned}$$

where $I_{n\times n}$ and $0_{n\times n}$ denote a $n \times n$ identity matrix and matrix of zeros, respectively. The quadratic performance index

$J(u) \in \mathbb{R}$ to be minimized subject to the constraints in (11) is

$$J(u) \triangleq \int_0^{\infty} \left(\frac{1}{2} z^T Q z + \frac{1}{2} u^T R u \right) \quad dt. \tag{12}$$

In (12), $Q \in \mathbb{R}^{2n \times 2n}$ and $R \in \mathbb{R}^{n \times n}$ are positive definite symmetric matrices to weight the influence of the states and (partial) control effort, respectively. Furthermore, the matrix Q can be broken into blocks as:

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}.$$

As stated in [8], [9], the fact that the performance index is only penalized for the auxiliary control $u(t)$ is practical since the gravity, Coriolis, and friction compensation terms in (8) can not be modified by the optimal design phase.

To facilitate the subsequent development, let $P(q) \in \mathbb{R}^{2n \times 2n}$ be defined as

$$P(q) = \begin{bmatrix} K & 0_{n \times n} \\ 0_{n \times n} & M \end{bmatrix} \tag{13}$$

where $K \in \mathbb{R}^{n \times n}$ denotes a gain matrix. Based on the development in Theorem 1 of ([8], [9]), if α_1, R, and K, introduced in (5), (12), and (13), satisfy the following algebraic relationships

$$K = K^T = -\frac{1}{2}\left(Q_{12} + Q_{12}^T\right) > 0 \tag{14}$$

$$Q_{11} = \alpha_1^T K + K \alpha_1, \tag{15}$$

$$R^{-1} = Q_{22}, \tag{16}$$

then $P(q)$ satisfies the Riccati differential equation, and the value function $V(z,t) \in \mathbb{R}$

$$V = \frac{1}{2} z^T P z$$

satisfies the HJB equation. Lemma 1 of ([8], [9]) can be used to conclude that the optimal control $u(t)$ that minimizes (12) subject to (11) is

$$u(t) = -R^{-1} B^T \left(\frac{\partial V(z,t)}{\partial z} \right)^T = -R^{-1} e_2. \tag{17}$$

V. RISE Feedback Control Development

In general, the bounded disturbance $\tau_d(t)$ and the nonlinear dynamics given in (8) are unknown, so the controller given in (9) can not be implemented. However, if the control input contains some method to identify and cancel these effects, then $z(t)$ will converge to the state space model in (11) so that $u(t)$ minimizes the respective performance index. As stated in the introduction, several results have explored this strategy using function approximation methods such as neural networks, where the tracking control errors converge to a neighborhood near the state space model yielding a type of approximate optimal controller. In this section, a control input is developed that exploits RISE feedback to identify the nonlinear effects and bounded disturbances to enable $z(t)$ to asymptotically converge to the state space model.

To develop the control input, the error system in (6) is premultiplied by $M(q)$ and the expressions in (1), (4), and (5) are utilized to obtain

$$Mr = -V_m e_2 + h + \tau_d + \alpha_2 M e_2 - \tau. \tag{18}$$

Based on the open-loop error system in (18), the control input is composed of the optimal control developed in (17), plus a subsequently designed auxiliary control term $\mu(t) \in \mathbb{R}^n$ as

$$\tau \triangleq \mu - u. \tag{19}$$

The closed-loop tracking error system can be developed by substituting (19) into (18) as

$$Mr = -V_m e_2 + h + \tau_d + \alpha_2 M e_2 + u - \mu. \tag{20}$$

To facilitate the subsequent stability analysis the auxiliary function $f_d(q_d, \dot{q}_d, \ddot{q}_d) \in \mathbb{R}^n$, which is defined as

$$f_d \triangleq M(q_d)\ddot{q}_d + V_m(q_d, \dot{q}_d)\dot{q}_d + G(q_d) + F(\dot{q}_d), \tag{21}$$

is added and subtracted to (20) to yield

$$Mr = -V_m e_2 + \bar{h} + f_d + \tau_d + u - \mu + \alpha_2 M e_2, \tag{22}$$

where $\bar{h}(q, \dot{q}, q_d, \dot{q}_d, \ddot{q}_d) \in \mathbb{R}^n$ is defined as

$$\bar{h} \triangleq h - f_d. \tag{23}$$

The time derivative of (22) can be written as

$$M\dot{r} = -\frac{1}{2}\dot{M}r + \tilde{N} + N_D - e_2 - R^{-1}r - \dot{\mu} \tag{24}$$

after strategically grouping specific terms. In (24), the unmeasurable auxiliary terms $\tilde{N}(e_1, e_2, r, t)$, $N_D(t) \in \mathbb{R}^n$ are defined as

$$\begin{aligned} \tilde{N} \triangleq\; & -\dot{V}_m e_2 - V_m \dot{e}_2 - \frac{1}{2}\dot{M}r + \dot{\bar{h}} \\ & + \alpha_2 \dot{M} e_2 + \alpha_2 M \dot{e}_2 + e_2 + \alpha_2 R^{-1} e_2 \end{aligned} \tag{25}$$

$$N_D \triangleq \dot{f}_d + \dot{\tau}_d.$$

Motivation for grouping terms into $\tilde{N}(e_1, e_2, r, t)$ and $N_D(t)$ comes from the subsequent stability analysis and the fact that the Mean Value Theorem, Property 3, and Property 4 can be used to upper bound the auxiliary terms as

$$\left\| \tilde{N}(t) \right\| \le \rho(\|y\|)\|y\|, \tag{26}$$

$$\|N_D\| \le \zeta_1, \qquad \left\| \dot{N}_D \right\| \le \zeta_2, \tag{27}$$

where $y(t) \in \mathbb{R}^{3n}$ is defined as

$$y(t) \triangleq [e_1^T \;\; e_2^T \;\; r^T]^T, \tag{28}$$

the bounding function $\rho(\|y\|) \in \mathbb{R}$ is a positive globally invertible nondecreasing function, and $\zeta_i \in \mathbb{R}$ $(i = 1, 2)$ denote known positive constants. Based on (24), the control term $\mu(t)$ is designed based on the RISE framework (see [14]–[16]) as

$$\begin{aligned} \mu(t) \triangleq\; & (k_s + 1)e_2(t) - (k_s + 1)e_2(0) \\ & + \int_0^t [(k_s + 1)\alpha_2 e_2(\sigma) + \beta sgn(e_2(\sigma))] d\sigma, \end{aligned} \tag{29}$$

where k_s, $\beta \in \mathbb{R}$ are positive constant control gains. The closed loop error systems for $r(t)$ can now be obtained by substituting the time derivative of (29) into (24) as

$$M\dot{r} = -\frac{1}{2}\dot{M}r + \tilde{N} + N_D - e_2 - R^{-1}r - (k_s+1)r - \beta sgn(e_2). \tag{30}$$

VI. Stability Analysis

Theorem 1: The controller given in (17) and (19) ensures that all system signals are bounded under closed-loop operation, and the tracking errors are regulated in the sense that

$$\|e_1(t)\|, \|e_2(t)\|, \|r(t)\| \to 0 \qquad as \qquad t \to \infty. \tag{31}$$

The boundedness of the closed loop signals and the result in (31) can be obtained provided the control gain k_s introduced in (29) is selected sufficiently large (see the subsequent stability analysis), and α_1, α_2 are selected according to the sufficient conditions

$$\lambda_{\min}(\alpha_1) > \frac{1}{2} \qquad \alpha_2 > 1, \tag{32}$$

where $\lambda_{\min}(\alpha_1)$ is the minimum eigenvalue of α_1, and β is selected according to the following sufficient condition:

$$\beta > \zeta_1 + \frac{1}{\alpha_2}\zeta_2, \tag{33}$$

where β was introduced in (29). Furthermore, $u(t)$ converges to an optimal controller that minimizes (12) subject to (11) provided the gain conditions given in (14)-(16) are satisfied.

Remark 1: The control gain α_1 can not be arbitrarily selected, rather it is calculated using a Lyapunov equation solver. Its value is determined based on the value of Q and R. Therefore Q and R must be chosen such that (32) is satisfied.

Proof: Let $\mathcal{D} \subset \mathbb{R}^{3n+1}$ be a domain containing $\Phi(t) = 0$, where $\Phi(t) \in \mathbb{R}^{3n+1}$ is defined as

$$\Phi(t) \triangleq [y^T(t) \quad \sqrt{O(t)}]^T. \tag{34}$$

In (34), the auxiliary function $O(t) \in \mathbb{R}$ is defined as

$$O(t) \triangleq \beta\|e_2(0)\| - e_2(0)^T N_D(0) - \int_0^t L(\tau)d\tau, \tag{35}$$

where the auxiliary function $L(t) \in \mathbb{R}$ is defined as

$$L(t) \triangleq r^T(N_D(t) - \beta sgn(e_2)) \tag{36}$$

where $\beta \in \mathbb{R}$ is a positive constant chosen according to the sufficient conditions in (33). As illustrated in [15], provided the sufficient conditions introduced in (33) are satisfied, the following inequality can be obtained

$$\int_0^t L(\tau)d\tau \leq \beta\|e_2(0)\| - e_2(0)^T N_D(0). \tag{37}$$

Hence, (37) can be used to conclude that $O(t) \geq 0$.

Let $V_L(\Phi, t) : \mathcal{D} \times [0, \infty) \to \mathbb{R}$ be a continuously differentiable positive definite function defined as

$$V_L(\Phi, t) \triangleq e_1^T e_1 + \frac{1}{2}e_2^T e_2 + \frac{1}{2}r^T M(q) r + O \tag{38}$$

which satisfies the following inequalities:

$$U_1(\Phi) \leq V_L(\Phi, t) \leq U_2(\Phi) \tag{39}$$

provided the sufficient conditions introduced in (33) are satisfied. In (39), the continuous positive definite functions $U_1(\Phi)$, and $U_2(\Phi) \in \mathbb{R}$ are defined as $U_1(\Phi) \triangleq \lambda_1\|\Phi\|^2$, and $U_2(\Phi) \triangleq \lambda_2(q)\|\Phi\|^2$, where λ_1, $\lambda_2(q) \in \mathbb{R}$ are defined as

$$\lambda_1 \triangleq \frac{1}{2}\min\{1, m_1\} \qquad \lambda_2(q) \triangleq \max\left\{\frac{1}{2}\bar{m}(q), 1\right\},$$

where m_1, $\bar{m}(q)$ are introduced in (2). After taking the time derivative of (38), $\dot{V}_L(\Phi, t)$ can be expressed as

$$\dot{V}_L(\Phi, t) = 2e_1^T\dot{e}_1 + e_2^T\dot{e}_2 + \frac{1}{2}r^T\dot{M}(q)r + r^T M(q)\dot{r} + \dot{O}$$

After utilizing (5), (6), (30), and substituting in for the time derivative of $O(t)$, $\dot{V}(\Phi, t)$ can be simplified as follows:

$$\dot{V}_L(\Phi, t) \leq -2e_1^T\alpha_1 e_1 + 2e_2^T e_1 + r^T\tilde{N}(t) - (k_s + 1 + \lambda_{\min}(R^{-1}))\|r\|^2 - \alpha_2\|e_2\|^2. \tag{40}$$

Based on the fact that

$$e_2^T e_1 \leq \frac{1}{2}\|e_1\|^2 + \frac{1}{2}\|e_2\|^2$$

the expression in (40) can be simplified as

$$\dot{V}_L(\Phi, t) \leq r^T\tilde{N}(t) - (k_s + 1 + \lambda_{\min}(R^{-1}))\|r\|^2 - (2\lambda_{\min}(\alpha_1) - 1)\|e_1\|^2 - (\alpha_2 - 1)\|e_2\|^2. \tag{41}$$

By using (26), the expression in (41) can be rewritten as

$$\dot{V}_L(\Phi, t) \leq -\lambda_3\|y\|^2 - \left[k_s\|r\|^2 - \rho(\|y\|)\|r\|\|y\|\right], \tag{42}$$

where $\lambda_3 \triangleq \min\{2\lambda_{\min}(\alpha_1) - 1, \alpha_2 - 1, 1 + \lambda_{\min}(R^{-1})\}$ and α_1 and α_2 are chosen according to the sufficient condition in (32). After completing the squares for the terms inside the brackets in (42), the following expression can be obtained

$$\dot{V}_L(\Phi, t) \leq -\lambda_3\|y\|^2 + \frac{\rho^2(\|y\|)\|y\|^2}{4k_s} \leq -U(\Phi), \tag{43}$$

where $U(\Phi) = c\|y\|^2$, for some positive constant c, is a continuous, positive semi-definite function that is defined on the following domain:

$$\mathcal{D} \triangleq \left\{\Phi \in \mathbb{R}^{3n+1} \mid \|\Phi\| \leq \rho^{-1}\left(2\sqrt{\lambda_3 k_s}\right)\right\}.$$

The inequalities in (39) and (43) can be used to show that $V_L(\Phi, t) \in \mathcal{L}_\infty$ in $\mathcal{D}$; hence, $e_1(t)$, $e_2(t)$, and $r(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Given that $e_1(t)$, $e_2(t)$, and $r(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$, standard linear analysis methods can be used to prove that $\dot{e}_1(t)$, $\dot{e}_2(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$ from (5) and (6). Since $e_1(t)$, $e_2(t)$, $r(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$, the assumption that $q_d(t)$, $\dot{q}_d(t)$, $\ddot{q}_d(t)$ exist and are bounded can be used along with (4)-(6) to conclude that $q(t)$, $\dot{q}(t)$, $\ddot{q}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Since $q(t)$, $\dot{q}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$, Property 3 can be used to conclude that $M(q)$, $V_m(q, \dot{q})$, $G(q)$, and $F(\dot{q}) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Thus from (1) and Property 4, we can show that $\tau(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Given that $r(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$, it can be shown that $\dot{\mu}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Since $\dot{q}(t)$, $\ddot{q}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$, Property 3 can be used to show that $\dot{V}_m(q,\dot{q})$, $\dot{G}(q)$, $\dot{F}(q)$ and $\dot{M}(q) \in \mathcal{L}_\infty$ in $\mathcal{D}$; hence, (30) can be used to show that $\dot{r}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Since $\dot{e}_1(t)$, $\dot{e}_2(t)$, $\dot{r}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$, the definitions for $U(y)$ and $z(t)$ can be used to prove that $U(y)$ is uniformly continuous in $\mathcal{D}$.

Let $\mathcal{S} \subset \mathcal{D}$ denote a set defined as follows:

$$\mathcal{S} \triangleq \left\{ \Phi(t) \subset \mathcal{D} \mid U_2(\Phi(t)) < \lambda_1 \left(\rho^{-1}\left(2\sqrt{\lambda_3 k_s}\right)\right)^2 \right\}. \tag{44}$$

The region of attraction in (44) can be made arbitrarily large to include any initial conditions by increasing the control gain k_s (i.e., a semi-global type of stability result) [15]. Theorem 8.4 of [21] can now be invoked to state that

$$c\left\|y(t)\right\|^2 \to 0 \qquad as \qquad t \to \infty \qquad \forall y(0) \in \mathcal{S}. \tag{45}$$

Based on the definition of $y(t)$, (45) can be used to conclude the results in (31) $\forall y(0) \in \mathcal{S}$.

Since $u(t) \to 0$ as $e_2(t) \to 0$ (see (17)), then (22) can be used to conclude that

$$\mu \to \bar{h} + f_d + \tau_d \qquad \text{as } r(t),\, e_2(t) \to 0. \tag{46}$$

The result in (46) indicates that the dynamics in (1) converge to the state-space system in (11). Hence, $u(t)$ converges to an optimal controller that minimizes (12) subject to (11) provided the gain conditions given in (14)-(16) are satisfied.

VII. Simulation Results

In order to examine the performance of the controller proposed in (19) a numerical simulation was performed. The simulation is based on the dynamics for a two-link robot:

$$\begin{aligned}\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} &= \begin{bmatrix} p_1+2p_3c_2 & p_2+p_3c_2 \\ p_2+p_3c_2 & p_2\end{bmatrix}\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{bmatrix} \\ &+ \begin{bmatrix} -p_3s_2\dot{q}_2 & -p_3s_2(\dot{q}_1+\dot{q}_2) \\ p_3s_2\dot{q}_1 & 0\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} \\ &+ \begin{bmatrix} f_{d_1} & 0 \\ 0 & f_{d_2}\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} + \begin{bmatrix}\tau_{d_1}\\ \tau_{d_2}\end{bmatrix},\end{aligned} \tag{47}$$

where $p_1 = 3.473\ kg \cdot m^2$, $p_2 = 0.196\ kg \cdot m^2$, $p_3 = 0.242\ kg \cdot m^2$, $f_{d_1} = 5.3\ Nm \cdot \sec$, $f_{d_2} = 1.1\ Nm \cdot \sec$, c_2 denotes $cos(q_2)$, s_2 denotes $sin(q_2)$ and τ_{d_1}, τ_{d_2} denote bounded disturbances defined as

$$\begin{aligned}\tau_{d_1} &= 0.1\ \sin(t) + 0.15\ \cos(3t)\\ \tau_{d_2} &= 0.15\ \sin(2t) + 0.1\ \cos(t).\end{aligned} \tag{48}$$

The desired trajectory is given as

$$q_{d_1} = q_{d_2} = \frac{1}{2}\ \sin(2t), \tag{49}$$

and the initial conditions of the robot were selected as

$$\begin{aligned} q_1(0) &= q_2(0) = 14.3\ deg \\ \dot{q}_1(0) &= \dot{q}_2(0) = 28.6\ deg/\sec. \end{aligned}$$

The weighting matrixes were chosen as

$$Q_{11} = \begin{bmatrix} 20 & 2 \\ 2 & 20 \end{bmatrix} \qquad Q_{12} = \begin{bmatrix} -4 & 5 \\ 3 & -6 \end{bmatrix}$$
$$Q_{22} = diag\left\{\ 35,\quad 35\ \right\}.$$

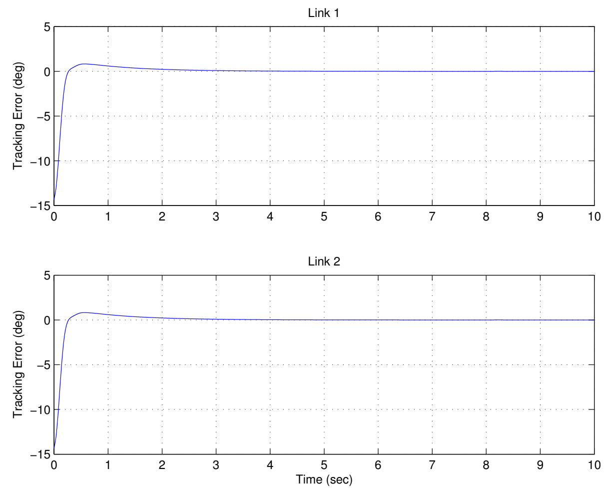

Fig. 1. The tracking errors for the controller developed in (19).

which using (14), (15), and (16) yielded the following values for K, α_1, and R

$$K = \begin{bmatrix} 4 & -4 \\ -4 & 6 \end{bmatrix} \qquad \alpha_1 = \begin{bmatrix} 8.1 & 5.6 \\ 5.6 & 5.4 \end{bmatrix}$$
$$R = diag\left\{ \frac{1}{35}, \quad \frac{1}{35} \right\}.$$

The control gains were selected as

$$\alpha_2 = 20 \qquad \beta = 20 \qquad k_s = 75.$$

The tracking errors from the control inputs are shown in Fig. 1 and Fig. 2, respectively. To show that the RISE feedback identifies the nonlinear effects and bounded disturbances, a plot of the difference is shown in Fig. 3. As this difference goes to zero, the dynamics in (1) converge to the state-space system in (11), and the controller becomes optimal. To test how the optimal controller performed for the unknown system compared to feedback linearized system in (11), $J(u)$ was calculated for each. For this calculation the contribution of RISE feedback term in (19) as well as the contribution of $h(q,\dot{q},q_d,\dot{q}_d,\ddot{q}_d)$ and $\tau_d(t)$ in (9) are not considered. This is due to the fact that it is the input $u(t)$ that minimizes (12). For the unknown system, $J(u)$ was calculated to be 43.32. For the feedback linearized system $J(u)$ was calculated to be 40.41. As expected, the perfectly feedback linearized system has a lower performance index, however the difference between the two values is less than 10%. These are preliminary results, and further simulations and an experimental study are required for more conclusive results.

VIII. Conclusion

A control scheme is developed for a class of nonlinear Euler-Lagrange systems that enables the generalized coordinates to asymptotically track a desired time-varying trajectory despite general uncertainty in the dynamics such as additive bounded disturbances and parametric uncertainty that does not have to satisfy a linear-in-the-parameters assumption. The

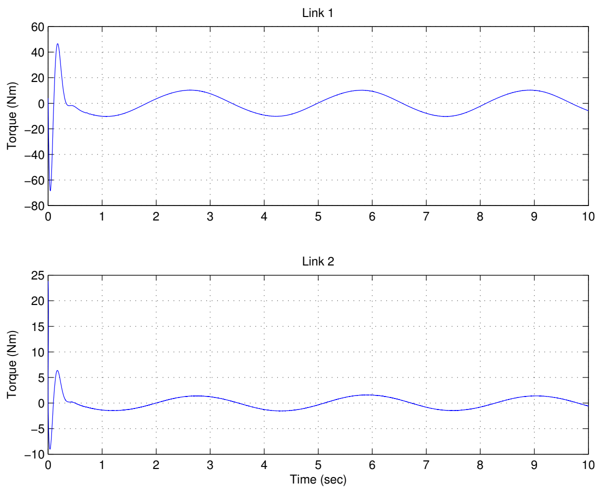

Fig. 2. The torques for the controller developed in (19)

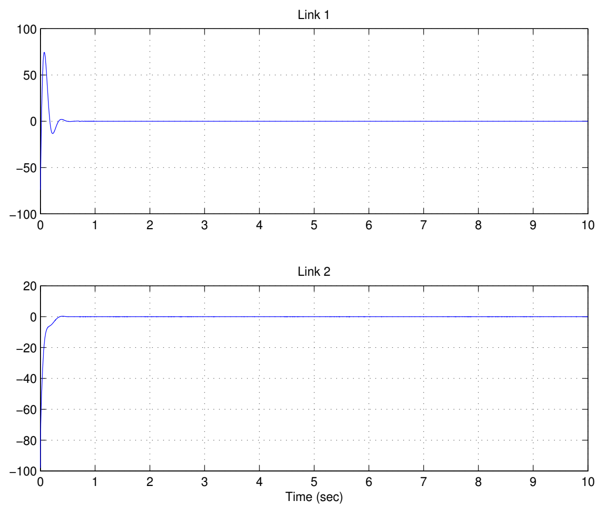

Fig. 3. The difference between the RISE feedback and the nonlinear effects and bounded disturbances.

main contribution of this work is that a RISE feedback method is augmented with an auxiliary control term that minimizes a quadratic performance index based on a HJB optimization scheme. Like the influential work in ([8]–[13], [18], [19]) the result in this effort initially develops the HJB optimization scheme based on a partially feedback linearized state-space model assuming exact knowledge of the dynamics. However, unlike previous results that use a neural network structure to approximate the uncertain dynamics (i.e., the state space model is approximated with a residual function reconstruction error), the result in this paper uses the implicit learning capabilities of the RISE control structure to learn the uncertain dynamics asymptotically. That is, the dynamics asymptotically converge to the state-space system that the HJB optimization scheme is based on. The implication is that the use of the RISE feedback structure compensates for the uncertain nonlinear dynamics yielding a state space system with a quadratic performance index that is optimized by an additional control element.

References

[1] R. Freeman and P. Kokotovic, "Optimal nonlinear controllers for feedback linearizable systems," in *Proc. of the American Controls Conf.*, Jun. 1995, pp. 2722–2726.

[2] Q. Lu, Y. Sun, Z. Xu, and T. Mochizuki, "Decentralized nonlinear optimal excitation control," *IEEE Trans. Power Syst.*, vol. 11, no. 4, pp. 1957–1962, Nov. 1996.

[3] M. Sekoguchi, H. Konishi, M. Goto, A. Yokoyama, and Q. Lu, "Nonlinear optimal control applied to STATCOM for power system stabilization," in *Proc. IEEE/PES Transmission and Distribution Conference and Exhibition*, Oct. 2002, pp. 342–347.

[4] V. Nevistic and J. A. Primbs, "Constrained nonlinear optimal control: a converse HJB approach," California Institute of Technology, Pasadena, CA 91125, Tech. Rep. CIT-CDS 96-021, 1996.

[5] J. A. Primbs and V. Nevistic, "Optimality of nonlinear design techniques: A converse hjb approach," California Institute of Technology, Pasadena, CA 91125, Tech. Rep. CIT-CDS 96-022, 1996.

[6] M. Krstic and H. Deng, *Stabilization of Nonlinear Uncertain Systems*. Springer, 1998.

[7] R. Johansson, "Quadratic optimization of motion coordination and control," *IEEE Trans. Autom. Control*, vol. 35, no. 11, pp. 1197–1208, Nov. 1990.

[8] Y. Kim, F. Lewis, and D. Dawson, "Intelligent optimal control of robotic manipulator using neural networks," *Automatica*, vol. 36, pp. 1355–1364, 2000.

[9] Y. Kim and F. Lewis, "Optimal design of CMAC neural-network controller for robot manipulators," *IEEE Trans. Syst., Man, Cybern. C*, vol. 30, no. 1, pp. 22–31, Feb. 2000.

[10] M. Abu-Khalaf and F. Lewis, "Nearly optimal HJB solution for constrained input systems using a neural network least-squares approach," in *Proc. IEEE Conf. on Decision and Control*, Dec. 2002, pp. 943–948.

[11] T. Cheng and F. Lewis, "Fixed-final time constrained optimal control of nonlinear systems using neural network HJB approach," in *Proc. IEEE Conf. on Decision and Control*, Dec. 2006, pp. 3016–3021.

[12] T. Cheng and F. L. Lewis, "Neural network solution for finite-horizon h-infinity constrained optimal control of nonlinear systems," in *Proc. IEEE Intl. Conf. on Control and Automation*, Jun. 2007, pp. 1966–1972.

[13] T. Cheng, F. Lewis, and M. Abu-Khalaf, "A neural network solution for fixed-final time optimal control of nonlinear systems," *Automatica*, vol. 43, pp. 482–490, 2007.

[14] Z. Qu and J. Xu, "Model-based learning controls and their comparisons using Lyapunov direct method," *Asian Journal of Control*, vol. 4, no. 1, pp. 99–110, Mar. 2002.

[15] B. Xian, D. M. Dawson, M. S. de Queiroz, and J. Chen, "A continuous asymptotic tracking control strategy for uncertain nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 49, no. 7, pp. 1206–1211, Jul. 2004.

[16] P. M. Patre, W. MacKunis, C. Makkar, and W. E. Dixon, "Asymptotic tracking for systems with structured and unstructured uncertainties," *IEEE Trans. Contr. Syst. Technol.*, vol. 16, no. 2, pp. 373–379, 2008.

[17] ——, "Asymptotic tracking for systems with structured and unstructured uncertainties," in *Proc. IEEE Conf. on Decision and Control*, San Diego, CA, 2006, pp. 441–446.

[18] M. Abu-Khalaf, J. Huang, and F. Lewis, *Nonlinear H-2/H-Infinity Constrained Feedback Control*. Springer, 2006.

[19] F. L. Lewis, *Optimal Control*. John Wiley & Sons, 1986.

[20] R. Ortega, A. Loría, P. J. Nicklasson, and H. J. Sira-Ramirez, *Passivity-based Control of Euler-Lagrange Systems: Mechanical, Electrical and Electromechanical Applications*. Springer, 1998.

[21] H. K. Khalil, *Nonlinear Systems*. New Jersey: Prentice-Hall, 2002.