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Optimal Control with Limited Control Actions and Lossy Transmissions

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Abstract— **We consider a class of non-standard networked control problems where (i) there is a limit on the number of times control signals can be transmitted to the plant, and (ii) the links over which the transmission takes place are lossy in the sense that there is a nonzero probability with which packets carrying control signals are dropped. The framework is that of discrete-time LQG optimal control where the control has access to noisy state measurements, and the objective is minimization of the expected value of a quadratic performance index under the nontraditional constraints introduced above. When there is a limitation on the number of control actions, we show that the optimal policy involves thresholding the optimal estimate of the state. When there is also a lossy link, and the control is allowed to receive acknowledgements from the plant as to whether the transmitted packets were received or not, we show that the optimal policy is again of the threshold type, involving off-line computation.**

I. INTRODUCTION

As wireless sensing and control become increasingly applicable in fields ranging from real time alarm systems to aeronautical guidance, the theoretical foundation in these areas has grown likewise [1]. In the past few years, new and nontraditional constraints in the implementation of systems have been introduced, with respect to how information is collected and how transmissions can be made [2]. The desire to exchange information reliably and efficiently in a remote and distributed setting validates the need for research that provides guidelines on how to do this.

Point to point estimation in the presence of noise has been well studied and can be considered fairly complete [3], even when the receiver does not know with certainty whether a signal is present or not [4]. However, constraints in such problems that until recently have not been considered are opening a fresh and challenging field for investigation. Problems with limitations in the number of times a channel may be used (such as in control and transmission/estimation schemes) have been introduced in [5]. Specifically, these problems deal with finite horizon problems, say of length N, in which a channel is used to communicate some information to minimize a cost function, but may only be used $M < N$ times [6], [7], [8]. Such a limitation arises naturally in many applications, from financial settings to sensor networks.

In addition to efficient power usage, extensive study has been conducted with respect to optimal design of networked control systems when physical links limit the reliability of information transfer. Indeed, in [1] and [9] the problem has been extensively studied for linear systems and under both TCP as well as UDP information structures. That is, control systems in which observation and control packets have nonzero probability of being lost are considered, along with information structures in which acknowledgement signals may or may not be present.

A number of related contributions have been made with respect to optimal control policies that must account for unreliability. Papers on the topic incorporate a variety of tools to handle the unreliability, including stability analysis and information theory [10], [11], [12]. The thresholding policies that one finds in the solutions to usage limitation problems, as in [6], also appear in event based control [13].

In this paper we are concerned with a new framework which utilizes both types of nontraditional constraints: usage of channels is limited and information packets may be dropped. We build on the analysis of control systems in which control actions may only be made a limited number of times, as introduced in [14]. We study this problem in the setting of networked control systems in which physical links may fail with nonzero probability. The model is made formal in Section II. We see in Section III that when the cost function involves a rank one matrix, a multi-dimensional control system under limited control usage can be studied with decision regions as subsets of $\mathcal{R}$ rather than subsets of $\mathcal{R}^n$, which simplifies the computation substantially. In Section IV, we then address the problem of optimal control usage in scalar systems with nonzero probability of control packet loss. Section V presents numerical results, and concluding remarks and discussion on future work can be found in Section VII.

II. PROBLEM FORMULATION

Consider a plant described by

$$x_{k+1} = Ax_k + \alpha_k Bu_k + w_k, \quad k = 0, 1, ...N-1,$$

where $x_k \in \mathcal{R}^n$ is the state, $u_k \in \mathcal{R}^n$ is the control, and $w_k \in \mathcal{R}^n$ is a zero-mean i.i.d. Gaussian process with positive definite covariance matrix Σ_w. Note that there are as many control channels as the dimension of the state; in view of this assumption, we take B to be invertible. The variable $\alpha_k \in \{0,1\}$ is an i.i.d. Bernoulli random variable with $P[\alpha_k = 0] = \alpha$, N denotes the decision horizon and the initial state is Gaussian. The measurement at time k is

$$y_k = x_k + v_k, \quad k = 0, 1, ...N-1,$$

where v_k, which models observation noise, is an i.i.d. Gaussian process with zero mean and covariance matrix Σ_v. We assume the noise processes w_k, v_k and the initial state x_0 to

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be independent. Now let I_k denote the information available to the controller at time k:

$$I_k = \{y_0^k, \alpha_0^{k-1}, u_0^{k-1}\}, k = 1, ..., N-1; I_0 = y_0$$

where y_0^k denotes the set of observations from time 0 to time k, and similar convention applies to α_0^{k-1} and u_0^{k-1}. This information structure reflects the TCP protocol - acknowledgements are provided to the controller to let it know whether a control action transmitted has been recieved or not. Consider the class of policies consisting of a sequence of functions $\pi = \mu_0, \mu_1, ..., \mu_{N-1}$, where μ_k maps I_k into the control space C_k, which are restricted such that the control can map I_k into $C_k = \mathcal{R}^n$ a limited number of times. For all other k, we have $C_k = 0$. Control policies of this form are called admissible and our goal is to find an admissible policy π that minimizes $J_\pi = E\left\{\sum_{k=0}^N x_k^T Q x_k\right\}$ with $Q \geq 0$.

Note that the restriction on control sets simply corresponds to a zero control for times during which control is not allowed to act. We assume that control is mapped into $\mathcal{R}^n$ only $M \leq N$ times. Also we do not include a direct penalty for control but there is an indirect penalty in terms of limitation on the number of times it can act.

III. SYSTEMS WITH VECTOR STATE SPACE AND LOSSLESS CONTROL

In this section, we present the optimal policy for problems in which $A = aI$ (but we take $a = 1$ w.l.o.g.), $\alpha = 0$ (reliable control) and $Q = gg^T$ for some $g \in \mathcal{R}^n$. Let s and t respectively denote the number of control actions left and the number of decision instances left. Given M and N and going backwards in time, t increases from $t = 1$ to $t = N$ while s takes values in $\max\{0, M-(N-t)\} \leq s \leq \min\{t, M\}$. Thus given t, N, and M such that $1 \leq M \leq N$, the maximal interval in which s can take values is $0 \leq s \leq t$.

As in [14], the solution is obtained by a dynamic programming argument starting with $t = 1$, and going backwards in time (or forward in t) to $t = N$. For each value of t, we consider the potential values s can take which are given above. From a particular decision and control state (s, t), we decide either to go to stage $(s-1, t-1)$ or $(s, t-1)$ depending on whether we decide to act at stage (s, t) or not. Similarly, we see that we must have arrived to (s, t) either from $(s, t+1)$ or $(s+1, t+1)$.

To proceed with this approach, we start with $t = 1$ and note that $0 \leq s \leq 1$. When $s = 0$, we must have $u_{(0,1)} = 0$, where $u_{(0,1)}$ is the optimal control input at this stage. Since $(0, 1)$ can only lead to (0,0), we can calculate the optimal cost-to-go from $(0, 1)$ and denote it as $J_{(0,1)}$. Then, we must consider stage (1,1), which can only lead to (0,0), and calculate the corresponding optimal control $u_{(1,1)}$ and the associated cost-to-go, $J_{(1,1)}$.

Continuing the process for higher values of t and s, it is possible to obtain expressions for optimal control and associated cost-to-go at an arbitrary stage (s,t). We write $J^{(0)}_{(s,t)}$ for the optimal cost-to-go when a control input is not used at stage (s,t) and $J^{(1)}_{(s,t)}$ when a control input is used. The control input for stage (s,t) will be written as $u_{(s,t)}$.

In the analysis to derive these expressions, which have been omitted due to page limitations but can be found in [8], an important property makes the development possible. By Lemma 3.1 of [14], we know that for every $k \in [0, N-1]$, $x_k - E\{x_k|I_k\}$ is independent of the control policy being used. We denote the error covariance matrix by $\Sigma_{k|k-1} := Cov(x_k - E\{x_k|y_0^{k-1}\})$. The evolution of $\Sigma_{k|k-1}$ is given by

$$\Sigma_{k+1|k} = \Sigma_{k|k} + \Sigma_w$$
$$\Sigma_{k|k} = \Sigma_{k|k-1} - \Sigma_{k|k-1}[\Sigma_{k|k-1} + \Sigma_v]^{-1}\Sigma_{k|k-1}$$

Another useful property is that the decision whether to transmit or not depends only on the *projection* of the best estimate of the current state onto a one dimensional manifold, making decision regions subsets of $\mathcal{R}$. The computational complexity is kept low, due to the fact that Q is a rank one matrix. If Q was of rank higher than 1, say m, we would be forced to choose decision regions as subsets of $\mathcal{R}^m$.

By induction we find from our analysis that the optimal control policy is a threshold policy on a projection of the best estimate of the plant state, which can be recursively generated by a Kalman filter. Furthermore, the threshold at time k is a function of four variables: N (the decision horizon), t_k (number of decision instances left, s_k (number of control actions left) and $\Sigma_{k|k-1}$ (the error covariance). The error covariance $\Sigma_{0|-1}$ is known and all other values can be iteratively calculated. For a given N the thresholds can be computed entirely offline using the procedure described in this section. On the other hand, the best estimates for state must be calculated online with a Kalman filter and initial condition $\hat{x}_{0|-1} = E\{x_0\}$.

Thus, starting with $\hat{x}_{0|-1} = E\{x_0\}$, $s_0 = M$, $t_0 = N$, the optimal control policy can be implemented by the following algorithm: For each k, $0 \leq k \leq N-1$,

1) Look up the threshold $\tau^+_{(s_k,t_k)}$ corresponding to the current stage from a table generated (further below).
2) Observe y_k and update the state estimate to $\hat{x}_{k|k}$ using the Kalman filter recursion.
3) Apply the control policy
$$u_{(s_k,t_k)} = \begin{cases} 0 & \text{if } |g^T\hat{x}_{k|k}| < \tau^+_{(s_k,t_k)} \\ -B^{-1}\hat{x}_{k|k} & \text{if } |g^T\hat{x}_{k|k}| \geq \tau^+_{(s_k,t_k)} \end{cases}$$
4) Update s_{k+1} according to whether a control is used and $t_{k+1} = t_k - 1$

We finally give the iterations to calculate the thresholds $\tau^+_{(s_k,t_k)}$ for a given horizon $N \geq 1$ and arbitrary pair of integers (s, t) such that $1 \leq s \leq t \leq N$. The cost-to-go functions can be written as

$$\begin{aligned} J^{(0)}_{(s,t)} &= 2E\{(g^T x_{N-t})^2|y_0^{N-t}\} + g^T\Sigma_w g + \Lambda_{(s,t-1)} \\ &\quad + \int_{|g^T\hat{x}_{N-t+1|N-t+1}| \leq \tau^+_{(s,t-1)}} \Delta_{(s,t-1)} \\ &\quad \times f^{(0)}_{g^T\hat{x}_{N-t+1|N-t+1}|y_0^{N-t}} d(g^T\hat{x}_{N-t+1|N-t+1}) \\ J^{(1)}_{(s,t)} &= E\{(g^T x_{N-t})^2|y_0^{N-t}\} + \Lambda_{(s,t)} \end{aligned}$$

where

$$f^{(0)}_{g^T\hat{x}_{N-t+1|N-t+1}|y_0^{N-t}} \sim N(g^T\hat{x}_{N-t|N-t}, \\ g^T\Sigma_{N-t+1|N-t}(\Sigma_{N-t+1|N-t}+\Sigma_v)^{-1}\Sigma_{N-t+1|N-t}g)$$

and for $1 < s < t \le N$, $\Lambda_{(s,t)}$ is defined by the recursion

$$\begin{aligned}\Lambda_{(s,t)} &= \Lambda_{(s-1,t-1)} + g^T\Sigma_{N-t|N-t}g + g^T\Sigma_w g \\ &+ \int_{|g^T\hat{x}_{N-t+1|N-t+1}|\le\tau^+_{(s,t-1)}} \Delta_{(s-1,t-1)} \\ &\times f^{(1)}_{g^T\hat{x}_{N-t+1|N-t+1}|y_0^{N-t}} d(g^T\hat{x}_{N-t+1|N-t+1})\end{aligned}$$

where $f^{(1)}_{g^T\hat{x}_{N-t+1|N-t+1}|y_0^{N-t}}$ is mean zero Gaussian with the same variance as $f^{(0)}_{g^T\hat{x}_{N-t+1|N-t+1}|y_0^{N-t}}$. For $1 \le t \le N$ we have $\Lambda_{(1,t)} = tg^T\Sigma_{N-t|N-t}g + \frac{t(t+1)}{2}g^T\Sigma_w g$. Define $\Delta_{(s,t)} := J^{(0)}_{(s,t)} - J^{(1)}_{(s,t)}$. Then for $1 \le s < t \le N$,

$$\begin{aligned}\Delta_{(s,t)} &= (g^T\hat{x}_{N-t|N-t})^2 + g^T\Sigma_{N-t|N-t}g + g^T\Sigma_w g \\ &+ \int_{|g^T\hat{x}_{N-t+1|N-t+1}|\le\tau^+_{(s,t-1)}} \Delta_{(s,t-1)} \\ &\times f^{(0)}_{g^T\hat{x}_{N-t+1|N-t+1}|y_0^{N-t}} d(g^T\hat{x}_{N-t+1|N-t+1}) \\ &+ \Lambda_{(s,t-1)} - \Lambda_{(s,t)}\end{aligned}$$

Note that $\Lambda_{(s,t)}$ is a sequence of real numbers while $\Delta_{(s,t)}$ is a sequence of functions. We must also have boundary conditions for $1 \le t \le N$:

$$\begin{aligned}\Lambda_{(t,t)} &= \Lambda_{(t-1,t-1)} + g^T\Sigma_{N-t|N-t}g + g^T\Sigma_w g \\ \Delta_{(t,t)}(g^T\hat{x}_{N-t|N-t}) &= (g^T\hat{x}_{N-t|N-t})^2\end{aligned}$$

with $\Lambda_{(0,0)} = 0$. We also have $\tau^+_{(t,t)} = 0$, $1 \le t \le N$ and for $1 \le s < t \le N$, the thresholds, $\tau^+_{(s,t)}$, are given by the positive solution of the nonlinear equation $\Delta_{(s,t)}(\tau^+_{(s,t)}) = 0$. In order to show that such a solution exists, we have the following result, which mirrors the main result of [14].

Proposition 1: *Let $N \ge 2$ be given. For $1 \le s < t \le N$, the sequence of functions $\Delta_{(s,t)}(u)$ are even,differentiable with a unique critical point at $u = 0$, i.e., $\left.\frac{\partial\Delta_{(s,t)}(u)}{\partial u}\right|_{u=0} = 0$. Furthermore, we have $\frac{\partial\Delta_{(s,t)}(u)}{\partial u} > 0$ if $u > 0$, and $\frac{\partial\Delta_{(s,t)}(u)}{\partial u} < 0$ if $u < 0$ so that the global minimum is achieved at $u = 0$. Also, the minimum of $\Delta_{(s,t)}(u)$ at critical point $u = 0$ is nonpositive, i.e., $\Delta_{(s,t)}(0) \le 0$.*

In the offline computation of thresholds, we start with $s = 1$ and increase t from 1 to N and determine $\tau^+_{(1,t)}$. Next, we increment s by 1 to $s = 2$ and increase t from 2 to N to determine $\tau^+_{(2,t)}$. We continue this process until $s = N$, at which point we stop since $\tau^+_{(N,N)} = 0$. The procedure allows us to determine thresholds for all (s,t) such that $1 \le s < t \le N$.

IV. SCALAR SYSTEMS WITH LOSSY CONTROL

Now let us consider problems in which $n = 1$ (scalar system), $\alpha \in (0,1)$, $Q = 1$, $\Sigma_w = \sigma_w^2$ and $\Sigma_v = \sigma_v^2$. First consider the following recursion with $K_{(0,0)} = 1$:

$$K_{(s,t)} = 1 + \alpha A^2 K_{(s-1,t-1)}; K_{(0,t)} = 1 + A^2 K_{(0,t-1)}$$

Now that there is a nonzero probability of dropped control, even when control action is taken, the cost-to-go must take into account that control could be lost. Let s and t denote respectively the number of control actions left and the number of decision instances left. We start with stage $(0,1)$ with $u_{(0,1)} = 0$ and

$$J_{(0,1)} = K_{(0,1)}E\{x^2_{N-1}|y_0^{N-1}\} + \sigma_w^2$$

When $s = 1$, (1,1) can only lead to stage (0,0) with the control $u_{(1,1)} = -AE\{x_{N-1}|y_0^{N-1}\}$ and cost-to-go

$$\begin{aligned}J_{(1,1)} &= K_{(1,1)}E\{x^2_{N-1}|y_0^{N-1}\} + \sigma_w^2 \\ &+ \bar{\alpha}A^2E\{(x_{N-1} - E\{x_{N-1}|y_0^{N-1}\})^2|y_0^{N-1}\}\end{aligned}$$

where α and $\bar{\alpha} = 1-\alpha$ have entered the equation due to the probability of dropped control. We again know that for every $k \in [0, N-1]$, $x_k - E\{x_k|I_k\}$ is independent of the control policy being used. Therefore we may write:

$$J_{(1,1)} = K_{(1,1)}E\{x^2_{N-1}|y_0^{N-1}\} + \bar{\alpha}A^2\sigma^2_{N-1|N-1} + \sigma_w^2$$

with $\sigma^2_{k|k}$ defined by $\sigma^2_{k|k} := E\{(x_k - E\{x_k|y_0^k\})^2|y_0^k\}$ and evolution given by

$$\begin{aligned}\sigma^2_{k+1|k} &= A^2\sigma^2_{k|k} + \sigma_w^2 \\ \sigma^2_{k|k} &= \sigma^2_{k|k-1} - \frac{(\sigma^2_{k|k-1})^2}{\sigma^2_{k|k-1} + \sigma_v^2}.\end{aligned}$$

Now let $t = 2$. When $s = 0$, $(0,2)$ can only lead to $(0,1)$ so we have $u_{(0,2)} = 0$ and

$$J_{(0,2)} = K_{(0,2)}E\{x^2_{N-2}|y_0^{N-2}\} + K_{(0,1)}\sigma_w^2 + \sigma_w^2$$

With $s = 1$, $(1,2)$ may either lead to $(1,1)$ or $(0,1)$, depending on whether control is applied or not. When control is applied we find the optimal policy by minimizing the quadratic cost-to-go function to arrive at:

$$u^{(0)}_{(1,2)} = 0; u^{(1)}_{(1,2)} = -AE\{x_{N-2}|y_0^{N-2}\}$$

and corresponding cost-to-go

$$\begin{aligned}J^{(0)}_{(1,2)} &= (1 + A^2K_{(1,1)})E\{x^2_{N-2}|y_0^{N-2}\} + \bar{\alpha}A^2\sigma^2_{N-1|N-1} \\ &+ (2 + \alpha A^2)\sigma_w^2 \\ J^{(1)}_{(1,2)} &= K_{(1,2)}E\{x^2_{N-2}|y_0^{N-2}\} + \bar{\alpha}A^2K_{(0,1)}\sigma^2_{N-2|N-2} \\ &+ (1 + K_{(0,1)})\sigma_w^2\end{aligned}$$

Compare the possible outcomes using $\Delta_{(1,2)}$:

$$\Delta_{(1,2)} = \bar{\alpha}A^2(\hat{x}^2_{N-2|N-2} + \sigma^2_{N-1|N-1} - \sigma^2_{N-1|N-2})$$

As a function of $E\{x_{N-2}|y_0^{N-2}\}$, $\Delta_{(1,2)}(E\{x_{N-2}|y_0^{N-2}\})$ has a unique minimum:

$$\Delta_{(1,2)}(0) = \bar{\alpha}A^2(\sigma^2_{N-1|N-1} - \sigma^2_{N-1|N-2}) \le 0$$

Since this is a quadratic function, $\Delta_{(1,2)}(\hat{x}_{N-2|N-2}) = 0$ is even and has two real roots, $\tau^+_{(1,2)} = -\tau^-_{(1,2)} \ge 0$.

If $s=2$, $(2,2)$ can only lead to (1,1) with the control law $u^{(1)}_{(2,2)} = -AE\{x_{N-2}|y_0^{N-2}\}$ and corresponding cost-to-go

$$J_{(2,2)} = K_{(2,2)}E\{x^2_{N-2}|y_0^{N-2}\} + \bar{\alpha}A^2K_{(1,1)}\sigma^2_{N-2|N-2} + \bar{\alpha}A^2\sigma^2_{N-1|N-1} + (1+K_{(1,1)})\sigma^2_w$$

Next let $t=3$, which implies that $0 \le s \le 3$. If $s=0$, we reason that $u_{(0,3)} = 0$ and

$$J_{(0,3)} = K_{(0,3)}E\{(x_{N-3})^2|y_0^{N-3}\} + \sum_{k=0}^{2} K_{(0,k)}\sigma^2_w$$

If $s=1$, $(1,3)$ may either lead to $(1,2)$ or $(0,2)$. In case we use control at $(1,3)$, we have the optimal policy $u^{(1)}_{(1,3)} = -AE\{x_{N-3}|y_0^{N-3}\}$ since it must minimize a quadratic cost function. The associated cost-to-go is

$$J^{(1)}_{(1,3)} = K_{(1,3)}E\{x^2_{N-3}|y_0^{N-3}\} + \bar{\alpha}A^2\sigma^2_{N-3|N-3} + \sum_{k=0}^{2} K_{(0,k)}\sigma^2_w$$

If no control is used at $(1,3)$ ($u^{(0)}_{(1,3)} = 0$), we need to average $J_{(1,2)}$ over the statistics of $E\{x_{N-2}|y_0^{N-2}\}$ given y_0^{N-3} to get the cost-to-go:

$$J^{(0)}_{(1,3)} = E\{(x_{N-3})^2|y_0^{N-3}\} + \int_{|\hat{x}_{N-2|N-2}|\le\tau^+_{(1,2)}} J^{(0)}_{(1,2)} f^{(0)}_{\hat{x}_{N-2|N-2}|y_0^{N-3}} + \int_{|\hat{x}_{N-2|N-2}|>\tau^+_{(1,2)}} J^{(1)}_{(1,2)} f^{(0)}_{\hat{x}_{N-2|N-2}|y_0^{N-3}}$$

where $f^{(0)}_{\hat{x}_{N-2|N-2}|y_0^{N-3}}$ is the conditional density function of $\hat{x}_{N-2|N-2}$ given the available information. It has the same parameters as in [14]:

$$f^{(0)}_{\hat{x}_{N-2|N-2}|y_0^{N-3}} \sim N(A\hat{x}_{N-3|N-3}, \frac{(\sigma^2_{N-2|N-3})^2}{\sigma^2_{N-2|N-3}+\sigma^2_v})$$

Substituting and rearranging gives

$$J^{(0)}_{(1,3)} = (1+A^2K_{(1,2)})E\{x^2_{N-3}|y_0^{N-3}\} + K_{(1,2)}\sigma^2_w + \bar{\alpha}A^2K_{(0,1)}\sigma^2_{N-2|N-2} + K_{(0,1)}\sigma^2_w + \sigma^2_w + \int_{|\hat{x}_{N-2|N-2}|\le\tau^+_{(1,2)}} \Delta_{(1,2)} f^{(0)}_{\hat{x}_{N-2|N-2}|y_0^{N-3}}$$

Define $\Delta_{(1,3)}$ and compare against zero. Substitution yields

$$\Delta_{(1,3)} = \bar{\alpha}A^2\hat{x}^2_{N-3|N-3} + \bar{\alpha}A^2K_{(0,1)}\sigma^2_{N-2|N-2} - \bar{\alpha}A^2K_{(0,1)}\sigma^2_{N-2|N-3} + \int_{|\hat{x}_{N-2|N-2}|\le\tau^+_{(1,2)}} \Delta_{(1,2)} f^{(0)}_{\hat{x}_{N-2|N-2}|y_0^{N-3}}$$

Yet again, the form of $\Delta_{(1,2)}$ dictates that $\Delta_{(1,3)}$ is even, achieves minimum at $\hat{x}_{N-3|N-3} = 0$ and is increasing to the right of the minimum.

Next let $s=2$ and note that $(2,3)$ only leads to $(2,2)$ or $(1,2)$. If control is not used and $u^{(0)}_{(2,3)} = 0$, we incur

$$J^{(0)}_{(2,3)} = (1+A^2K_{(2,2)})E\{x^2_{N-3}|y_0^{N-3}\} + \bar{\alpha}A^2K_{(1,1)}\sigma^2_{N-2|N-2} + \bar{\alpha}A^2\sigma^2_{N-1|N-1} + (K_{(2,2)}+K_{(1,1)}+1)\sigma^2_w.$$

On the other hand, if we do apply control at $(2,3)$ we must choose $u^{(1)}_{(2,3)}$ to minimize

$$J^{(1)}_{(2,3)} = E\{x^2_{N-3}|y_0^{N-3}\} + \alpha\left\{\int_{-\infty}^{\infty} J^{(1)}_{(1,2)} f^{(0)}_{\hat{x}_{N-2|N-2}|y_0^{N-3}} + \int_{|\hat{x}_{N-2|N-2}|\le\tau^+_{(1,2)}} \Delta_{(1,2)} f^{(0)}_{\hat{x}_{N-2|N-2}|y_0^{N-3}}\right\} + \bar{\alpha}\min_{u^{(1)}_{(2,3)}}\left\{\int_{|\hat{x}_{N-2|N-2}|\le\tau^+_{(1,2)}} \Delta_{(1,2)} f^{(1)}_{\hat{x}_{N-2|N-2}|y_0^{N-3}} + \int_{-\infty}^{\infty} J^{(1)}_{(1,2)} f^{(1)}_{\hat{x}_{N-2|N-2}|y_0^{N-3}}\right\}$$

where $f^{(1)}_{\hat{x}_{N-2|N-2}|y_0^{N-3}}$ is the conditional density function of $\hat{x}_{N-2|N-2}$ given the available information and given that the control input is nonzero *and is not dropped*. The distribution $f^{(1)}_{\hat{x}_{N-2|N-2}|y_0^{N-3}}$ is Gaussian, but depends on the input. Performing the necessary calculations reveals that

$$f^{(1)}{}_{\hat{x}_{N-2|N-2}|y_0^{N-3}} \sim N(A\hat{x}_{N-3|N-3} + u^{(1)}_{(2,3)}, \frac{(\sigma^2_{N-2|N-3})^2}{\sigma^2_{N-2|N-3}+\sigma^2_v})$$

Substituting the optimal policy $u^{(1)}_{(2,3)} = -AE\{x_{N-3}|y_0^{N-3}\}$ and simplifying gives

$$J^{(1)}_{(2,3)} = K_{(2,3)}E\{x^2_{N-3}|y_0^{N-3}\} + \bar{\alpha}A^2K_{(1,2)}\sigma^2_{N-3|N-3} + \bar{\alpha}A^2K_{(0,1)}\sigma^2_{N-2|N-2} + (K_{(1,2)}+K_{(0,1)}+1)\sigma^2_w + \bar{\alpha}\int_{|\hat{x}_{N-2|N-2}|\le\tau^+_{(1,2)}} \Delta_{(1,2)} f^{(1)}_{\hat{x}_{N-2|N-2}|y_0^{N-3}} + \alpha\int_{|\hat{x}_{N-2|N-2}|\le\tau^+_{(1,2)}} \Delta_{(1,2)} f^{(0)}_{\hat{x}_{N-2|N-2}|y_0^{N-3}}$$

We next derive $\Delta_{(2,3)}$. Substitution yields

$$\Delta_{(2,3)} = (1+A^2K_{(2,2)}-K_{(2,3)})(\hat{x}^2_{N-3|N-3}+\sigma^2_{N-3|N-3}) - \bar{\alpha}A^2K_{(1,2)}\sigma^2_{N-3|N-3} + \bar{\alpha}A^2(K_{(1,1)}-K_{(0,1)})\sigma^2_{N-2|N-2} + \bar{\alpha}A^2\sigma^2_{N-1|N-1} + (K_{(2,2)}+K_{(1,1)}-K_{(1,2)}-K_{(0,1)})\sigma^2_w - \alpha\int_{|\hat{x}_{N-2|N-2}|\le\tau^+_{(1,2)}} \Delta_{(1,2)} f^{(0)}_{\hat{x}_{N-2|N-2}|y_0^{N-3}} - \bar{\alpha}\int_{|\hat{x}_{N-2|N-2}|\le\tau^+_{(1,2)}} \Delta_{(1,2)} f^{(1)}_{\hat{x}_{N-2|N-2}|y_0^{N-3}}$$

If the regularity condition of Proposition 2 given below is satisfied, $\Delta_{(2,3)}$ is even, obtains minimum at $\hat{x}_{N-3|N-3} = 0$ and is increasing for $\hat{x}_{N-3|N-3} > 0$. Finally, we let $s=3$

and conclude, after optimizing, that optimal policy $u^{(1)}_{(3,3)} = -AE\{x_{N-3}|y_0^{N-3}\}$ incurs cost-to-go

$$\begin{aligned}J_{(3,3)} &= K_{(3,3)}E\{x^2_{N-3}|y_0^{N-3}\} + \bar{\alpha}A^2K_{(2,2)}\sigma^2_{N-3|N-3}\\ &+ \bar{\alpha}A^2K_{(1,1)}\sigma^2_{N-2|N-2} + \bar{\alpha}A^2\sigma^2_{N-1|N-1}\\ &+ (1 + K_{(1,1)} + K_{(2,2)})\sigma^2_w\end{aligned}$$

We continue this procedure for $t = 4, 5, ..., N$ and by induction we see that the optimal control policy is a threshold policy on a projection of the best estimate of the plant state, which can be recursively generated by a Kalman filter. Furthermore, the threshold at time k is a function of four variables: N (the decision horizon), t_k (number of decision instances left, s_k (number of control actions left) and $\sigma^2_{k|k-1}$ (the error covariance). The error covariance $\sigma^2_{0|-1} = E\{(x_0 - E\{x_0\})^2\}$ is known and all other values can be iteratively calculated. For a given N the thresholds can be computed entirely offline using the procedure described in this section. On the other hand, the best estimates for state must be calculated online with a Kalman filter and initial condition $\hat{x}_{0|-1} = E\{x_0\}$.

Thus, starting with $\hat{x}_{0|-1} = E\{x_0\}$, $s_0 = M$, $t_0 = N$ the optimal control policy can be implemented by the following algorithm: For each k in $0 \le k \le N-1$,

1) Look up the threshold $\tau^+_{(s_k,t_k)}$ corresponding to the current stage from the table. The regularity condition of Proposition 2 below must be satisfied at each stage (s,t) with $1 < s < t \le N$
2) Observe y_k and update the state estimate to $\hat{x}_{k|k}$ using the Kalman filter recursion.
3) Apply the control policy

$$u_{(s_k,t_k)} = \begin{cases} 0 & \text{if } |\hat{x}_{k|k}| < \tau^+_{(s_k,t_k)} \\ -A\hat{x}_{k|k} & \text{if } |\hat{x}_{k|k}| \ge \tau^+_{(s_k,t_k)} \end{cases}$$

4) Update s_{k+1} according to whether a control is used and $t_{k+1} = t_k - 1$

We finally give the iterations to calculate the thresholds $\tau^+_{(s_k,t_k)}$ for a given horizon $N \ge 1$ and arbitrary pair of integers (s,t) such that $1 \le s \le t \le N$. The cost-to-go functions can be written as

$$\begin{aligned}J^{(0)}_{(s,t)} &= (1 + A^2K_{(s,t-1)})E\{x^2_{N-t}|y_0^{N-t}\} + K_{(s,t-1)}\sigma^2_w\\ &+ \Lambda_{(s,t-1)} + \int_{|\hat{x}_{N-t+1|N-t+1}| \le \tau^+_{(s,t-1)}} \Delta_{(s,t-1)}\\ &\times f^{(0)}_{\hat{x}_{N-t+1|N-t+1}|y_0^{N-t}} d\hat{x}_{N-t+1|N-t+1}\\ &+ \int_{-\infty}^{\infty} F_{(s,t-1)}(\hat{x}_{N-t+1|N-t+1})\\ &\times f^{(0)}_{\hat{x}_{N-t+1|N-t+1}|y_0^{N-t}} d\hat{x}_{N-t+1|N-t+1}\end{aligned}$$

$$J^{(1)}_{(s,t)} = K_{(s,t)}E\{x^2_{N-t}|y_0^{N-t}\} + \Lambda_{(s,t)} + F_{(s,t)}(\hat{x}_{N-t|N-t})$$

where

$$f^{(0)}_{\hat{x}_{N-t+1|N-t+1}|y_0^{N-t}} \sim N(A\hat{x}_{N-t|N-t}, \frac{(\sigma^2_{N-t+1|N-t})^2}{\sigma^2_{N-t+1|N-t} + \sigma^2_v})$$

and $f^{(1)}_{\hat{x}_{N-t+1|N-t+1}|y_0^{N-t}}$ is mean zero Gaussian with the same variance as $f^{(0)}_{\hat{x}_{N-t+1|N-t+1}|y_0^{N-t}}$. For $1 < s < t \le N$,

$$\begin{aligned}\Lambda_{(s,t)} &= \Lambda_{(s-1,t-1)} + \bar{\alpha}A^2K_{(s-1,t-1)}\sigma^2_{N-t|N-t} +\\ &+ K_{(s-1,t-1)}\sigma^2_w + \alpha\int_{|\hat{x}_{N-t+1|N-t+1}| \le \tau^+_{(s,t-1)}} \Delta_{(s,t-1)}\\ &\times f^{(1)}_{\hat{x}_{N-t+1|N-t+1}|y_0^{N-t}} d\hat{x}_{N-t+1|N-t+1}\\ &+ \bar{\alpha}\int_{-\infty}^{\infty} F_{(s,t-1)}(\hat{x}_{N-t+1|N-t+1})\\ &\times f^{(1)}_{\hat{x}_{N-t+1|N-t+1}|y_0^{N-t}} d\hat{x}_{N-t+1|N-t+1}\end{aligned}$$

$$\begin{aligned}F_{(s,t)} &= \alpha\int_{|\hat{x}_{N-t+1|N-t+1}| \le \tau^+_{(s-1,t-1)}} \Delta_{(s-1,t-1)}\\ &\times f^{(0)}_{\hat{x}_{N-t+1|N-t+1}|y_0^{N-t}} d\hat{x}_{N-t+1|N-t+1}\\ &+ \alpha\int_{-\infty}^{\infty} F_{(s-1,t-1)}(\hat{x}_{N-t+1|N-t+1})\\ &\times f^{(0)}_{\hat{x}_{N-t+1|N-t+1}|y_0^{N-t}} d\hat{x}_{N-t+1|N-t+1}\end{aligned}$$

For $s = 1$ and $1 \le t \le N$, we have $F_{(1,t)} = 0$, $\Lambda_{(1,t)} = \bar{\alpha}A^2K_{(0,t-1)}\sigma^2_{N-t|N-t} + \sum_{n=0}^{t-1} K_{(0,n)}\sigma^2_w$.

Define $\Delta_{(s,t)} := J^{(0)}_{(s,t)} - J^{(1)}_{(s,t)}$. Then for $1 \le s < t \le N$,

$$\begin{aligned}\Delta_{(s,t)}(\hat{x}_{N-t|N-t}) &= (1 + A^2K_{(s,t-1)} - K_{(s,t)})\\ &\times (\hat{x}_{N-t|N-t} + \sigma^2_{N-t|N-t}) + K_{(s,t-1)}\sigma^2_w\\ &+ \Lambda_{(s,t-1)} - \Lambda_{(s,t)} + \int_{|\hat{x}_{N-t+1|N-t+1}| \le \tau^+_{(s,t-1)}} \Delta_{(s,t-1)}\\ &\times f^{(0)}_{\hat{x}_{N-t+1|N-t+1}|y_0^{N-t}} d\hat{x}_{N-t+1|N-t+1}\\ &+ \int_{-\infty}^{\infty} F_{(s,t-1)}(\hat{x}_{N-t+1|N-t+1})\\ &\times f^{(0)}_{\hat{x}_{N-t+1|N-t+1}|y_0^{N-t}} d\hat{x}_{N-t+1|N-t+1} - F_{(s,t)}\end{aligned}$$

Note that $\Lambda_{(s,t)}$ is a sequence of real numbers while $\Delta_{(s,t)}$ is a sequence of functions. We must also have boundary conditions for $1 \le t \le N$

$$\begin{aligned}\Lambda_{(t,t)} &= \Lambda_{(t-1,t-1)} + \bar{\alpha}A^2K_{(t-1,t-1)}\sigma^2_{N-t|N-t}\\ &+ K_{(t-1,t-1)}\sigma^2_w, \Lambda_{(0,0)} = 0\\ \Delta_{(t,t)} &= \bar{\alpha}A^2K_{(t-1,t-1)}\hat{x}^2_{N-t|N-t}; F_{(t,t)} = 0\end{aligned}$$

We also have $\tau^+_{(t,t)} = 0$, $1 \le t \le N$ and for $1 \le s < t \le N$, the thresholds, $\tau^+_{(s,t)}$, are given by the positive solution of the nonlinear equation $\Delta_{(s,t)}(\tau^+_{(s,t)}) = 0$. In order to show that such a solution exists, we need the following result, whose proof is not given due to page limitations.

Proposition 2: *Let $N \ge 2$ be given, and a regularity condition be satisfied for each (s,t) with $1 < s \le t \le N$:*

$$\begin{aligned}&2A^2(K_{(s,t-1)} - \alpha K_{(s-1,t-1)})\\ &\qquad + \frac{\alpha A^2}{\sigma^2}(\Delta_{(s-1,t-1)}(0) + F_{(s-t,t-1)}(0)) > 0\end{aligned}$$

where $\sigma^2 = \frac{(\sigma^2_{N-t+1|N-t})^2}{\sigma^2_{N-t+1|N-t}+\sigma^2_v}$. *Then for* $1 \le s < t \le N$, *the sequence of functions* $\Delta_{(s,t)}(u)$ *are even, differentiable with a unique critical point at* $u = 0$, *i.e.,* $\left.\frac{\partial \Delta_{(s,t)}(u)}{\partial u}\right|_{u=0} = 0$. *Furthermore, we have* $\frac{\partial \Delta_{(s,t)}(u)}{\partial u} > 0$ *if* $u > 0$, *and* $\frac{\partial \Delta_{(s,t)}(u)}{\partial u} < 0$ *if* $u < 0$ *so that the global minimum is achieved at* $u = 0$. *Also, the minimum of* $\Delta_{(s,t)}(u)$ *at critical point* $u = 0$ *is nonpositive, i.e.,* $\Delta_{(s,t)}(0) \le 0$

In the offline computation of thresholds, we start with $s = 1$ and increase t from 1 to N and determine $\tau^+_{(1,t)}$. Next, we increment s by 1 to $s = 2$ and increase t from 2 to N to determine $\tau^+_{(2,t)}$. We continue this procedure until $s = N$, at which point we stop since $\tau^+_{(N,N)} = 0$. The process allows us to determine thresholds for all (s,t) such that $1 \le s < t \le N$. Before each threshold calculation, we must check that the regularity condition is satisfied for (s,t) in the range $1 < s \le t \le N$. Additionally, the calculation of each $J^{(1)}_{(s,t)}$ requires that the regularity condition is satisfied for $\Delta_{(s-1,t-1)}$. It is easily seen that the condition is satisfied for (i) $s = t$ and (ii) $\alpha = 0$ (as well as some neighborhood of $\alpha = 0$, by continuity). For $s = 1$, no condition needs to be satisfied.

V. A NUMERICAL EXAMPLE

Suppose we have a scalar system with $A = 1$, $N = 10$, $\sigma^2_w = \sigma^2_v = 1$ and x_0 with mean zero and variance 1. It can be verified that the regularity condition holds for these parameters and *any* choice of α, and as stated in Proposition 2 the plots are even, achieve minimum at zero and increasing to the right of the origin. We plot $\Delta_{(2,3)}$ as a function of $\hat{x}_{7|7}$ for $\alpha = 0, 0.1, 0.2$ in Fig 1. Note that as the probability of control action loss increases, it is optimal to be more conservative with control actions. This can be seen by comparing the roots of $\Delta_{(2,3)}$, which function as thresholds for deciding whether to transmit or not.

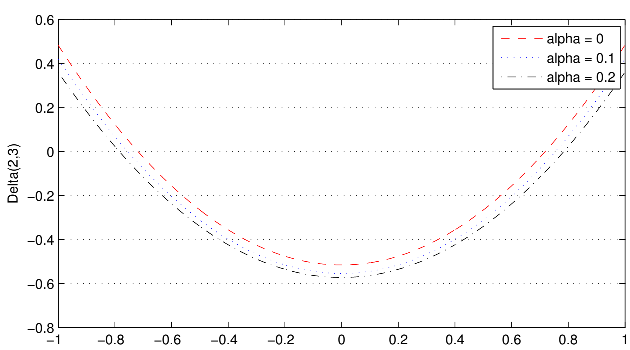

Fig. 1. Graph of $\Delta_{(2,3)}$ for different values of α

VI. CONCLUSIONS AND FUTURE WORK

In this paper we have obtained the optimal control policies for two classes of control problems in which there are limits on the number of control actions. In the first class the system is multidimensional but there are as many control inputs as the dimension of the state and the system output that is penalized is scalar. In the second class, we have also allowed lossiness in the transmission of control signals, but have considered scalar plants. For both problems the optimal controls are threshold policies.

Although this paper has considered special cases of the general framework of Section I, it should be noted that the techniques used can be applied to any variant of the proposed problem type. In the most general case, with no restriction on the structure of cost matrix Q or dimension of state, it turns out that the thresholds must be made on $\hat{x}_{k|k}$, which results in decision regions being subsets of $\mathcal{R}^n$. Even in the case of $Q = gg^T$, but with A not a multiple of I, the solution will not be in terms of the *projection* of $\hat{x}_{k|k}$ onto g.

One may also consider a more general problem statement in which not only control packets are dropped, but observations as well. In this case, we may no longer keep track of $\sigma^2_{k|k-1}$ offline. As new observations arrive, we must update the error covariance accordingly and so thresholds must also be recomputed every time an observation is made.

There are several avenues for future work that is related to what is presented here. For example, one may wish to gain insight into the regularity condition and show under what conditions it would be true. Also, the information pattern studied can be switched to UDP [9], a more difficult scenario. Finally, one could look into distributed control problems in which agents may communicate with each other only a limited number of times.

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