Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Lyapunov characterization of Zeno behavior in hybrid systems

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Abstract— **Necessary and sufficient Lyapunov characterizations of a particular type of asymptotic stability in hybrid dynamical systems are given. This type involves uniform bounds on the amount of ordinary time, but not the number of jumps, it takes solutions to converge to the asymptotically stable set. Connections of this asymptotic stability concept to Zeno behavior are explored. Necessary and sufficient conditions for Zeno and uniform Zeno stability are shown.**

I. INTRODUCTION

Hybrid systems are rich in dynamical phenomena not encountered in classical dynamical systems. One such phenomenon, which has captured the attention of many researchers, is Zeno behavior. For a description see, for example, [25], [12]. Roughly speaking, Zeno solutions to a hybrid system are those that experience an infinite number of jumps in a finite amount of ordinary time. The existence of such solutions brings up questions like "How does one simulate Zeno solutions?" [17], [18] and "(How) should one try to get past Zeno times?" [3], [26], [8], [6]. An additional line of research aims to characterize when Zeno behavior is present or not present in a hybrid system. Work in this direction includes [21], [16], [24], [4], [1], [11], [5], [2], [19].

Most of the results that characterize Zeno behavior revolve around so-called Zeno equilibria, which are asymptotically stable points that attract Zeno solutions. When Zeno solutions are linked to asymptotic stability, like in [5], it is natural to consider Lyapunov characterizations of Zeno-type convergence. This was done in the recent work [19] and it is also the nature of our work here. Our results are inspired by [19, Theorem 1] which provides Lyapunov-like sufficient conditions for the existence of Zeno behavior, leading to necessary and sufficient conditions for Zeno behavior in a class of Lagrangian hybrid systems. Based on recent converse Lyapunov theorems for hybrid systems [9], [10], we provide necessary and sufficient Lyapunov conditions for a certain type of uniform Zeno asymptotic stability. We will show that the Lyapunov-like sufficient conditions in [19, Theorem 1] imply the Lyapunov conditions we enumerate.

Our necessary and sufficient Lyapunov conditions address asymptotically stable compact sets for which the solutions' length of ordinary time required to converge to the set decreases to zero uniformly as the initial conditions approach the set. (We call this "uniformly small ordinary time", i.e., USOT, stability.) We relate the Lyapunov characterization of USOT stability to Zeno behavior, including "truly" or "genuinely" Zeno behavior. There is a strong connection between USOT stability and finite-time asymptotic stability in purely continuous-time setting. As in that setting (see, for example, [22]), we use ordinary time as a component in the construction of smooth Lyapunov functions.

In what follows, we review the hybrid systems framework used here and certain aspects of stability theory for such systems. Then we present our necessary and sufficient conditions for USOT stability. Next, we relate USOT stability to Zeno behaviors. At the end of the paper, we relate our Lyapunov conditions to those used in [19, Theorem 1].

II. BACKGROUND

A. Hybrid systems

Let $F, G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be set-valued mappings and $C, D \subset \mathbb{R}^n$ be sets. We consider hybrid systems of the form

$$\mathcal{H}: \quad \left\{ \begin{array}{ll} \dot{x} \in F(x) & x \in C \ , \\ x^+ \in G(x) & x \in D \ . \end{array} \right. \tag{1}$$

For more background on hybrid systems in this framework, including a description of how to view hybrid automata in this framework, see [7], [13], and [14].

A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if $E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ ... \leq t_J$. It is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ...J\})$ is a compact hybrid time domain. Equivalently, E is a hybrid time domain if E is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the "last" interval possibly of the form $[t_j, T)$ with T finite or $T = +\infty$. A *hybrid arc* is a function ϕ whose domain $\operatorname{dom} \phi$ is a hybrid time domain and such that for each $j \in \mathbb{N}$, $t \to \phi(t, j)$ is locally absolutely continuous on $I_j := \{t \,|\, (t, j) \in \operatorname{dom} \phi\}$. A hybrid arc ϕ is *complete* if its domain, $\operatorname{dom} \phi$, is unbounded.

A hybrid arc ϕ is a *solution to the hybrid system* $\mathcal{H}$ if $\phi(0, 0) \in C \cup D$ and

(S1) for all $j \in \mathbb{N}$ such that the interval I_j has nonempty interior and for almost all $t \in I_j$,

$$\phi(t, j) \in C, \quad \dot{\phi}(t, j) \in F(\phi(t, j));$$

(S2) for all $(t, j) \in \operatorname{dom} \phi$ such that $(t, j+1) \in \operatorname{dom} \phi$,

$$\phi(t, j) \in D, \quad \phi(t, j+1) \in G(\phi(t, j)).$$

A solution ϕ is *maximal* if there does not exist a solution ψ with $\operatorname{dom} \phi \subset \operatorname{dom} \psi$, $\operatorname{dom} \phi \neq \operatorname{dom} \psi$, $\phi(t, j) = \psi(t, j)$ for all $(t, j) \in \operatorname{dom} \phi$. Complete solutions are maximal.

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Structural properties of solutions to $\mathcal{H}$, like (appropriately understood) sequential compactness of the space of solutions and outer/upper semicontinuous dependence of solutions on initial conditions, were established in [14] under the following assumptions (simplified slightly for this paper):[1]

Assumption 2.1:

(A1) *C and D are closed subsets of $\mathbb{R}^n$;*

(A2) *$F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $F(x)$ is nonempty and convex for all $x \in C$;*

(A3) *$G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $G(x)$ is nonempty for all $x \in D$.*

B. Asymptotic stability, robustness, and Lyapunov functions

The work [14] included results on uniformity of asymptotic stability and its robustness, and made possible the general converse Lyapunov theorems for hybrid systems in [9], [10]. We recall key results from these papers below. First, a definition: compact set $\mathcal{A} \in \mathbb{R}^n$ is *pre-asymptotically stable* (pre-AS) for the hybrid system (1) if:

(a) for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for each solution ϕ to (1) with $|\phi(0,0)|_{\mathcal{A}} \leq \delta$ one has $|\phi(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$;

(b) each solution ϕ to (1) is bounded, and if it is complete, then also $|\phi(t,j)|_{\mathcal{A}} \to 0$ as $t+j \to \infty$, $(t,j) \in \operatorname{dom}\phi$.

Above and in what follows, with some abuse of notation we write $|\cdot|_{\mathcal{A}}$ for the distance from the set $\mathcal{A}$, in the Euclidean norm. That norm itself will be denoted by $|\cdot|$.

Note that the definition of pre-asymptotic stability does not insist on completeness of solutions. This is motivated by the fact that Lyapunov inequalities have no bearing on completeness (or even existence) of solutions. Not insisting on completeness also justifies adding the prefix "pre" to the name. Also note that, to simplify the presentation, we are implicitly using *global* pre-asymptotic stability, in that *each* solution must satisfy (b). (Recall that there are no solutions from points outside $C \cup D$.) Local pre-asymptotic stability can be converted to global pre-asymptotic stability by intersecting C and D with any compact subset of the basin of attraction. Otherwise, behavior on the entire basin of attraction can be considered, like in [10].

When discussing robustness of pre-asymptotic stability of a compact set $\mathcal{A}$, by an *admissible perturbation radius* we will understand any continuous function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is positive on $\mathbb{R}^n \setminus \mathcal{A}$. A compact set $\mathcal{A}$ is *robustly pre-asymptotically stable* for the hybrid system (1) if there exists an admissible perturbation radius ρ such that $\mathcal{A}$ is pre-asymptotically stable for the hybrid system

$$\mathcal{H}_\rho : \quad \begin{cases} \dot{x} \in F_\rho(x) & x \in C_\rho, \\ x^+ \in G_\rho(x), & x \in D_\rho, \end{cases}$$

given by the data

$$F_\rho(x) = \operatorname{con} F\left(x + \rho(x)\mathbb{B}\right) + \rho(x)\mathbb{B},$$

$$C_\rho = \{x \mid x + \rho(x)\mathbb{B} \cap C \neq \emptyset\},$$

$$G_\rho(x) = \bigcup_{y \in G(x+\rho(x)\mathbb{B})} y + \rho(y)\mathbb{B},$$

$$D_\rho = \{x \mid x + \rho(x)\mathbb{B} \cap D \neq \emptyset\}.$$

Results of [14], [10] imply that pre-asymptotic stability is automatically robust.

Theorem 2.2: *Under Assumption 2.1, a compact set $\mathcal{A} \subset \mathbb{R}^n$ is pre-asymptotically stable for* (1) *if and only if it is robustly pre-asymptotically stable.*

Results of [9], [10] imply the following theorem.

Theorem 2.3: *Under Assumption 2.1, a compact set $\mathcal{A} \subset \mathbb{R}^n$ is pre-asymptotically stable for* (1) *if and only if there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that*

$$\alpha_1\left(|x|_{\mathcal{A}}\right) \leq V(x) \leq \alpha_2\left(|x|_{\mathcal{A}}\right) \qquad \forall x \in \mathbb{R}^n,$$

$$\langle \nabla V(x), f\rangle \leq -V(x) \qquad \forall\ x \in C,\ f \in F(x),$$

$$V(g) \leq e^{-1}V(x) \qquad \forall\ x \in D,\ g \in G(x).$$

III. Necessary and sufficient Lyapunov conditions for uniformly small ordinary time pre-asymptotic stability

This section analyzes a particular type of pre-asymptotic stability, in which the "ordinary time" it takes a solution to reach a pre-AS compact set $\mathcal{A}$ decreases to zero with initial conditions approaching $\mathcal{A}$. A detailed definition is below. It will be convenient to use the following object: given a set $X \subset \mathbb{R}^n$ and a hybrid arc ϕ, let

$$T_X(\phi) := \sup\left\{t \mid \exists j \text{ s.t. } (t,j) \in \operatorname{dom}\phi,\ \phi(t,j) \in X\right\}.$$

Definition 3.1: *[USOT pre-AS] A compact set $\mathcal{A} \subset \mathbb{R}^n$ is called* uniformly small ordinary time pre-asymptotically stable *for the hybrid system $\mathcal{H}$ if the following hold:*

(i) *$\mathcal{A}$ pre-asymptotically stable, and*

(ii) *for each $\varepsilon > 0$ there exists $\delta > 0$ such that every solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \leq \delta$ satisfies $T_{\mathbb{R}^n \setminus \mathcal{A}}(\phi) \leq \varepsilon$.*

We start with a set of Lyapunov-Krasovskii-LaSalle conditions that are sufficient for USOT pre-AS. We note that sufficient conditions of a very different nature, relying on homogeneity properties of the data of a hybrid system, are presented in a companion paper [15].

Proposition 3.2: *Suppose that $\mathcal{H}$ satisfies Assumption 2.1 and let $\mathcal{A} \subset \mathbb{R}^n$ be compact. The set $\mathcal{A}$ is USOT pre-AS if there exist $\sigma > 0$, $p \in [0,1)$, a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on an open set containing $C \setminus \mathcal{A}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that*

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \quad \forall x \in C \cup D \cup G(D)$$

$$\begin{array}{rclll} \langle \nabla V(x), f\rangle & \leq & -\sigma V(x)^p & \forall x \in C \setminus \mathcal{A}\ , & f \in F(x)\ , \\ V(g) & \leq & V(x) & \forall x \in D\ , & g \in G(x)\ , \end{array}$$

and every discrete and complete solution (i.e., one with domain $\{0\} \times \mathbb{N}$) converges to $\mathcal{A}$.

[1] The set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* if for every convergent sequence of x_i's and every convergent sequence of $y_i \in F(x_i)$, $\lim y_i \in F(\lim x_i)$. F is *locally bounded* if for every compact $K \subset \mathbb{R}^n$ there exists a compact $K' \subset \mathbb{R}^n$ such that $F(K) \subset K'$. Similarly for G.

The necessary Lyapunov conditions for USOT pre-AS in the theorem below constitute our main result. The conditions are also sufficient, but stronger than those in Proposition 3.2 since the Lyapunov function decreases at jumps.

Theorem 3.3: *[Lyapunov characterization of USOT pre-AS] Suppose that a hybrid system $\mathcal{H}$ satisfies Assumption 2.1. For a compact set $\mathcal{A} \subset \mathbb{R}^n$ to be USOT pre-AS for $\mathcal{H}$ it is both necessary and sufficient that there exist a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is continuously differentiable on $\mathbb{R}^n \setminus \mathcal{A}$ and functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that*

$$\alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A}) \qquad \forall x \in \mathbb{R}^n$$

$$\begin{array}{rcll} \langle \nabla V(x), f \rangle & \leq & -1 & \forall x \in C \setminus \mathcal{A}\,, \quad f \in F(x) \\ V(g) - V(x) & \leq & -\alpha_3(V(x)) & \forall x \in D\,, \quad\;\; g \in G(x)\,. \end{array}$$

Example 3.4: [*The ubiquitous bouncing ball*] Consider the hybrid system with data

$$F(x) = \begin{bmatrix} x_2 \\ -g \end{bmatrix}, \qquad C = \left\{ x \in \mathbb{R}^2 : x_1 \geq 0 \right\},$$
$$G(x) = \begin{bmatrix} 0 \\ -\gamma x_2 \end{bmatrix}, \quad D = \left\{ x \in \mathbb{R}^2 : x_1 = 0\,, x_2 \leq 0 \right\},$$

where $g > 0$, $\gamma \in [0, 1)$. Let $k > \sqrt{2}(1+\gamma)/(1-\gamma)$ and let V be continuously differentiable on $\mathbb{R}^2 \setminus \{0\}$ and such that, for all x in D and an open set containing $C \setminus \{0\}$,

$$V(x) = g^{-1} \left(x_2 + k\sqrt{\frac{1}{2}x_2^2 + g x_1} \right) .$$

This choice is inspired by [19]. It is simple to verify that it satisfies the condition on f in Theorem 3.3. The condition on G comes from the lower bound on k which gives $\gamma|x_2| + \frac{k\gamma|x_2|}{\sqrt{2}} < -|x_2| + \frac{k|x_2|}{\sqrt{2}}$ for all $x_2 \neq 0$. □

Proving the necessity part of Theorem 3.3 includes showing that USOT pre-AS is a robust property. That is, if a compact set is USOT pre-AS for $\mathcal{H}$, then there exist an admissible perturbation radius ρ such that $\mathcal{A}$ is also USOT pre-AS for $\mathcal{H}_\rho$. One could ask whether "finite ordinary time pre-asymptotic stability" is also robust. This turns out to be false, as the following example illustrates.

Example 3.5: In the xy-plane, consider a hybrid system $\mathcal{H}$ with

$$F(x, y) = (y, 0), \quad C = \{(x, y) \,|\, y \geq x \geq 0\},$$
$$G(x, y) = (0, x - y), \quad D = \{(x, y) \,|\, x \geq y \geq 0\}.$$

Then $\mathcal{A} = (0, 0)$ is pre-AS. Moreover, every solution to $\mathcal{H}$ reaches $\mathcal{A}$ in finite amount of hybrid time, and consequently, in finite amount of ordinary time. Indeed, solutions starting in C may flow for up to one unit of time, after which they jump to $\mathcal{A}$. Solutions from D jump either to $\mathcal{A}$, or to $(0, y)$ with $y > 0$, after which they flow for one unit of time and jump to $\mathcal{A}$. In particular, $T_{\mathbb{R}^2\setminus\mathcal{A}}(\phi) \leq 1$ for each solution ϕ to $\mathcal{H}$. However, there are solutions starting arbitrarily close to $\mathcal{A}$ with $T_{\mathbb{R}^2\setminus\mathcal{A}}(\phi) = 1$, so $\mathcal{A}$ is not USOT pre-AS.

Now consider an arbitrary admissible perturbation radius ρ. In particular, $\rho(x, y) > 0$ for each $x = y > 0$, and thus for each such (x, y), $G_\rho(x, y)$ contains $(0, z)$ with $z > 0$. This leads to solutions, starting arbitrarily close to $\mathcal{A}$ in fact, that experience infinitely many intervals of flow of length 1. For such solutions ϕ, $T_{\mathbb{R}^2\setminus\mathcal{A}}(\phi) = \infty$. □

IV. Necessary and sufficient conditions for uniform Zeno stability

The goal of this section is to develop conditions, with a Lyapunov component, that are necessary and sufficient for the existence of a Zeno equilibrium or compact set. This motivation and the results of the previous section lead us to consider definitions of Zeno behavior that incorporate pre-AS and USOT pre-AS. Indirectly, we compare the definition we use to other definitions that have been used in the literature.

Often a hybrid arc ϕ is called *Zeno* if it is complete but

$$\sup_t \operatorname{dom} \phi := \sup\{t \in \mathbb{R}_{\geq 0} \,|\, \exists j \text{ s.t. } (t, j) \in \operatorname{dom} \phi\}$$

is finite. In short, ϕ is Zeno if it experiences infinitely many jumps in finite (ordinary) time. In such terminology, the "tail" of the hybrid arc, or even the whole arc itself, may consist of infinitely many instantaneous jumps. While this definition of Zeno behavior can serve as the basis for necessary and sufficient conditions, we choose to focus on a subset of these behaviors, called "truly Zeno" behaviors, as characterized by the following definition. (Cf. [19] for example.)

Definition 4.1: *[Zeno arc] A hybrid arc ϕ is* Zeno *if*

(i) *ϕ is complete,*
(ii) $\sup_t \operatorname{dom} \phi < \infty$,
(iii) *there does not exist j such that $(\sup_t \operatorname{dom} \phi, j) \in \operatorname{dom} \phi$.*

It is straightforward, from (i) and (iii) above, that each Zeno arc ϕ satisfies $\sup_t \operatorname{dom} \phi > 0$. Note that $T_{\mathbb{R}^n}(\phi) = \sup_t \operatorname{dom} \phi$, and so any Zeno arc satisfies $0 < T_{\mathbb{R}^n}(\phi) < \infty$.

Nonexistence of Zeno solutions is not robust. Indeed, replacing the flow map in Example 3.5 with $F(x, y) = (1, 0)$ leads to a system with no Zeno solutions. For the modified system $\mathcal{A}$ is USOT pre-AS, in contrast to the original system. Moreover, arbitrarily small perturbations lead to Zeno solutions. This can be argued along the lines used in Example 3.5.

Below we consider two forms of stability in the presence of Zeno solutions: Zeno stability and uniform Zeno stability.

A. Zeno stability

Definition 4.2: *[Zeno stability] A compact set $\mathcal{A} \subset \mathbb{R}^n$ is called* Zeno asymptotically stable *for the hybrid system $\mathcal{H}$ if the following hold:*

(i) *$\mathcal{A}$ is pre-asymptotically stable, and*
(ii) *there exists $\varepsilon > 0$ such that every maximal solution ϕ to $\mathcal{H}$ with $|\phi(0, 0)|_\mathcal{A} \in (0, \varepsilon]$ is Zeno and*

$$T_{\mathbb{R}^n}(\phi) = T_{\mathbb{R}^n\setminus\mathcal{A}}(\phi).$$

In words, a Zeno asymptotically stable compact set is one that is pre-asymptotically stable with trajectories converging toward the set using a finite amount of ordinary time, but never actually reaching a final ordinary time and never actually reaching $\mathcal{A}$. In fact, we have:

Lemma 4.3: *Suppose that $\mathcal{H}$ satisfies Assumption 2.1, a compact set $\mathcal{A}$ is pre-AS, and ϕ is a Zeno solution to $\mathcal{H}$. Then $T_{\mathbb{R}^n}(\phi) = T_{\mathbb{R}^n\setminus\mathcal{A}}(\phi)$ if and only if $\phi(t,j) \notin \mathcal{A}$ for all $(t,j) \in \operatorname{dom}\phi$.*

Most other definitions of Zeno stability in the literature are for an equilibrium point, say the origin, and insist that the hybrid system admits no flowing solutions from the origin by imposing $f(0) \neq 0$. See, for example, [5], [2], [19]. Our Zeno stability definitions make no assumption about the solutions starting in $\mathcal{A}$, other than that $\mathcal{A}$ is forward invariant. One may have no flowing solutions in $\mathcal{A}$ and not have Zeno stability; see Example 4.4 below. Conversely, one may have flowing solutions in $\mathcal{A}$ yet have the stronger *uniform* Zeno stability; we show this in Example 4.6 with the definition of uniform Zeno stability coming in Definition 4.7. The phenomenon reported in Example 4.4 was indicated in the introduction of [19] when discussing [5, Proposition 1].

Example 4.4: [*Behavior in $\mathcal{A}$ does not predict Zeno stability, I*] Consider the hybrid system with the data

$$F(x) = -x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} , \; G(x) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$$

$$C = \{x : x_1 \geq 0 \; , \; x_2 \geq 0\} \; , \; D = \{x : x_1 = 0\} \; .$$

Note that F is continuous and $F(0) \neq 0$. Consider the function $V(x) = x_1 + x_2$ which, when restricted to $C \cup D$, is positive definite with respect to the origin and radially unbounded. We have $\langle \nabla V(x), F(x)\rangle = -x_1 - 1 - x_2 + 1 = -V(x)$ and, for $x \in D$, $V(G(x)) = x_2 = x_2 + x_1 = V(x)$. It follows that, for each solution ϕ, $V(\phi(t,j)) = \exp(-t)V(\phi(0,0))$. Thus, the only way that trajectories can converge to the origin is if t is unbounded. It follows that the origin is not Zeno stable. However, the origin is asymptotically stable. This follows by applying the invariance principle of [23] since there are no solutions starting outside of the origin that keep V constant. □

Proposition 4.5: *Suppose that $\mathcal{H}$ satisfies Assumption 2.1. Let the compact set $\mathcal{A} \subset \mathbb{R}^n$ be pre-asymptotically stable for $\mathcal{H}$. Then, $\mathcal{A}$ is Zeno asymptotically stable for $\mathcal{H}$ if and only if the following two conditions hold:*

(a) **(Positive yet bounded time of flow near $\mathcal{A}$)** *there exists $\varepsilon > 0$ such that every maximal solution ϕ with $|\phi(0,0)|_{\mathcal{A}} \in (0,\varepsilon]$ satisfies*

$$0 < T_{\mathbb{R}^n\setminus\mathcal{A}}(\phi) < \infty;$$

(b) **(No backward flow out of $\mathcal{A}$)** *there does not exist an absolutely continuous $x : [0,\varepsilon] \to \mathbb{R}^n$ with $\varepsilon > 0$ such that of $\dot{x}(t) \in -F(x(t))$, $x(t) \in C$ for almost all $t \in [0,\epsilon]$ and $x(0) \in \mathcal{A}$ while $x(\epsilon) \notin \mathcal{A}$.*

Example 4.6: [*Behavior in $\mathcal{A}$ does not predict Zeno stability, II*] This example shows that the flow and jump maps can be linear (including zero at zero) and still the origin is Zeno stable. In fact, in this example the origin is uniformly Zeno stable, a property defined below that is stronger than Zeno stability. Consider the hybrid system $\mathcal{H}$ with data

$$F(x) = \begin{bmatrix} 0 \\ x_1 \end{bmatrix} \; , \; G(x) = \begin{bmatrix} 0.5x_1 \\ 0 \end{bmatrix}$$
$$C = \left\{x : 0 \leq x_2 \leq x_1^3\right\} \; , \; D = \left\{x : x_1 \geq 0 \; , \; x_2 = x_1^3\right\} \; .$$

It is straightforward to check that the conditions of Proposition 4.5 hold. Next, consider the Lyapunov function $V(x) = x_1^3 - 0.5x_2$ which, when restricted to the set $C \cup D$, is positive definite with respect to the origin and radially unbounded. We have $\langle \nabla V(x), F(x)\rangle = -0.5x_1 \leq -0.5V(x)^{1/3}$ for all $x \in C$ and $V(G(x)) = 0.5^3x_1^3 = 0.25V(x)$ for all $x \in D$. It then follows from Proposition 3.2 that the origin is (uniformly) Zeno stable. Homogeneity plays a strong role in this example. For more information see [15]. □

B. Uniform Zeno stability

Definition 4.7: *[Uniform Zeno stability] A compact set $\mathcal{A} \subset \mathbb{R}^n$ is called* uniformly Zeno asymptotically stable *for the hybrid system $\mathcal{H}$ if the following hold:*

(i) *$\mathcal{A}$ is Zeno asymptotically stable, and*
(ii) *for each $\varepsilon > 0$ there exists $\delta > 0$ such that every maximal solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \leq \delta$ satisfies $T_{\mathbb{R}^n\setminus\mathcal{A}}(\phi) \leq \varepsilon$.*

In words, uniform Zeno stability is Zeno stability plus USOT pre-AS, i.e., the amount of ordinary time in the domain of solutions converge to zero uniformly with the initial condition distance to the set $\mathcal{A}$. The gap between Zeno stability and uniform Zeno stability is illustrated below.

Example 4.8: [*The gap between Zeno stability and uniform Zeno stability, I*] Take any decreasing, to 0, sequence of positive numbers a_i, $i = 1, 2, \ldots$. Consider a hybrid system in the xy-plane with $\mathcal{A} = [0,1] \times \{0\}$ and the data

$$C = \mathcal{A} \cup \bigcup_{i=1}^{\infty} \{(x,y) \,|\, x \in [0,1], \; a_{2i-1}x \geq y \geq a_{2i}x\} \,,$$

$$D = \mathcal{A} \cup \bigcup_{i=1}^{\infty} \{(x,y) \,|\, x \in [0,1], \; y = a_{2i}x\} \,,$$

$$F(x,y) = (0, -x^2),$$

$$G(x, a_{2i}x) = (x, a_{2i+1}x), \; i = 1, 2, \ldots, \; G(x,0) = (x,0).$$

Let $c = \sum_{i=1}^{\infty}(a_{2i-1} - a_{2i})$. The solution ϕ from the initial point (x, a_1x) converges to $(x,0)$ and is Zeno, with Zeno time $T_{\mathbb{R}^n}(\phi) = cx/x^2 = c/x$. Similarly, solutions with initial points $(x,y) \in C \cup D$ converge to $(x,0)$ and have smaller Zeno times. Clearly, $\mathcal{A}$ is Zeno asymptotically stable but not uniformly Zeno asymptotically stable. □

Example 4.9: [*The gap between Zeno stability and uniform Zeno stability, II*] The data of this example, denoted $(\widetilde{f}, \widetilde{C}, \widetilde{g}, \widetilde{D})$, is a modification of the data (f, C, g, D) in Example 4.4. In particular, $\widetilde{C} \subset C$ and $D \subset \widetilde{D}$ are constructed by removing parts of the flows set C (but not the origin) and adding them to the jump set D. The jump map satisfies $\widetilde{g}(x) = g(x)$ for all $x \in D$ and is extended to $\widetilde{D}$ as discussed below. The flow map satisfies $\widetilde{f}(x) = f(x)$ for all $x \in \widetilde{C}$. In particular, $\widetilde{f}(0) \neq 0$. Define

$$\begin{aligned}
\mathcal{P} &:= \left\{x \in \mathbb{R}^2 : 0 \le x_2 \le 2x_1 \ , \ 0 \le x_1 \le 2x_2\right\} \\
\mathcal{R}_i &:= \left\{x \in \mathbb{R}^2 : \frac{3}{4}2^{-i}\varepsilon \le x_1 + x_2 \le 2^{-i}\right\} \cap \mathcal{P} \\
\widetilde{C} &:= \overline{C \setminus \left(\cup_{i \in \mathbb{Z}_{\ge 0}} \mathcal{R}_i\right)} \\
\widetilde{D} &:= \overline{D \cup \left(\cup_{i \in \mathbb{Z}_{\ge 0}} \mathcal{R}_i\right)} \ .
\end{aligned}$$

Note that the sets $\mathcal{R}_i$, $i \in \mathbb{Z}_{\ge 0}$ are closed, disjoint, and do not intersect D. Now let

$$\widetilde{g}(x) = \begin{bmatrix} 2^{-(i+1)} \\ 0 \end{bmatrix} \qquad \forall x \in \mathcal{R}_i$$

and note that, with $\widetilde{g}(x) = g(x)$ for all $x \in D$, $\widetilde{g}$ is continuous since $g(0) = 0$ and $\widetilde{g}(x) \to 0$ as $x \to 0$. Note that, for each $x \in \widetilde{D}\setminus\{0\}$, $\widetilde{g}(x) \notin D$. It can be shown that the origin is asymptotically stable, again using $V(x) = x_1 + x_2$ and the invariance principle.

Now, take the sequence of initial conditions $x_i = 2^{-1}\begin{bmatrix} 0.75 \\ 0 \end{bmatrix}$. Solutions from such initial conditions evolve like the solutions in Example 4.4 until they reach $\mathcal{R}_{i+1}$. Thus, the function V will start with the value $0.75(2^{-i})$ and will evolve according to

$$V_1(\phi_i(t,j)) = \exp(-t)V_1(x_i) = \exp(-t)0.75(2^{-i})$$

at least until t is such that $\exp(-t)0.75(2^{-i}) = 2^{-(i+1)}$, i.e., at least until $\exp(-t) = 2/3$, equivalently, $t = \ln(1.5)$. Since the sequence x_i converges to the origin, this establishes that the origin is not uniformly Zeno stable.

To see that the origin is Zeno stable, we will establish that each solution ϕ has $T_{\mathbb{R}^n}(\phi)$ bounded. Then the conditions of Proposition 4.5 can be verified to establish Zeno stability. First note that each solution eventually reaches some set $\mathcal{R}_i$, with i arbitrarily large, where it must jump to the point

$$x_i := \begin{bmatrix} 2^{-(i+1)} \\ 0 \end{bmatrix} .$$

So, it is enough to show that from each such initial condition solutions are Zeno. Note that the time to pass from such an initial condition to the line $x_1 = x_2$ (if missing $\mathcal{R}_{i+1}$) is upper bounded by $2^{-(i+2)}$. So, for i sufficiently large, these initial conditions must hit $\mathcal{R}_{i+1}$ with flowing time not more than $2^{-(i+2)}$. Repeating this argument, it follows that, for each solution ϕ with $\phi(0,0) = x_i$, with i sufficiently large, $T_{\mathbb{R}^n}(\phi)$ is bounded by

$$\sum_{i \in \mathbb{Z}_{\ge 0}} 2^{-(i+2)} = 1/2 \ .$$

This establishes Zeno stability. □

A combination of Theorem 3.3 and Proposition 4.5 yields the following.

Corollary 4.10: *Let $\mathcal{A} \subset \mathbb{R}^n$ be compact. Then, $\mathcal{A}$ is uniformly Zeno asymptotically stable for $\mathcal{H}$ if and only if:*

(a) *conditions (a) and (b) of Proposition 4.5 hold, and*
(b) *there exist a function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ that is continuously differentiable on $\mathbb{R}^n\setminus\mathcal{A}$ and class-$\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$ such that*
  (i) *for all $x \in \mathbb{R}^n$,*
  $$\alpha_1(|x|_{\mathcal{A}}) \le V(x) \le \alpha_2(|x|_{\mathcal{A}}),$$
  (ii) *for all $x \in C\setminus\mathcal{A}$, $f \in F(x)$,*
  $$\langle \nabla V(x), f\rangle \le -1;$$
  (iii) *for all $x \in D$, $g \in G(x)$,*
  $$V(g) \le V(x) - \alpha_3(V(x)).$$

V. THE SUFFICIENT CONDITIONS OF [19]

Below we give a set of sufficient conditions for the Lyapunov conditions in Corollary 4.10. Then we relate those to sufficient conditions given in [19].

Proposition 5.1: *If there exist $\lambda \in [0,1)$, a continuous function $W : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on $\mathbb{R}^n \setminus \mathcal{A}$ and $\mathcal{K}_\infty$ functions $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$ such that*

$$\begin{array}{ll}
\widetilde{\alpha}_1(|x|_{\mathcal{A}}) \le W(x) \le \widetilde{\alpha}_2(|x|_{\mathcal{A}}) & x \in C \cup D \cup G(D) \\
\langle \nabla W(x), f\rangle \le 0 & \forall x \in C \ , \ f \in F(x) \\
V(g) \le \lambda V(x) & \forall x \in D \ , \ g \in G(x)
\end{array} \qquad (2)$$

constants $a, b, c > 0$, and a continuous function $B : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on an open set containing $C\setminus\mathcal{A}$ such that

$$\langle \nabla B(x), f\rangle \le -c \qquad \forall x \in C \setminus \mathcal{A} \ , \ f \in F(x)$$

while

$$|B(x)| \le b(W(x))^a \qquad \forall x \in C \cup D \cup G(D)$$

then, with $\sigma > (1+\lambda^a)/(1-\lambda^a)$, the functions

$$\begin{aligned}
V(x) &:= \frac{1}{c}\left[B(x) + \sigma b(W(x))^a\right] \\
\alpha_1(s) &:= \frac{1}{c}(\sigma - 1)b\widetilde{\alpha}_1(s)^a \\
\alpha_2(s) &:= \frac{1}{c}(\sigma + 1)b\widetilde{\alpha}_2(s)^a \\
\alpha_3(s) &:= \left(1 - \frac{\sigma+1}{\sigma-1}\lambda^a\right)s
\end{aligned}$$

satisfy item (b) of Corollary 4.10.

Now let ℓ be a positive integer and consider the "periodic" hybrid system

$$\left.\begin{array}{rcl} \dot{\xi} & \in & F_k(\xi) \\ \dot{k} & = & 0 \end{array}\right\} \qquad \xi \in C_k$$
$$\left.\begin{array}{rcl} \xi^+ & \in & G_k(\xi) \\ k^+ & = & \mathrm{mod}_\ell(k) + 1 \end{array}\right\} \qquad \xi \in D_k$$

Let $\widetilde{\mathcal{A}}$ be a compact set to which ξ should converge.

Proposition 5.2: *If there exist $\lambda_\circ \in [0,1)$, a family of continuous functions $W_k : \mathbb{R}^n \to \mathbb{R}$ that are continuously differentiable on $\mathbb{R}^n \setminus \widetilde{\mathcal{A}}$, and class-$\mathcal{K}_\infty$ functions $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$ such that, for each $k \in \{1, \ldots, \ell\}$,*

(i) *for all* $\xi \in C_k \cup D_k \cup G_k(D_k)$,

$$\widetilde{\alpha}_1(|\xi|_{\widetilde{\mathcal{A}}}) \leq W_k(\xi) \leq \widetilde{\alpha}_2(|\xi|_{\widetilde{\mathcal{A}}}),$$

(ii) *for all* $\xi \in C_k \setminus \widetilde{\mathcal{A}}$, $f \in F_k(\xi)$,

$$\langle \nabla W_k(x), f \rangle \leq 0;$$

(iii) *for all* $\xi \in D_k$, $g \in G_k(\xi)$,

$$V_{\ \mathrm{mod}_\ell(k)+1}(g) \leq V_k(\xi);$$

an integer $k^* \in \{1, \dots, n\}$ *such that*

(iv) *for all* $\xi \in D_k$, $g \in G_{k^*}(\xi)$,

$$V_{\ \mathrm{mod}_\ell(k^*)+1}(g) \leq \lambda_\circ V_{k^*}(\xi);$$

positive constants a, b, and c, and a family of continuous functions $B_k : \mathbb{R}^n \to \mathbb{R}$ *with* B_k *continuously differentiable on an open set containing* $C_k \backslash \widetilde{\mathcal{A}}$ *such that*

(v) *for all* $\xi \in C_k \setminus \widetilde{\mathcal{A}}$, $f \in F_k(\xi)$,

$$\langle \nabla B_k(\xi), f \rangle \leq -c;$$

(vi) *for all* $\xi \in C_k \cup D_k \cup G_k(D_k)$

$$|B_k(\xi)| \leq b(W_k(\xi))^a;$$

then the conditions of the Proposition 5.1 hold with $\mathcal{A} := \widetilde{\mathcal{A}} \times \{1, \dots, \ell\}$, $x := (\xi, k)$, $C := \{(\xi, k) : \xi \in C_k\}$, $D := \{(\xi, k) : \xi \in D_k\}$,

$$F(x) := \begin{bmatrix} F_k(\xi) \\ 0 \end{bmatrix}, \quad G(x) := \begin{bmatrix} G_k(\xi) \\ \mathrm{mod}_\ell(k) + 1 \end{bmatrix},$$

$B(x) = B_k(\xi)$, *and* $W(x) := \rho_k W_k(\xi)$ *where, for each* $k \in \{1, \dots, \ell\}$, $\rho_k > 0$ *and* $\rho_{\ \mathrm{mod}_\ell(k)+1} = \gamma_k \rho_k$ *where* $\gamma_k = \lambda_\circ^{1/\ell}$ *for* $k \in \{1, \dots, \ell\}, k \neq k^*$ *and* $\gamma_{k^*} = \lambda_\circ^{-1+1/\ell}$.

Remark 5.3: This conditions of Proposition 5.2 align with the conditions used in [19]. In particular, [19, EC1] corresponds to (i) in Proposition 5.2; [19, EC2] corresponds to (ii) in Proposition 5.2, [19, EC3+C2] provides a special case of (vi); [19, EC4] corresponds to (v); [19, ED1] corresponds to (iii); and [19, C1] corresponds to (iv). □

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