Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

An $\mathcal{S}$-regularity approach to the robust analysis of descriptor models

Bilal Sari, Olivier Bachelier and Driss Mehdi

***Abstract*— This paper exploits the notion of $\mathcal{S}$-regularity of a matrix pencil to propose insights for the robustness analysis of descriptor models of the form $E\dot{x} = Ax$ (or $Ex_{k+1} = Ax_k$ for the discrete case) subject to norm-bounded LFT (Linear Fractional Transform)-based uncertainties on both matrices A and E. The property to be studied is the robust $\mathcal{D}$-admissibility (robust $\mathcal{D}$-stability togeteher with robust regularity and robust impulse freeness). All the proposed conditions are expressed in terms of *strict* LMI (Linear Matrix Inequalities). Two techniques are proposed and numerically compared.**

***Index Terms*— Descriptor systems, $\mathcal{S}$-regularity, Robust $\mathcal{D}$-stability, Robust $\mathcal{D}$-admissibility, *strict* LMI.**

I. INTRODUCTION

It is now well admitted that systems of the form $E\dot{x} = Ax$ (or $Ex_{k+1} = Ax_k$ for the discrete case), that are called singular systems, descriptor systems, generalized systems or implicit systems, and so on, are of great interest for the modelling of many practical devices (interconnected systems, electrical networks, robotics). For conciseness, rather than to quote many references, we urge the interested reader to examine [21], [7] and some references therein.

As for conventional models for which $E = I$ (or at least E is non singular), the $\mathcal{D}$-stability, *i.e.* the clustering of the eigenvalues of (E, A) in some region $\mathcal{D}$ of the complex plane, is of high importance to analyze the transient behaviour of the system, particularly to assess asymptotic stability. But it does not suffice in the singular case. Two other properties have to hold, namely the regularity (existence of a unique solution to the generalized state-space equation) and impulse freeness (meaning that the infinite eigenvalues of the pencil induce no impulsive terms in the response even when the control signals are not smooth [7]). When these three properties hold, the model is said to be $\mathcal{D}$-admissible.

It is important to derive some simple tools that enable the designer to test whether an exactly known pencil is $\mathcal{D}$-admissible. For admissibility test, a big focus has been put on the generalized Lyapunov equations [13], [20], [9]. Though very interesting, some of those approaches require the systems to be transformed into equivalent forms, which is not desirable in an uncertain context. In the presence of uncertainties, the use of *strict* LMI (Linear Matrix Inequalities [5]) might be prefered. In that sense, one of the first steps was made in [6]. The advantage of LMI is also that they can easily enable ones to extend the results to various clustering regions $\mathcal{D}$.

The authors are with the University of Poitiers, Laboratoire d'Automatique et d'Informatique Industrielle (LAII), de l'École Supérieure d'Ingénieurs de Poitiers (ESIP), Bâtiment de Mécanique, 40, Av. du Recteur Pineau, 86022 POITIERS CEDEX, France, Olivier.Bachelier@univ-poitiers.fr

Many contributions deal with robust analysis or control of descriptor models, particularly through (unfortunately not necessarily *strict*) LMI approach: see [22], the seminal work of Masubuchi ([16], [17] and the references therein), and many others. But very few really consider uncertainty on E. Let us quote [15] where A is however precisely known or [14] for interval matrices. But the best insights can actually be found in [12], [18].

In this paper, we make an extensive use of the notion of $\mathcal{S}$-regularity inspired from that of $\partial\mathcal{D}$-regularity [2], [3] and of some versions of the so-called S-procedure (see [5], [11], [19], [10] and the references therein) to derive *strict* LMI (sufficient) conditions for a descriptor model subject to norm-bounded LFT (Linear Fractional Transform)-based uncertainties on both matrices A and E (that are general enough to encompass many uncertainties encountered in practice). Our purpose is to propose a tool, useful at once, as simple as possible, that can be a basis for many other future works.

The paper is organized as follows. The next section is dedicated to the mathematical problem statement, including basic definitions, the description of the uncertain matrix pencil, the formulation of the considered regions and the actual condition to be checked by LMI. Section 3 proposes a first reasoning to derive a sufficient *strict* LMI condition. We insist on the importance of the strictness of LMI from a computational point of view. This part of our contribution. Another reasoning is followed in Section 4 and yields another condition. In section 5, a discussion is led about singular systems and their properties (regularity, impulse freenees) to analyse our conditions through the lens of those fundamental properties. The two conditions are numerically compared on an example in Section 6 before to conclude.

Notations: M' is the transpose conjugate of M. $\bar{\sigma}(M)$ is the maximum singular value of M. I and 0 are identity and zero matrices of appropriate dimensions respectively. In matrix inequalities, < 0, > 0, ≤ 0 and ≥ 0 must be understood in the sense of Löwner (sign definition of matrices). $\mathbf{i}$ is the imaginary unit and $\otimes$ denotes the Kronecker product.

II. Preliminaries and Problem statement

A. Basic definitions

In this section, we propose various definitions of properties for matrix pencils. The reasonings in the paper are mainly followed on matrix pencils whereas the descriptor models are only considered in Section 5.

Definition 1: Let (E, A) be a matrix pencil where $\{A; E\} \in \mathbb{C}^{n\times n}$. We denote by $\lambda(E, A)$ the generalized spectrum of the pencil (E, A) defined by

$$\lambda(E, A) = \{\lambda \in \mathbb{C} : \det(\mathbb{A}(\lambda)) = 0\}, \quad (1)$$

with $\mathbb{A}(\lambda) = E\lambda - A$, and the elements of $\lambda(E, A)$ are referred to as the eigenvalues of (E, A).

Definition 2: (inspired from [2]) Let (E, A) be a matrix pencil where $\{A; E\} \in \mathbb{C}^{n\times n}$. Also let $\mathcal{S}$ be any subset of the complex plane. The pencil (E, A) is said to be

- $\mathcal{S}$-regular if $\lambda(E, A) \cap \mathcal{S} = \emptyset$,
- $\mathcal{S}$-singular otherwise.

Definition 3: Let (E, A) be a matrix pencil where $\{A; E\} \in \mathbb{C}^{n\times n}$. The pencil (E, A) is said to be

- regular if there exists $\mathcal{S} \neq \emptyset$ such that (E, A) is $\mathcal{S}$-regular,
- singular otherwise.

It has to be noticed that in the remaining part of the paper, $\mathrm{rank}(E) = r \le n$, meaning that some eigenvalues of (E, A) might not be finite.

B. Formulation of the uncertain pencil

In our reasonings, the matrices E and A are actually uncertain and comply with

$$\left\{\begin{array}{l} A = D_A + C_A\bar{\Delta}_A B_A,\\ E = D_E + C_E\bar{\Delta}_E B_E, \end{array}\right. \quad (2)$$

with

$$\left\{\begin{array}{ll} \bar{\Delta}_A = \Delta_A(I - A_A\Delta_A)^{-1}, & \Delta_A \in \mathbf{\Delta}_A = \{\Delta_A : \bar{\sigma}(\Delta_A) \le \sqrt{\gamma_A^{-1}}\},\\ \bar{\Delta}_E = \Delta_E(I - A_E\Delta_E)^{-1}, & \Delta_E \in \mathbf{\Delta}_E = \{\Delta_E : \bar{\sigma}(\Delta_E) \le \sqrt{\gamma_E^{-1}}\}, \end{array}\right. \quad (3)$$

where $\gamma_A > 0$ and $\gamma_E > 0$ are scalar numbers. The structure of the uncertain matrices A and E is the so-called LFT (Linear Fractional Transform)-based uncertainty and the matrices Δ_A and Δ_E are both norm-bounded *i.e.* $\mathbf{\Delta}_A$ and $\mathbf{\Delta}_E$ are bounded balls of matrices. It is possible to consider more sophisticated sets $\mathbf{\Delta}_A$ and $\mathbf{\Delta}_E$ but we here restrict our analysis to balls of matrices for the sake of conciseness.
We define the uncertainty Δ as

$$\Delta = \{\Delta_A; \Delta_E\} \in \mathbf{\Delta} = \mathbf{\Delta}_A \times \mathbf{\Delta}_E. \quad (4)$$

Assumption 1: The uncertainty domain $\mathbf{\Delta}$ is assumed to be *implicitly well posed*:

(i) $\det(I - A_A\Delta_A) \neq 0$ and $\det(I - A_E\Delta_E) \neq 0$ over $\mathbf{\Delta}$;
(ii) $\mathrm{rank}(E) = r \le n \quad \forall \Delta_E \in \mathbf{\Delta}_E$.

Assumption (i) is the classical well posedness of LFT forms and the term *implicit* refers to assumption (ii) which is the only new concept introduced here.

Remark 1: The uncertainty on E can be useful in practice, for instance to take uncertain inertias into account.

C. Formulation of $\mathcal{S}$

In the paper, the set $\mathcal{S}$ is defined by

$$\mathcal{S} = \left\{ s \in \mathbb{C} : \begin{bmatrix} s\\ 1\end{bmatrix}' R \begin{bmatrix} s\\ 1\end{bmatrix} = 0; \quad \& \right.$$
$$\left. \begin{bmatrix} s\\ 1\end{bmatrix}' \Phi_h \begin{bmatrix} s\\ 1\end{bmatrix} \ge 0, \quad \forall h \in \{1, \dots, \bar{h}\} \right\}, \quad (5)$$

where R and Φ_h, $h = 1, \dots, \bar{h}$ are 2×2 Hermitian matrices. This kind of description is borrowed from [3] following insights proposed in [11], [10]. Special sets can be emphasized:

- Imaginary axis: $\bar{h} = 0$; $R = \begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}$;
- Unit circle: $\bar{h} = 0$; $R = \begin{bmatrix} 1 & 0\\ 0 & -1\end{bmatrix}$;
- Right half plane: $\bar{h} = 1$;$R = 0$; $\Phi_1 = \begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}$;
- Exterior of the unit disc: $\bar{h} = 1$;$R = 0$; $\Phi_1 = \begin{bmatrix} 1 & 0\\ 0 & -1\end{bmatrix}$.

It is possible to extend the class of sets $\mathcal{S}$ by considering matrices R and Φ_h in $\mathbb{C}^{2d\times 2d}$ with $d \ge 1$ but, once again, for the sake of conciseness, here, d can only equal 1. Nevertheless, such a descrition encompasses the boundaries of many so-called EEMI-regions [4], or even some of those regions themselves.

D. Problem Statement

Consider an uncertain matrix pencil (E, A) that complies with the uncertainty defined in §II-B and Assumption 1. Also let a set $\mathcal{S}$ be described as in §II-C. This work aims at finding *strict* LMI conditions such that (E, A) remains $\mathcal{S}$-regular for any $\Delta \in \mathbf{\Delta}$. After having proposed two techniques to handle this problem in the next two sections, we discuss about their usefulness in the study of descriptor systems in Section 5.

III. Some "augmented LFT" solution

In this part, we transform the original problem that consists in analysing the spectrum (E, A) into another one that consists in analysing the spectrum of an augmented pencil subject to an agumented LFT-based uncertainty. This transformation can seem rather classical. However, a motivation of this work is to show that this description seems to lead to more conservative results than those proposed in the next section. Nevertheless, for clarity, we have to present this calculation that leads to (8). One has to satisfy

$$\det(\mathbb{A}(\lambda, \Delta)) \neq 0 \, \forall \{\lambda; \Delta\} \in \mathcal{S} \times \mathbf{\Delta} \quad (6)$$
$$\Leftrightarrow \det((D_E + C_E\bar{\Delta}_E B_E)\lambda - D_A - C_A\bar{\Delta}_A B_A) \neq 0$$

Note that owing to Assumption 1, $\det(-(I - A_E\Delta_E)) \neq 0$ which leads to

$$\det((D_E + C_E\bar{\Delta}_E B_E)\lambda - D_A - C_A\bar{\Delta}_A B_A)\det(-(I - A_E\Delta_E)) \neq 0$$
$$\det \begin{bmatrix} (D_E + C_E\bar{\Delta}_E B_E)\lambda - D_A - C_A\bar{\Delta}_A B_A & 0\\ 0 & -(I - A_E\Delta_E)\end{bmatrix} \neq 0$$
$$\Leftrightarrow \det\left(\begin{bmatrix} I & C_E\Delta_E(I - A_E\Delta_E)^{-1}\\ 0 & I\end{bmatrix} \times \right.$$

$$\begin{bmatrix} D_E\lambda - D_A - C_A\bar{\Delta}_A B_A & C_E\Delta_E \\ B_E\lambda & -(I - A_E\Delta_E) \end{bmatrix} \times$$

$$\left.\begin{bmatrix} I & 0 \\ (I - A_E\Delta_E)^{-1}B_E\lambda & I \end{bmatrix}\right) \neq 0$$

$$\det\left(\begin{bmatrix} D_E\lambda - D_A - C_A\bar{\Delta}_A B_A & C_E\Delta_E \\ B_E\lambda & -(I - A_E\Delta_E) \end{bmatrix}\right) \neq 0$$

$$\Leftrightarrow \det(\tilde{A}(\lambda,\Delta)) \neq 0,\ \forall\{\lambda;\Delta\} \in \mathcal{S} \times \mathbf{\Delta} \tag{7}$$

with

$$\tilde{A}(\lambda,\Delta) = \overbrace{\begin{bmatrix} D_E & 0 \\ B_E & 0 \end{bmatrix}}^{\tilde{E}}\lambda -$$

$$\left(\overbrace{\begin{bmatrix} D_A & 0 \\ 0 & I \end{bmatrix}}^{D_{\tilde{A}}} + \overbrace{\begin{bmatrix} C_A & -C_E \\ 0 & -A_E \end{bmatrix}}^{C_{\tilde{A}}}\overbrace{\begin{bmatrix} \Delta_A & 0 \\ 0 & \Delta_E \end{bmatrix}}^{\tilde{\Delta}} \times\right.$$

$$\left.\left(I - \overbrace{\begin{bmatrix} A_A & 0 \\ 0 & 0 \end{bmatrix}}^{A_{\tilde{A}}}\overbrace{\begin{bmatrix} \Delta_A & 0 \\ 0 & \Delta_E \end{bmatrix}}^{\tilde{\Delta}}\right)^{-1}\overbrace{\begin{bmatrix} B_A & 0 \\ 0 & I \end{bmatrix}}^{B_{\tilde{A}}}\right). \tag{8}$$

Let us notice that we recover an LFT-based uncertainty in which $\tilde{\Delta}$ is block-diagonal (the bad point) and $\tilde{E}$ is not uncertain (the good point). It is clear that $\det(\tilde{E}\lambda - D_{\tilde{A}}) \neq 0$ (the nominal augmented model has to be $\mathcal{S}$-regular otherwise why considering the uncertain case) and therefore (7) is equivalent to

$$\det(I - (\tilde{E}\lambda - D_{\tilde{A}})^{-1}C_{\tilde{A}}\tilde{\Delta}(I - A_{\tilde{A}}\tilde{\Delta})^{-1}B_{\tilde{A}}) \neq 0$$

$$\Leftrightarrow \det(I - B_{\tilde{A}}(\tilde{E}\lambda - D_{\tilde{A}})^{-1}C_{\tilde{A}}\tilde{\Delta}(I - A_{\tilde{A}}\tilde{\Delta})^{-1}) \neq 0$$

$$\Leftrightarrow \det(I - (A_{\tilde{A}} + B_{\tilde{A}}(\tilde{E}\lambda - D_{\tilde{A}})^{-1}C_{\tilde{A}})\tilde{\Delta}) \neq 0$$

(since $\det(I - A_{\tilde{A}}\tilde{\Delta}) \neq 0$), that one can write

$$\det(I - \tilde{G}(\lambda)\tilde{\Delta}) \neq 0,\ \forall\{\lambda;\Delta\} \in \mathcal{S} \times \mathbf{\Delta},$$

with $\tilde{G}(\lambda) = A_{\tilde{A}} + B_{\tilde{A}}(\tilde{E}\lambda - D_{\tilde{A}})^{-1}C_{\tilde{A}}$. The previous difference holds if and only if

$$\begin{bmatrix} \tilde{\Delta} \\ I \end{bmatrix}' \tilde{Q} \begin{bmatrix} \tilde{\Delta} \\ I \end{bmatrix} < 0,\ \forall\{\lambda;\Delta\} \in \mathcal{S} \times \mathbf{\Delta}, \tag{9}$$

with

$$\tilde{Q} = \begin{bmatrix} \tilde{G}'(\lambda) \\ I \end{bmatrix} (-I) \begin{bmatrix} \tilde{G}'(\lambda) \\ I \end{bmatrix}'. \tag{10}$$

Define $\hat{\mathbf{\Delta}}$ as the set of all matrices $\hat{\Delta}$ such that

$$\begin{bmatrix} \hat{\Delta} \\ I \end{bmatrix}' \tilde{\Psi} \begin{bmatrix} \hat{\Delta} \\ I \end{bmatrix} \geq 0, \tag{11}$$

where

$$\tilde{\Psi} = \text{blockdiag}(-\gamma_A I; -\gamma_E I, I) \tag{12}$$

It is clear that $\tilde{\Delta}$ lies in a set $\tilde{\mathbf{\Delta}}$ that is strictly contained in $\hat{\mathbf{\Delta}}$. Then, the S-procedure as proposed in [5] can be applied to claim that (9) holds **if** there exists a scalar $\tau > 0$ such that

$$\tilde{Q} + \tau\tilde{\Psi} < 0,\ \forall\lambda \in \mathcal{S}, \tag{13}$$

which, by virtue of Finsler's lemma (that can also be seen as a special case of S-procedure), is equivalent to

$$\begin{bmatrix} I \\ \tilde{G}(\lambda) \end{bmatrix}' \tilde{\Psi} \begin{bmatrix} I \\ \tilde{G}(\lambda) \end{bmatrix} < 0,\ \forall\lambda \in \mathcal{S}. \tag{14}$$

The above inequality can also be written

$$\begin{bmatrix} I \\ (\tilde{E}\lambda - D_{\tilde{A}})^{-1}C_{\tilde{A}} \end{bmatrix}' \tilde{\Theta} \begin{bmatrix} I \\ (\tilde{E}\lambda - D_{\tilde{A}})^{-1}C_{\tilde{A}} \end{bmatrix} < 0,\ \forall\lambda \in \mathcal{S}, \tag{15}$$

where

$$\tilde{\Theta} = \begin{bmatrix} I & 0 \\ A_{\tilde{A}} & B_{\tilde{A}} \end{bmatrix}' \tilde{\Psi} \begin{bmatrix} I & 0 \\ A_{\tilde{A}} & B_{\tilde{A}} \end{bmatrix}. \tag{16}$$

At this stage, it is possible to apply the generalized Kalman-Popov-Yakubovich (KYP) lemma proposed in [10] (yet another application of a generalized version of the S-procedure), with slight adaptations as in [3] (in order to encompass the case $\bar{h} > 1$), to claim that (15) holds if and only if there exist an Hermitian matrix P and $\bar{h}$ Hermitian positive definite matrices Q_h, $h = 1, \ldots, \bar{h}$, such that

$$\begin{bmatrix} C_{\tilde{A}} & D_{\tilde{A}} \\ 0 & \tilde{E} \end{bmatrix}' (R \otimes P + \sum_{h=1}^{\bar{h}} \Phi_h \otimes Q_h) \begin{bmatrix} C_{\tilde{A}} & D_{\tilde{A}} \\ 0 & \tilde{E} \end{bmatrix} + \tilde{\Theta} < 0. \tag{17}$$

The previous reasoning is summarized by the next theorem.

Theorem 1: Let an uncertain matrix pencil (E, A) comply with the uncertainty described in §II-B and Assumption 1. Also let a set $\mathcal{S}$ be described as in §II-C. (E, A) is robustly $\mathcal{S}$-regular **if** there exist an Hemitian matrix P and $\bar{h}$ Hermitian positive definite matrices Q_h, $h = 1, \ldots, \bar{h}$, such that the *strict* LMI (17) holds.

The conservativeness in the above theorem is due to the 2-block diagonal structure of $\tilde{\Delta}$ which makes the S-procedure be pessimistic from (13) to (9) or in other words, it is due to the fact that $\tilde{\mathbf{\Delta}} \subset \hat{\mathbf{\Delta}}$ but $\tilde{\mathbf{\Delta}} \neq \hat{\mathbf{\Delta}}$.

IV. SOME "NON-AUGMENTED LFT" SOLUTION

The term "non-augmented LFT" solution refers to the fact that in this section, we preserve the two LFT-based uncertainties of §II-B (the one on E and the other one on A) without augmenting the size of the uncertainty matrix. A contribution is to emphasize the fact that this second technique seems to be less conservative than the previously presented one.

It is clear that the uncertain pencil (E, A) can be $\mathcal{S}$-regular only if (E, D_A) is $\mathcal{S}$-regular (if the property does not hold for the nominal part of A, it is no use going further). So, necessarily, $\det(E\lambda - D_A) \neq 0$ for any $\{\lambda; \Delta_E\} \in \mathcal{S} \times \mathbf{\Delta}_E$ and then (6) is equivalent to

$$\det(I - (E\lambda - D_A)^{-1}C_A\bar{\Delta}_A B_A) \neq 0$$

$$\Leftrightarrow \det(I - B_A(E\lambda - D_A)^{-1}C_A\bar{\Delta}_A) \neq 0$$

$$\Leftrightarrow \det(I - B_A(E\lambda - D_A)^{-1}C_A\Delta_A(I - A_A\Delta_A)^{-1}) \neq 0. \tag{18}$$

From Assumption 1, it comes $\det((I - A_A\Delta_A)^{-1}) \neq 0$ which enables ones to write

$$\det(I - A_A\Delta_A - B_A(E\lambda - D_A)^{-1}C_A\Delta_A)\det((I - A_A\Delta_A)^{-1}) \neq 0$$

$$\Leftrightarrow \det(I - A_A\Delta_A - B_A(E\lambda - D_A)^{-1}C_A\Delta_A) \neq 0$$

$$\Leftrightarrow \det(I - [A_A + B_A(E\lambda - D_A)^{-1}C_A]\Delta_A) \neq 0 \tag{19}$$

that one can write

$$\det(I - G(\lambda, \Delta_E)\Delta_A) \neq 0, \forall\{\lambda;\Delta\} \in \mathcal{S} \times \mathbf{\Delta}. \tag{20}$$

Taking the definition of $\mathbf{\Delta}_A$ into account, the previous inequality is equivalent to

$$\inf_{\lambda\in\mathcal{S}}\{\inf_{\Delta_A}\{\bar{\sigma}(\Delta_A) : \det(I - G(\lambda,\Delta_E)\Delta_A) = 0\}\} > \sqrt{\gamma_A^{-1}},\ \forall\Delta_E \in \mathbf{\Delta}_E$$

$$\Leftrightarrow \{\sup_{\lambda\in\mathcal{S}}(\mu_{\mathbb{C}}(G(\lambda,\Delta_E))\}^{-1} > \sqrt{\gamma_A^{-1}},\ \forall\Delta_E \in \mathbf{\Delta}_E \tag{21}$$

where $\mu_{\mathbb{C}}$ denotes the structured singular value introduced in [8]. The above inequality is equivalent to

$$\bar{\sigma}(G(\lambda,\Delta_E)) < \sqrt{\gamma_A} \Leftrightarrow G'(\lambda,\Delta_E)G(\lambda,\Delta_E) < \gamma_A,\ \forall \Delta_E \in \mathbf{\Delta}_E, \quad (22)$$

which can also be written

$$\left[\begin{array}{c} (E\lambda - D_A)^{-1}C_A \\ I \end{array}\right]' \Theta \left[\begin{array}{c} (E\lambda - D_A)^{-1}C_A \\ I \end{array}\right] < 0,\ \forall \Delta_E \in \mathbf{\Delta}_E, \quad (23)$$

where

$$\Theta = \left[\begin{array}{c} B'_A \\ A'_A \end{array}\right]' \left[\begin{array}{c} B'_A \\ A'_A \end{array}\right] - \left[\begin{array}{cc} 0 & 0 \\ 0 & \gamma_A I \end{array}\right]. \quad (24)$$

As in the previous section, applying the generalized KYP lemma [10] with adaptations of [3] enables ones to see that (23) holds if and only if there exist an Hermitian matrix $P(\Delta_E)$ and $\bar{h}$ Hermitian positive definite matrices $Q_h(\Delta_E)$ (note that these matrices depend on Δ_E) such that

$$\Theta + \left[\begin{array}{cc} D_A & C_A \\ E & 0 \end{array}\right]' M(\Delta_E) \left[\begin{array}{cc} D_A & C_A \\ E & 0 \end{array}\right] < 0,\ \forall \Delta_E \in \mathbf{\Delta}_E, \quad (25)$$

where

$$M(\Delta_E) = R \otimes P(\Delta_E) + \sum_{h=1}^{\bar{h}} \Phi_h \otimes Q_h(\Delta_E). \quad (26)$$

Recalling that $E = D_E + C_E\Delta_E(I - A_E\Delta_E)^{-1}B_E$, one can deduce the next LFT:

$$\mathbf{G}_E = \left[\begin{array}{cc} D_A & C_A \\ E & 0 \end{array}\right] = \mathcal{D}_E + \mathcal{C}_E\Delta_E(I - \mathcal{A}_E\Delta_E)^{-1}\mathcal{B}_E =$$

$$\mathcal{D}_E + \mathcal{C}_E(I - \Delta_E\mathcal{A}_E)^{-1}\Delta_E\mathcal{B}_E, \quad (27)$$

with

$$\left[\begin{array}{c|c} \mathcal{A}_E & \mathcal{B}_E \\ \hline \mathcal{C}_E & \mathcal{D}_E \end{array}\right] = \left[\begin{array}{c|cc} A_E & B_E & 0 \\ \hline 0 & D_A & C_A \\ C_E & D_E & 0 \end{array}\right]. \quad (28)$$

Therefore, inequality (25) can be written

$$\left[\begin{array}{c} \mathbf{G}_E \\ I \end{array}\right]' \left[\begin{array}{cc} M(\Delta_E) & 0 \\ 0 & \Theta \end{array}\right] \left[\begin{array}{c} \mathbf{G}_E \\ I \end{array}\right] < 0. \quad (29)$$

At this stage, we introduce "some" degree of conservativeness by noting that (29) holds for some $M(\Delta_E)$ **if** it holds for some constant M. In other words, matrices P and Q_h are no longer assumed to depend on Δ_E. So (29) holds if

$$\overbrace{\left[\begin{array}{c} (I - \Delta_E\mathcal{A}_E)^{-1}\Delta_E\mathcal{B}_E \\ I \end{array}\right]'}^{\mathbf{N}'(\Delta_E)} \mathbf{\Theta} \overbrace{\left[\begin{array}{c} (I - \Delta_E\mathcal{A}_E)^{-1}\Delta_E\mathcal{B}_E \\ I \end{array}\right]}^{\mathbf{N}(\Delta_E)} < 0. \quad (30)$$

with

$$\mathbf{\Theta} = \left[\begin{array}{c} \mathcal{C}'_E \\ \mathcal{D}'_E \end{array}\right] M \left[\begin{array}{c} \mathcal{C}'_E \\ \mathcal{D}'_E \end{array}\right]' + \left[\begin{array}{cc} 0 & 0 \\ 0 & \Theta \end{array}\right]. \quad (31)$$

Notice that the columns of $\mathbf{N}(\Delta_E)$ span the kernel of of the substitution associated with

$$\left[\begin{array}{cc} I & -\Delta_E \end{array}\right] \overbrace{\left[\begin{array}{cc} I & 0 \\ \mathcal{A}_E & \mathcal{B}_E \end{array}\right]}^{F_E} = \left[\begin{array}{cc} (I - \Delta_E\mathcal{A}_E) & -\Delta_E\mathcal{B}_E \end{array}\right] \quad (32)$$

Also notice that $\mathbf{\Delta}_E$ can be defined as

$$\mathbf{\Delta}_E = \left\{\Delta_E : \left[\begin{array}{c} \Delta_E \\ I \end{array}\right]' \Psi_E \left[\begin{array}{c} \Delta_E \\ I \end{array}\right] \geq 0\right\}, \quad (33)$$

with

$$\Psi_E = \left[\begin{array}{cc} -\gamma_E I & 0 \\ 0 & I \end{array}\right]. \quad (34)$$

Taking these facts and the compactness of $\mathbf{\Delta}_E$ into account, we apply the full block S-procedure in the version proposed in [19] to claim that (30) holds if and only if there exists $\tau_E > 0$ such that

$$\tau_E F'_E \Psi_E F_E + \mathbf{\Theta} < 0, \quad (35)$$

which can also be written as in (36). Indeed, the previous reasoning is summarized as follows.

Theorem 2: Let an uncertain matrix pencil (E, A) comply with the uncertainty described in §II-B and Assumption 1. Also let a set $\mathcal{S}$ be described as in §II-C. (E, A) is robustly $\mathcal{S}$-regular **if** there exist an Hermitian matrix P, $\bar{h}$ Hermitian positive definite matrices Q_h, $h = 1, \ldots, \bar{h}$ and a scalar τ_E such that

$$\tau_E \left[\begin{array}{ccc} I & 0 & 0 \\ A_E & B_E & 0 \end{array}\right]' \Psi_E \left[\begin{array}{ccc} I & 0 & 0 \\ A_E & B_E & 0 \end{array}\right] +$$

$$\left[\begin{array}{ccc} 0 & B_A & A_A \\ 0 & 0 & I \end{array}\right]' \Psi_A \left[\begin{array}{ccc} 0 & B_A & A_A \\ 0 & 0 & I \end{array}\right] +$$

$$\left[\begin{array}{ccc} 0 & D_A & C_A \\ C_E & D_E & 0 \end{array}\right]' M \left[\begin{array}{ccc} 0 & D_A & C_A \\ C_E & D_E & 0 \end{array}\right] < 0 \quad (36)$$

where Ψ_E is given by (34) and

$$M = R \otimes P + \sum_{h=1}^{\bar{h}} \Phi_h \otimes Q_h, \quad (37)$$

$$\Psi_A = \left[\begin{array}{cc} I & 0 \\ 0 & -\gamma_A I \end{array}\right]. \quad (38)$$

The conservativeness is clearly due to the fact that the matrices are considered constant whereas they should depend on Δ_E. We only conjecture that (36) is less conservative than (17). We will illustrate it on an example in Section 6 but before, we will discuss about the interest of such results for the robust analysis of descriptor models.

It also has to be mentioned that Theorem 2 can be reduced, as a special case, when applied to conventional non descriptor models, to the LMI version of the so-called *Bounded Real Lemma* [1], which is used to compute the $\mathcal{H}_\infty$-norm of a realization.

V. ROBUST $\mathcal{D}$-ADMISSIBILITY OF DESCRIPTOR MODELS

We are here interested in the robust analysis of descriptor models of the form

$$E\dot{x} = Ax \quad (39)$$

(or

$$Ex_{k+1} = Ax_k \quad (40)$$

for the discrete case) where the matrices E and A comply with (2). It is well known (see [7] and the references therein) that the poles of such a model are the eigenvalues of the pencil (E, A), including finite and infinite ones. The system reponse contains a term with modes related to finite poles (as for conventional models where $E = I$) and another term with modes associated to infinite poles. As for usual models, the transient behaviour of the first term is strongly related to the location of the finite poles in the complex plane. For this reason, $\mathcal{D}$-stability (root-clustering in some region $\mathcal{D} \subset \mathbb{C}$) is of interest. But two other apsects have to be considered. The descriptor model should be regular (meaning that there is only one solution to the state equation: some kind of well posedness of the model) and it should be impulse free (meaning that the "infinite term" in the response does not convey impulses present in the control signals). We recall some classical definitions.

Definition 4: The model (39) or (40) is said to be $\mathcal{D}$-stable if the finite eigenvalues of (E, A) lie inside some region $\mathcal{D}$.

Definition 5: The model (39) or (40) is said to be $\mathcal{D}$-admissible if it is $\mathcal{D}$-stable, regular and impulse free.

For conventional models, $\mathcal{D}$-stability is known to be related to $\mathcal{S}$-regularity [2], [3]. Indeed, when $\mathcal{S}$ is the outside of $\mathcal{D}$ then $\mathcal{D}$-stability is the same as $\mathcal{S}$-regularity. A more frequent case is when $\mathcal{S}$ is the boundary of $\mathcal{D}$. Then $\mathcal{D}$-stability is a special case of $\mathcal{S}$-regularity for which all the poles are located on only one side of $\mathcal{S}$. The same reasoning can nearly be followed with descriptor systems. However, one has to be very careful with infinite poles. When $\mathcal{S}$ or $\mathcal{D}$ is unbounded, it is possible that the system be $\mathcal{S}$-singular because of infinite poles although they should not be considered for $\mathcal{D}$-stability. A solution is then to define $\mathcal{D}$ or $\mathcal{S}$ not only as a subset of $\mathbb{C}$ but as a subset of $\mathbf{C}$ where $\mathbf{C}$ is disc centred around the origin and of radius ω possibly very large. In this case, $\mathcal{S}$ might be only part of the boundary of $\mathcal{D}$, the remaining part being bounded by the frontier of $\mathbf{C}$. The finite poles necessarily lie inside $\mathbf{C}$ provided that ω is large enough and that Assumption 1.(ii) holds. The reason is that if the generalized order $r = \text{rank}(E)$ remains constant, a finite pole cannot become infinite (or the other way around) unless under infinite uncertainty (which shall reasonably not be considered) so it remains inside $\mathbf{C}$. This is of special interest when one considers Hurwitz stability for which the boundary $\mathcal{S}$ is the imaginary axis (thus unbounded). Roughly speaking, $\mathcal{S}$-regularity tests fail just because of infinite poles that can belong to $\mathcal{S}$ (actually, it is a bit more complicated: see [10]). In this case, it is possible to rather consider a long segment on the imaginary axis $[-\mathbf{i}\omega; \mathbf{i}\omega]$. Such a descritpion is allowed by (5) with the choice

$$\bar{h} = 1; \quad R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \Phi_1 = \begin{bmatrix} -1 & 0 \\ 0 & \omega^2 \end{bmatrix}. \tag{41}$$

This is exactly the idea of the finite frequency KYP lemma [11] and the connection with descriptor models is well highlighted in [10]. This case will be presented in the numerical illustration.

For a bounded $\mathcal{D}$ (*e.g.* a disc: then $\mathcal{S}$ is a circle), the problem is simpler. The finite poles lie inside $\mathcal{D}$ and the infinite poles outside. $\mathbf{C}$ is not required. Schur stability is then handled with $\mathcal{S}$ equaling the unit circle.

From the above discussion, it is clear that $\mathcal{D}$-stability can be tackled. The only additional assumption is that the nominal pencil (D_E, D_A) should be $\mathcal{D}$-stable.

Regularity is actually not a real problem. The regularity of a descriptor model is the regularity of the associated pencil as introduced in Defintion 3. So if $\mathcal{S}$-regularity is assessed for some non empty $\mathcal{S}$, regularity is proven.

Impulse freeness is completely related to Assumption 1.(ii). Actually, this property holds when the number of finite poles equals r [7], [21]. Since the number of finite poles does not change, it is clear that the sum of all geometric multiplicities of finite poles is preserved. This sum must equal r to ensure impulse freeness. Thus, Assumption 1.(ii) preserves the impulse freeness (provided (D_E, D_A) is impulse free of course). This assumption is then fundamental. But, in practice, it is not a drastic contraint because the rank deficiency of E is often due to structural properties of the model that are still valid in the presence of uncertainties. The only exception might be when the descriptor model results from the idealization of a "singularly perturbed" system.

As a conclusion of the above discussion, when (D_E, D_A) is $\mathcal{D}$-admissible, Theorems 1 and 2 can be used to analyze the robust $\mathcal{D}$-admissibility of (E, A).

VI. NUMERICAL ILLUSTRATION

The uncertain model is as follows:

$$\left[\begin{array}{c|c} A_A & B_A \\ \hline C_A & D_A \end{array}\right] =$$

$$\left[\begin{array}{ccc} 0.2140 & 0.3200 & 0.7266 \\ 0.6435 & 0.9601 & 0.4120 \\ \hline 0.2259 & 0.2091 & 0.5678 \\ 0.5798 & 0.3798 & 0.7942 \\ 0.7604 & 0.7833 & 0.0592 \\ 0.5298 & 0.6808 & 0.6029 \\ 0.6405 & 0.4611 & 0.0503 \end{array}\right| \cdots$$

$$\left.\begin{array}{ccccc} 0.4154 & 0.8744 & 0.7680 & 0.9901 & 0.4387 \\ 0.3050 & 0.0150 & 0.9708 & 0.7889 & 0.4983 \\ \hline 5.8413 & 13.4301 & 30.1742 & 27.2534 & 17.8494 \\ 5.0562 & -0.2859 & 15.8285 & 12.2772 & 7.0206 \\ -6.5957 & -6.863 & -24.2345 & -15.9162 & -9.7204 \\ 10.3767 & 11.4091 & 30.2249 & 20.4394 & 15.8384 \\ -16.0828 & -18.9503 & -47.4443 & -40.9319 & -30.7603 \end{array}\right] \tag{42}$$

$$\left[\begin{array}{c|c} A_E & B_E \\ \hline C_E & D_E \end{array}\right] =$$

$$\left[\begin{array}{cccc} 0.3295 & 0.6649 & 0.3830 & 0.6992 \\ 0.3090 & 0.6973 & 0.9834 & 0.3874 \\ 0.7329 & 0.5721 & 0.7906 & 0.0419 \\ 0.3944 & 0.5467 & 0.3867 & 0.2193 \\ \hline 0.3878 & 0.4480 & 0.4513 & 0.2346 \\ 0.7009 & 0.4883 & 0.9235 & 0.2231 \\ 0.0214 & 0.1904 & 0.7002 & 0.5491 \\ 0.7556 & 0.0708 & 0.1335 & 0.9363 \\ 0 & 0 & 0 & 0 \end{array}\right| \cdots$$

$$\left.\begin{array}{ccccc} 0.7847 & 0.1604 & 0.8695 & 0.3693 & 0 \\ 0.0862 & 0.7363 & 0.9474 & 0.5299 & 0 \\ 0.3433 & 0.0798 & 0.1366 & 0.2513 & 0 \\ 0.2559 & 0.4901 & 0.0385 & 0.2309 & 0 \\ \hline 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{43}$$

This model is such that $n = 5$ and $r = 4$. We assume that $\sqrt{\gamma_E^{-1}} = 0.001$. The nominal model $(D_E; D_A)$ is stable in the continuous sense, regular and impulse free. In other words, it admissible. Indeed, its nominal finite poles (the finite eigenvalues (D_E, D_A)) are

$$\{-7.0657; -5.0683; -4.7079; -1.1385\} \tag{44}$$

with the last nominal pole at infinity. Since the number of finite poles equals r, the nominal model is impulse free.

We apply Theorems 1 and 2 with the choice (41) and $\omega = 10000$ in order to test robust admissibility. Moreover, when solving LMI (17) and (36), we minimize γ_A. We obtain the respective robust stability bounds:

- Theorem 1 $\Rightarrow \bar{\sigma}(\Delta_A) \leq \sqrt{\gamma_A^{-1}} = 0.1255$;
- Theorem 2 $\Rightarrow \bar{\sigma}(\Delta_A) \leq \sqrt{\gamma_A^{-1}} = 0.3246$.

It is clear from these values that Theorem 2 seems to provide a far less pessimistic bound than Theorem 1. Indeed, by plotting several random uncertain models respecting the bounds on $\bar{\sigma}(\Delta_A)$ and $\bar{\sigma}(\Delta_E)$ in both cases, we can appreciate the weak conservatism of Theorem 2 (Figure 1). of course, it is difficult to estimate this conservatism if there is. If we could do this, we would know the ideal bound.

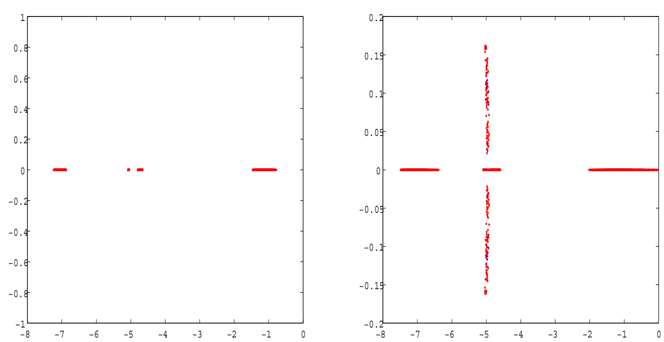

Fig. 1. Pole migration corresponding to obtained bounds (Th. 1 (left) and Th. 2 (right))

VII. Conclusion

In this paper we have proposed two simple *strict* LMI condtions for the robust $\mathcal{S}$-regularity of a pencil (E, A) when both matrices A and E are subject to norm-bounded LFT-based uncertainties and for a very large choice of set $\mathcal{S}$. We have shown on an example that the second condition seemed better and that the systematic use of a "big LFT" might not always be suitable. We have also explained how these conditions could be used to analyze the robust $\mathcal{D}$-admissibility of a continuous or discrete descriptor model. We insist on the compatibility of our conditions with the strong results on the conventional models.

As future investigations, we would like to consider more general uncertainty structures and try to give a better appreciation of the conservativeness induced by Theorem 2. We would also like to exploit the obtained condition in a design context, which is not straightforward.

References

[1] P. Apkarian, P Gahinet. A Linear Matrix Inequality approach to H_∞ control. *International Journal of Robust and Nonlinear Control*, 4:421-448, 1994.

[2] O. Bachelier, D. Henrion, B. Pradin, and D. Mehdi. Robust Matrix Root-Clustering of a Matrix in Intersections or Unions of Subregions. *SIAM Journal of Control and Optimization*, 43(3):1078–1093, 2004.

[3] O. Bachelier and D. Mehdi. Robust Matrix Root-Clustering through Extended KYP Lemma. *SIAM Journal of Control and Optimization*, 45(1):368–381, 2006.

[4] J. Bosche, O. Bachelier, and D. Mehdi. An approach for robust matrix root-clustering analysis in a union of regions. *IMA Journal of Mathematical Control and Information*, 22:227–239, 2005.

[5] S. Boyd, L. El Ghaoui, E. Féron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. Volume 15 of SIAM Studies in Applied Mathematics, USA, 1994.

[6] M. Chaabane, O. Bachelier, M. Souissi, and D. Mehdi. Stability and stabilization of continuous descriptor systems: an LMI approach. *Mathematical Problems in Engineering*, 2006:1–15, 2006.

[7] L. Dai. *Singular Control Systems*. Springer-Verlag, 1989.

[8] J. C. Doyle. Analysis of feedback systems with structured uncertainties. *IEE, Proc., Pt. D*, 129(6):242–250, November 1982.

[9] J. Y. Ishihara and M. H. Terra. On the Lyapunov theorem for singular systems. *IEEE Transactions on Automatic Control*, 47(11):1926–1930, 2002.

[10] T. Iwasaki and S. Hara. Generalized KYP lemma: unified frequency domain inequalities with design applications. *IEEE Transactions on Automatic Control*, 50(1):41–59, January 2005.

[11] T. Iwasaki, G. Meinsma, and M. Fu. Generalized S−procedure and finite frequency KYP lemma. *Mathematical Problems in Engineering*, 6:305–320, 2000.

[12] T. Iwasaki and G. Shibata. LPV system analysis via quadratic separator for uncertain implicit systems. *IEEE Transactions on Automatic Control*, 46(8):1195–1208, August 2001.

[13] F. L. Lewis. A survey of linear singular systems. *Circuits, Systems, and Signal Processing*, 5(1):3–36, 1986.

[14] C. Lin, J. Lam, J. L. Wang, and G.-H. Yang. Analysis on robust stability for interval descriptor systems. *Systems & Control Letters*, 42:267–278, 2001.

[15] C. Lin, J. L. Wang, G.-H. Yang, and J. Lam. Robust stabilization via state feedback for descriptor systems with uncertainties in the derivative matrix. *International Journal of Control*, 73(5):407–415, 2000.

[16] I. Masubuchi. Dissipativity inequalities for continuous-time descriptor systems with applications to synthesis of control gains. *Systems & Control Letters*, 55:158–164, 2005.

[17] I. Masubuchi. Output feedback controller synthesis for descriptor systems satisfying closed-loop dissipativity. *Automatica*, 43:339–345, 2007.

[18] D. Peaucelle, D. Arzelier, D. Henrion, and F. Gouaisbault. Quadratic separation for feedback connection of an uncertain matrix and an implicit linear transformation. *Automatica*, 43:796–804, 2007.

[19] C. W. Scherer. LPV control and full block multipliers. *Automatica*, 37:361–375, 2001.

[20] K. Takaba, N. Morihira, and T. A. Katayama. A generalized Lyapunov theorem for descriptor system. *Systems and Control Letters*, 24(1):49–51, 1995.

[21] G. C. Verghese, B. C. Lévy, and T. Kailath. A generalized state-space for singular systems. *IEEE Transactions on Automatic Control*, 26(4):361–375, August 1981.

[22] S. Xu and J. Lam. Robust stability and stabilization of discrete singular systems: an equivalent characterization. *IEEE Transactions on Automatic Control*, 49(4):568–574, 2004.