Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Interconnection and Damping Assignment Passivity–Based Control: Static vs Dynamic State–Feedback

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Abstract

Interconnection and Damping Assignment Passivity–Based Control (IDA–PBC) is a technique that regulates the behavior of nonlinear systems assigning a desired (Port Hamiltonian) structure to the closed–loop. This basic idea, introduced eight years ago, has turned out to be very successful and has provided solutions to a wide variety of physical problems. Although IDA–PBC is originally formulated as a static state–feedback technique it can easily be reformulated to use dynamic controllers. A natural question that arises is whether it is possible to extend the realm of applicability of the method by considering dynamic controllers. More precisely, is the set of plants that is stabilizable with static state–feedback IDA–PBC smaller than the one stabilizable with dynamic IDA–PBC? The main contribution of this paper is to prove that the answer to this question is, unfortunately, negative.

Notation All vectors defined in the paper are *column* vectors. For a scalar function $H : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, $(x,\zeta) \mapsto H$ we define the operators $\nabla_x H(x,\zeta) := \frac{\partial H(x,\zeta)}{\partial x}$ and $\nabla_\zeta H(x,\zeta) := \frac{\partial H(x,\zeta)}{\partial \zeta}$, their repeated application is denoted in the standard way, i.e., $\nabla_{x\zeta} H(x,\zeta) := \frac{\partial^2 H(x,\zeta)}{\partial x \partial \zeta}$, while we use $\nabla^2 H(x,\zeta)$ for the Hessian (with respect to (x,ζ)). When clear from the context the subindex of the operator ∇ is omitted. For the distinguished element $p_\star \in \mathbb{R}^q$ and a function $w : \mathbb{R}^q \to \mathbb{R}^s$, we denote the constant vector $w(p_\star) =: w_\star$.

1. Background material and problem formulation

IDA–PBC was introduced in [5, 4] as a procedure to stabilize a desired equilibrium for physical systems described by Port–Hamiltonian (PH) models, see also [1, 2, 3, 6] for closely related approaches. The procedure can be easily extended to systems of the form

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$, $m < n$, is the control action and $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are smooth functions with rank $\{g(x)\} = m$. The basic idea of IDA–PBC is to transform, via static state–feedback, the system (1) into a PH system with some desired energy function. The main result of IDA–PBC is summarized in the following proposition whose proof may be found in [7].

Proposition 1 *Consider the system (1). Define $g^\perp : \mathbb{R}^n \to \mathbb{R}^{(n-m)\times n}$ to be a full–rank left annihilator of $g(x)$, i.e., $g^\perp(x)g(x) = 0$, and* rank $\{g^\perp(x)\} = n - m$. *Let $x_\star \in \mathbb{R}^n$ be an assignable equilibrium, i.e.,*

$$x_\star \in \{x \in \mathbb{R}^n \mid g^\perp(x) f(x) = 0\}.$$

Assume there exists a matrix $F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and a function $H : \mathbb{R}^n \to \mathbb{R}$ such that the following holds

(A1) The matching equation

$$g^\perp(x) f(x) = g^\perp(x) F(x) \nabla H(x) \tag{2}$$

is satisfied.

Then the system (1) in closed–loop with the static state–feedback

$$u = [g^\top(x)g(x)]^{-1} g^\top(x)\{F(x)\nabla H(x) - f(x)\}, \tag{3}$$

takes the PH form

$$\dot{x} = F(x)\nabla H(x). \tag{4}$$

Furthermore, if:[1]

(A2) $F(x) + F^\top(x) \le 0$.

(A3) $(\nabla H)_\star = 0$ and $(\nabla^2 H)_\star > 0$.

Then $x_\star$ is a (locally) stable *equilibrium with Lyapunov function $H(x)$.*

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[1] Conditions (A2) and (A3) ensure that $\dot{H} \le 0$ and $x_\star$ is a strict minimizer of $H(x)$, respectively.

The question that we want to study in this paper is whether we can extend the class of systems for which IDA–PBC is applicable considering a dynamic controller instead of a simple static state–feedback as done above. More precisely, we consider an extended system

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} f(x)+g(x)\hat{u}(x,\zeta) \\ \hat{w}(x,\zeta) \end{bmatrix}$$

where $\zeta \in \mathbb{R}^p$, and $\hat{u} : \mathbb{R}^{n+p} \to \mathbb{R}^n$ and $\hat{w} : \mathbb{R}^{n+p} \to \mathbb{R}^p$ are functions to be defined in such a way that

$$\begin{bmatrix} f(x)+g(x)\hat{u}(x,\zeta) \\ \hat{w}(x,\zeta) \end{bmatrix} = \tilde{F}(x,\zeta) \begin{bmatrix} \nabla_x \tilde{H}(x,\zeta) \\ \nabla_\zeta \tilde{H}(x,\zeta) \end{bmatrix}$$

for some matrix $\tilde{F} : \mathbb{R}^{n+p} \to \mathbb{R}^{(n+p)\times(n+p)}$, such that $\tilde{F}(x,\zeta)+\tilde{F}^\top(x,\zeta) \leq 0$, and a function $\tilde{H} : \mathbb{R}^{n+p} \to \mathbb{R}$, which has a minimum at a point $(x_\star,\zeta_\star)$, for some $\zeta_\star \in \mathbb{R}^p$.

2. IDA–PBC with dynamic extension

To answer the question posed in the previous section we must extend Proposition 1 to consider dynamic (state–feedback) controllers, which leads to the following.

Proposition 2 *Consider the system (1) with $g^\perp(x)$ and $x_\star$ as defined in Proposition 1. Assume there exists a positive integer p, matrices $\tilde{F}_1 : \mathbb{R}^{n+p} \to \mathbb{R}^{n\times n}$, $\tilde{F}_2 : \mathbb{R}^{n+p} \to \mathbb{R}^{n\times p}$ and a function $\tilde{H} : \mathbb{R}^{n+p} \to \mathbb{R}$ such that the following holds*

(B1) The extended matching equation

$$\begin{aligned} g^\perp(x)f(x) &= g^\perp(x)[\tilde{F}_1(x,\zeta)\nabla_x\tilde{H}(x,\zeta)+ \\ &+ \tilde{F}_2(x,\zeta)\nabla_\zeta\tilde{H}(x,\zeta)] \end{aligned} \quad (5)$$

is satisfied.

Consider the p–dimensional dynamic state–feedback controller

$$\begin{aligned} \dot{\zeta} &= \tilde{F}_3(x,\zeta)\nabla_x\tilde{H}(x,\zeta)+\tilde{F}_4(x,\zeta)\nabla_\zeta\tilde{H}(x,\zeta) \\ u &= [g^\top(x)g(x)]^{-1}g^\top(x)\{\tilde{F}_1(x,\zeta)\nabla_x\tilde{H}(x,\zeta)+ \\ &+ \tilde{F}_2(x,\zeta)\nabla_\zeta\tilde{H}(x,\zeta)-f(x)\}, \end{aligned} \quad (6)$$

with arbitrary matrices $\tilde{F}_3 : \mathbb{R}^{n+p} \to \mathbb{R}^{p\times n}$ and $\tilde{F}_4 : \mathbb{R}^{n+p} \to \mathbb{R}^{p\times p}$

Then the system (1) in closed–loop with the controller (6) takes the PH form

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \tilde{F}(x,\zeta) \begin{bmatrix} \nabla_x \tilde{H}(x,\zeta) \\ \nabla_\zeta \tilde{H}(x,\zeta) \end{bmatrix} \quad (7)$$

where

$$\tilde{F}(x,\zeta) := \begin{bmatrix} \tilde{F}_1(x,\zeta) & \tilde{F}_2(x,\zeta) \\ \tilde{F}_3(x,\zeta) & \tilde{F}_4(x,\zeta) \end{bmatrix}.$$

Furthermore, if:[2]

(B2) [3]

$$\tilde{F}(x,\zeta)+\tilde{F}^\top(x,\zeta) \leq 0. \quad (8)$$

(B3) $(\nabla\tilde{H})_\star = 0$ *and* $(\nabla^2\tilde{H})_\star > 0$, *for some* $\zeta_\star \in \mathbb{R}^p$.

Then $(x_\star,\zeta_\star)$ is a (locally) stable *equilibrium with Lyapunov function $\tilde{H}(x,\zeta)$.*

Proof. Multiplying (1) on the left by the full rank matrix $\begin{bmatrix} g^\top(x) \\ g^\perp(x) \end{bmatrix}$ yields

$$\begin{bmatrix} g^\top(x) \\ g^\perp(x) \end{bmatrix} \dot{x} = \begin{bmatrix} g^\top(x)[f(x)+g(x)u] \\ g^\perp(x)f(x) \end{bmatrix}.$$

Replacing the control u defined in (6) and the extended matching equation (5) yields the first n rows of (7). The last p rows follow immediately from the first equation in (6) and the definition of $\tilde{F}(x,\zeta)$.

Now, in view of (8), we have that $\dot{\tilde{H}} \leq 0$, and the stability claim is established invoking (B2) and Lyapunov's second method.

3. Main result

Comparing Propositions 1 and 2 it is clear that to know whether dynamic extension enlarges the class of plants which are stabilizable via IDA–PBC we have to compare the sets of solutions of the partial differential equations (2) and (5)—subject to the constraints, (A2), (A3) and (B2), (B3), respectively. We show in the proposition below that (2) admits a solution if and only if there exists a solution for (5). This, unfortunately, proves that the answer to the question is negative.

Proposition 3 *Consider the system (1) with $g^\perp(x)$ and $x_\star$ as defined in Proposition 1. Then the following statements are equivalent:*

(S1) There exists a matrix $F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and a function $H : \mathbb{R}^n \to \mathbb{R}$ such that conditions (A1)–(A3) of Proposition 1 hold.

(S2) There exists a positive integer p, matrices $\tilde{F}_1 : \mathbb{R}^{n+p} \to \mathbb{R}^{n\times n}$, $\tilde{F}_2 : \mathbb{R}^{n+p} \to \mathbb{R}^{n\times p}$ and a function $\tilde{H} : \mathbb{R}^{n+p} \to \mathbb{R}$ such that conditions (B1)–(B3) of Proposition 2 hold.

Consequently, the system (1) is stabilizable via static state–feedback IDA–PBC if and only if it is stabilizable via dynamic (state–feedback) IDA–PBC.

[2] Conditions (B2) and (B3) ensure that $\dot{\tilde{H}} \leq 0$ and $(x_\star,\zeta_\star)$ is a strict minimizer of $H(x,\zeta)$, respectively.

[3] Notice that if $\tilde{F}_1(x,\zeta)+\tilde{F}_1^\top(x,\zeta) \leq 0$, B2 is trivially satisfied, for any $F_2(x,\zeta)$, setting $F_3(x,\zeta) = -F_2^\top(x,\zeta)$ and choosing any $F_4(x,\zeta)$ verifying $F_4(x,\zeta)+F_4^\top(x,\zeta) \leq 0$.

Proof. ((S1) $\Rightarrow$ (S2))[4] Assume conditions (A1)–(A3) of Proposition 1 hold. Select any positive integer p and fix

$$\tilde{F}(x,\zeta) = \begin{bmatrix} F(x) & 0_{n\times p} \\ 0_{p\times n} & \tilde{F}_4(x,\zeta) \end{bmatrix}$$

with $F_4 : \mathbb{R}^{n+p} \to \mathbb{R}^{p\times p}$ verifying $F_4(x,\zeta) + F_4^\top(x,\zeta) \leq 0$. The proposed matrix $\tilde{F}(x,\zeta)$ clearly verifies (B2) of Proposition 2. It is easy to see that, with this choice of $\tilde{F}(x,\zeta)$, the extended matching equation (5) admits a solution of the form

$$\tilde{H}(x,\zeta) = H(x) + \bar{H}(\zeta),$$

for any $\bar{H} : \mathbb{R}^p \to \mathbb{R}$. Hence, verifying (B1). Furthermore, selecting $\bar{H}(\zeta)$ such that $(\nabla\bar{H})_\star = 0$ and $(\nabla^2\bar{H})_\star > 0$, for some $\zeta_\star \in \mathbb{R}^p$ then (B3) of Proposition 2 clearly holds.

((S2) $\Rightarrow$ (S1)) Assume conditions (B1)–(B3) of Proposition 2 hold for some positive integer p. Now, since (2) and (5) have the same left hand side one obtains

$$\begin{aligned} & g^\perp(x)[F(x)\nabla H(x) - \tilde{F}_1(x,\zeta)\nabla_x\tilde{H}(x,\zeta) - \\ - \quad & \tilde{F}_2(x,\zeta)\nabla_\zeta\tilde{H}(x,\zeta)] = 0. \end{aligned} \tag{9}$$

We now construct functions $F(x)$ and $H(x)$ that satisfy (9)—hence (A1)—and conditions (A2), (A3) of Proposition 1. First, notice that, in view of (B3), one has

$$[\nabla_\zeta\tilde{H}(x,\zeta)]_\star = 0, \qquad \det\{[\nabla_{\zeta\zeta}\tilde{H}(x,\zeta)]_\star\} > 0.$$

Therefore, application of the Implicit Function Theorem [8] to the function $\nabla_\zeta\tilde{H}(x,\zeta)$ proves the existence of a function $\gamma : \mathbb{R}^n \to \mathbb{R}^p$ such that

$$[\nabla_\zeta\tilde{H}(x,\zeta)]|_{\zeta=\gamma(x)} = 0 \tag{10}$$

in some open neighborhood of $(x_\star,\zeta_\star)$.[5] Notice, also, that $\zeta_\star = \gamma_\star$. Replacing (10) in (9) yields

$$g^\perp(x)[F(x)\nabla H(x) - \tilde{F}_1(x,\gamma(x))W(x)] = 0, \tag{11}$$

where, to simplify the notation, the function $W : \mathbb{R}^n \to \mathbb{R}^n$

$$W(x) := \nabla_x\tilde{H}(x,\gamma(x)), \tag{12}$$

has been defined. Now, select $F(x) = \tilde{F}_1(x,\gamma(x))$ that, in view of (B2), necessarily satisfies (A2). Replacing $F(x)$ in (11) yields

$$g^\perp(x)F(x)[\nabla H(x) - W(x)] = 0.$$

A function $H(x)$ that satisfies the equation above is given as

$$H(x) := \tilde{H}(x,\gamma(x)).$$

Indeed, it follows from (10) that $\nabla H(x) = W(x)$. To complete the proof it only remains to show that the function $H(x)$ verifies condition (A3). For, note that

$$(\nabla H)_\star = W_\star = 0, \qquad (\nabla^2 H)_\star = (\nabla W)_\star > 0$$

where (B3), (12) and the fact that

$$\nabla W(x) = \nabla_{xx}[\tilde{H}(x,\gamma(x))],$$

have been used.

4. Concluding remarks

IDA–PBC is a popular (static state–feedback) technique for stabilization of nonlinear systems via energy–shaping. We have presented in this paper a new version of this technique that incorporates an (arbitrary) dynamic extension. We have shown that, for the purposes of (local Lyapunov) stabilization, no advantage is gained with this extension. More precisely, Proposition 3 shows that stabilizability via static state–feedback IDA–PBC is equivalent to stabilizability via dynamic IDA–PBC. This has been established proving that the existence of a local solution of the key matching equation (subject to the constraints that ensure stability) in the static case is equivalent to the solution (of the corresponding equation) in the dynamic case.

The proof of the equivalence is constructive, but relies on application of the Implicit Function Theorem, from which the local nature of our result is inherited. One consequence of this is that the domain of stability of the equilibrium for static feedback may be smaller that the one obtained using dynamic feedback. This is the case, for example, if dynamic feedback yields a global result while equation (10) admits only a local solution.

It is well–known that the main stumbling block for the application on IDA–PBC is the need to solve the set of partial differential equations defined by the matching equations.[6] The negative result regarding dynamic extension reported in this note is a modest contribution towards the understanding of this complicated problem, and certainly falls short from exhausting the various issues involved in it. Two questions that remain open, and are currently under investigation, are the following.

- It is clear from the proof of Proposition 3 that the input matrix $g(x)$ does not play any role, and $g^\perp(x)$ could have been removed from the matching equations. A similar remark applies to the sign–definiteness conditions of the matrices $F(x)$ and $\tilde{F}(x,\zeta)$. How can we incorporate these additional elements in the analysis?

[4] This implication is trivial and is given only for completeness.

[5] For the sake of brevity the reference to this neighborhood is omitted in the sequel, keeping in mind that the analysis is restricted to this neighborhood.

[6] This is, of course, endemic to all constructive design procedures for stabilization of general nonlinear systems.

- The version of IDA–PBC considered here does not presume any particular structure for the desired energy function. On the other hand, it has been shown that for some classes of systems, for instance, mechanical, it is convenient to "parameterize" the solutions—see e.g. [2, 6, 7]. Studying the effect of a dynamic extension in that case leads to a problem different from the one studied here.

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