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# Simple Conditions for Arbitrarily Close Deployment and Equilibrium Paths and Quasistationary Deployment

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*Abstract* — **Sufficient conditions for the state space trajectory of a system of ordinary differential equations (ODEs) to be arbitrarily close to an equilibrium path and for the evolution of the system to be quasistationary are proved using basic topology and mathematical analysis concepts. The conditions are less conservative than previously derived ones based on Lyapunov functions.**

## I. Introduction

In [1] a method which uses equilibrium paths was proposed for the control of nonlinear autonomous ODEs,

$$\dot{x} = \frac{dx}{dt} = f(x,u), \qquad x \in X \subset R^n,\ \ u \in U \subset R^m,\ \ t \in T \subset R. \tag{1}$$

Here $f(x,u)$ is a function of class $C^k$ on $X \times U$ ($k$>0), $x$, $u$, $t$ are the state, control vectors, time, whereas $X$, $U$, and $T$ are open sets in the $n$, $m$, and one dimensional real spaces, respectively. Note that for uniqueness of the initial value problem associated with (1), $f$ should be just Lipschitz in $x$.

The key idea is to control (1) such that its state space trajectory is close to an equilibrium path obtained by solving

$$0 = f(x,u). \tag{2}$$

In [1] several advantages of such an approach for practical applications are discussed in detail.

Assume that (2) can be solved for $x$ as a *continuous function* of $u$:

$$x = g(u), \quad f(g(u),u) = 0, \quad g : U_e \to X_e. \tag{3}$$

One situation when this is guaranteed is when the implicit function theorem applies: if $(x_i, u_i)$ is a solution of (2) and $J_i = \frac{\partial f}{\partial x}(x_i, u_i)$ is *not singular*, then there exist an open set $U_e$ and an *unique* function $g$ of class $C^k$ on $U_e$, such that (3) holds and $x_i = g(u_i)$. However the implicit function theorem is not necessary for (2) to have a solution like in (3).

In the above $U_e$ is the *largest domain* (open and connected set in $U$) in which (2) can be solved for $x$ as in (3). The $g$-image of $U_e$, $X_e$, is called the *equilibrium set* and it is also connected, but not necessarily open. Let $(x_f, u_f)$, $u_f \in U_e$, be another solution of (3), and $u_e(s)$ be a curve in $U_e$ parameterized by $s \in [0, \tau]$ which connects $u_i$ and $u_f$, with $u_e(0) = u_i$, $u_e(\tau) = u_f$. Then $u_e(s)$ is $g$-mapped onto an *equilibrium path,* $x_e(s) = g(u_e(s))$, with $x_e(0) = x_i$, $x_e(\tau) = x_f$.

The problem of interest is to control the evolution of the system between the two equilibrium states, $(x_i, u_i)$ and $(x_f, u_f)$, such that its state space trajectory - which will also be referred to as the *deployment path* - is close to the equilibrium path, as illustrated in Fig. 1. For this purpose the following strategy was proposed in [1]. The controls are fixed at $u_i$ and when the transition begins, at $t=0$, they start to vary along $u_e$, $u(t) = u_e(t)$, $t \in [0, \tau] \subset T$. When $t$ reaches $\tau$ the controls are frozen at the final desired value:

$$u(t) = \begin{cases} u_i, t < 0 \\ u_e(t), 0 \le t \le \tau \\ u_f, t > \tau \end{cases} \tag{4}$$

The deployment path, $x_d(t)$, is the solution of

$$\dot{x}_d = f(x_d, u(t)), \quad x_d(0) = x_i. \tag{5}$$

If the final equilibrium, $x_f$, is asymptotically stable for the "frozen", autonomous system, $\dot{x} = \frac{dx}{dt} = f(x, u_f)$, and $x_d(\tau)$ belongs to the region of attraction of $x_f$, then the system's trajectory will settle down, asymptotically in time, to the desired final value, $x_f$. The asymptotical stability of the final equilibrium, $x_f$, is essential for the application of this methodology.

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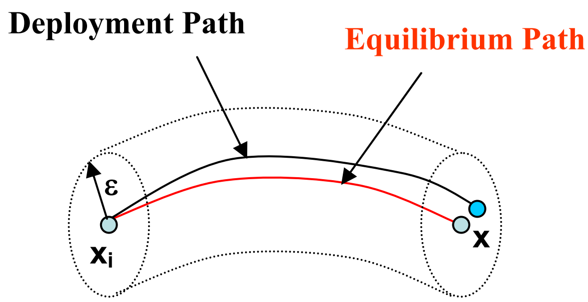


Fig. 1: Deployment and Equilibrium Paths

This methodology was easily extended to include piecewise constant controls [2], by discretizing the equilibrium path and using the resulting controls to drive (1).

In [1] several research questions were formulated with respect to this control strategy. Two of them are addressed here. Firstly, sufficient conditions for the state space trajectory to be close to the equilibrium path are proved. Secondly, sufficient conditions for the deployment to be quasistationary are proved. The proofs are constructive, result in piecewise constant controls, and use only basic concepts from topology and mathematical analysis. Hence, the conditions presented herein are less conservative than previous ones derived in [8] using Lyapunov functions.

## II. Related Work

Of the popular nonlinear systems control techniques the one which resembles most the idea presented in the above is sliding mode control (see [3, 4, 5]) where the system's state space trajectory is confined to a selected manifold. Control design reduces to finding the control law such that the system's trajectory approaches and then stays on the manifold. Sliding mode control has a major advantage in that it has good robustness properties. The main disadvantage is the chattering induced by the repeated switching in the control law. In order to alleviate chattering "sliding mode with boundary layer" techniques have been proposed, in which the system is allowed to evolve within a given distance from the desired manifold (see [6]). The thickness of the boundary layer required to eliminate chattering depends on the magnitude of the switching gain used. A controller with high switching gain produces high amplitude of chattering and needs a thicker boundary layer. On the other hand, the switching gain value depends on the system's uncertainties bounds: for a system with large uncertainties a thicker boundary layer is needed to eliminate chattering and the control system is changing to one without sliding mode.

The proposed strategy is different from sliding mode in several essential features. Firstly, the manifold which is used in the proposed strategy is an equilibrium manifold (path), which, under mild conditions, is *persistent under modeling perturbations.* For example if the Jacobian of the "frozen system" is not singular along the equilibrium path, this path has no stationary bifurcations [7]. The persistence under perturbations property results in several advantages, a key one being good robustness properties of the control method.

Secondly, the system's state space trajectory is not required to stay on a manifold. It must only be sufficiently close to the equilibrium path. If this path is persistent under perturbations then, under modeling errors of a certain size, the system's trajectory will remain close to the perturbed equilibrium path. This scenario resembles the "sliding mode with boundary layer" technique, however there are further fundamental differences which will be shortly revealed.

Thirdly, the controls are allowed to take value only in the equilibrium path's control set. As shown in the following, this facilitates satisfaction of the condition that the system's state space trajectory is close to this equilibrium path.

## III. Close Deployment and Equilibrium Paths

A key question is under what conditions $x_d(t)$ is *arbitrarily* close to the equilibrium path, $x_e(t)$, $t \in [0, \tau]$.

Let the *error function* be defined as

$$E(t) = x_d(t) - x_e(t) = x_d(t) - g(u_e(t)). \tag{6}$$

Khalil [8] addressed a very similar problem under the topic of slowly varying systems. Theorem 9.3 and Lemma 9.8 in [8] easily lead to the following result: if $f(x,u)$ is sufficiently smooth (at least of class $C^l$), its derivatives satisfy certain conditions (see [8]), (2) has a branch, $x_e(u)$, of class $C^l$ of isolated solutions, exponentially stable uniformly in $u$ for the "frozen system", and the system is driven by an *arbitrary* control of class $C^l$ such that $\|\dot{u}(t)\| \le \varepsilon$, then the error, $E(t)$, is bounded by a term proportional to $\varepsilon$. Note: the "frozen system" is obtained by fixing $u$ in (1).

This shows that $E(t)$ can be made arbitrarily small if the controls variation is sufficiently slow along the equilibrium path (in (4), $\|\dot{u}_e(t)\| \le \varepsilon$ ). This was the idea pursued in [1]. Khalil's results indicate that arbitrarily close equilibrium and deployment paths can be achieved with other controls also, which are not generated using the equilibrium path, because $u(t)$ can be arbitrary. Because the proofs rely heavily on Lyapunov techniques, which are known to be conservative, the result is rather restrictive, as seen in the above.

In this paper a more general result (Theorem 1 presented next) is proved which differs from Khalil's results in three aspects. Firstly, the proof uses only basic concepts from topology and mathematical analysis and does not rely on Lyapunov functions. Hence the conditions are less conservative. Secondly, it is not required that the equilibrium points of the equilibrium path are exponentially stable uniformly in $u$; asymptotical stability is shown to be sufficient. This is important because the exponential stability requirement is rather stringent and it is not always met in practical applications. Consider the following system:

$$\dot{x} = -(x-u)^3 \tag{7}$$

whose equilibria, $x_e(u)=u$, are (globally) asymptotically stable but not exponentially stable, as it can be ascertained by investigating the solution of the initial value problem,

$$x(t) = \frac{x_0 - u}{\sqrt{2(x_0-u)^2 t + 1}} + u. \tag{8}$$

Thirdly, the controls are not required to be of class $C^l$ as in Khalil's result; piecewise constant controls are sufficient.

**Theorem 1:** If the equilibrium path is *asymptotically stable uniformly in u* for (1), then, for $\forall \varepsilon > 0$ there exists a piecewise constant control $u(t)$ *obtained using the equilibrium path* such that the distance between the corresponding segments of the deployment and equilibrium paths is less than $\varepsilon$. Thus the deployment and equilibrium paths are *arbitrarily close*, $E(t) < \varepsilon$, $t \in [0, \tau]$.

Note: the equilibrium path, $x_e(u)$, is asymptotically stable uniformly in $u$ for (1) if for $\forall \varepsilon > 0$ there exists $\delta > 0$ such that for *any* equilibrium solution ($y, u_y$) on the equilibrium path, if $\|x_a - y\| < \delta$ then $\|x(x_a,t) - y\| < \varepsilon$ and $\lim_{t\to\infty} x(x_a,t) = y$, where $x(x_a,t)$ is the solution of $\dot{x} = f(x,u_y), x(0) = x_a$.

**Proof:** Consider an arbitrary $\varepsilon > 0$. Let the equilibrium path segment which is interior to the circle of radius $\varepsilon/4$ centered at $x_i = x_0$ be called $C_1$. Let $(x_1,u_1)$ be an arbitrary equilibrium solution on $C_1$ and $x_{e01}$ an arbitrary point on the segment of the equilibrium path between $x_0$ and $x_1$, called $x_e^{01}$. Clearly $\max_{x_{e01}\in x_e^{01}} \|x_1 - x_{e01}\| < \varepsilon/2$.

Consider that the following control is applied:

$$u^{01}(t) = \begin{cases} u_i = u_0, t \le 0 \\ u_1, t \in [0 \quad T_1]. \end{cases} \tag{9}$$

Because $x_1$ is stable, there exists $0 < \delta_1 < \varepsilon/2$ such that for $\forall x_a$ with $\|x_1 - x_a\| < \delta_1$, $\|x_d^{a1}(t) - x_1\| < \varepsilon/2$ where $x_d^{a1}(t)$ is the solution of $\dot{x} = f(x,u_1), x(0) = x_a$. If $\|x_1 - x_0\| < \delta_1$ then the choice $x_a = x_0$ leads to $\|x_d^{01}(t) - x_1\| < \varepsilon/2$, where $x_d^{01}(t)$ is called the "01" segment of the deployment path. A justification that $x_1$ can be selected such that $\|x_1 - x_0\| < \delta_1$ must be given. Indeed, if none of the points on $C_1$ can be selected such that $\|x_1 - x_0\| < \delta_1$, then $\forall x_1 \in C_1$, $\delta_1(x_1,\varepsilon) < \|x_1 - x_0\|$ leading to $\lim_{x_1\to x_0} \delta_1 = 0$. This cannot be true since $C_1$ is an asymptotically stable branch for (1).

Now apply the triangle inequality to get

$$\begin{aligned} &\|x_d^{01}(t) - x_{e01}\| \le \|x_d^{01}(t) - x_1\| + \|x_1 - x_{e01}\| \\ &< \varepsilon/2 + \max_{x_{e01}\in x_e^{01}} \|x_1 - x_{e01}\| < \varepsilon, \forall t \in [0 \quad T_1]. \end{aligned} \tag{10}$$

Thus $\forall \varepsilon > 0$, $(x_1,u_1)$ can be selected such that the distance between the two segments of the deployment and equilibrium paths, $x_d^{01}(t)$ and $x_e^{01}$, respectively, is smaller than $\varepsilon$. Recall that asymptotical stability of $x_1$ means that for $\forall \varepsilon_1 > 0$ $T_1$ can be selected (*sufficiently large*) such that $\|x_d^{01}(T_1) - x_1\| < \varepsilon_1$. The situation is depicted in Figure 2.

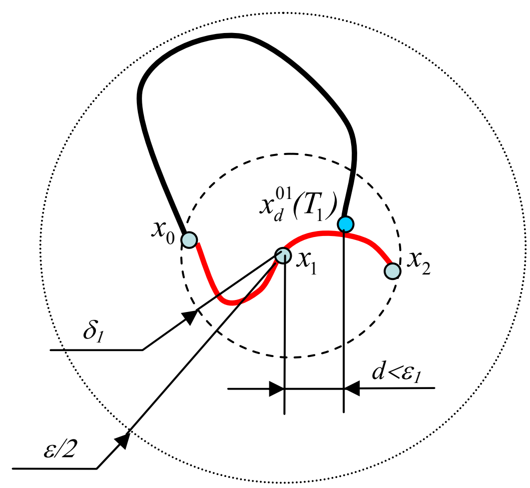


Figure 2: Deployment (black) and equilibrium (red) paths segments are within $\varepsilon$ distance from each other.

The process continues: let $(x_2,u_2)$ be an equilibrium solution on the equilibrium path in the direction of $(x_f,u_f)$. Like in the case of $x_1$, there exists a segment of the equilibrium path, $C_2$, such that $\forall x_2 \in C_2$ $\max_{x_{e12}\in x_e^{12}} \|x_2 - x_{e12}\| < \varepsilon/2$, where $x_{e12}$ is an arbitrary point on the segment of the equilibrium path between $x_1$ and $x_2$, denoted by $x_e^{12}$. Consider the control

$$u^{02}(t) = \begin{cases} u_0, t \le 0 \\ u_1, t \in [0 \quad T_1] \\ u_2, t \in [T_1 \quad T_2]. \end{cases} \tag{11}$$

Because $(x_2, u_2)$ is stable, there exists $0 < \delta_2 < \varepsilon/2$ such that if $\|x_2 - x_d^{01}(T_1)\| < \delta_2$, $\|x_d^{12}(t) - x_2\| < \varepsilon/2$. It is important to prove that $\|x_2 - x_d^{01}(T_1)\| < \delta_2$ is possible. Indeed, the triangle inequality yields (see Figure 2):

$$\begin{aligned} &\|x_2 - x_d^{01}(T_1)\| \le \|x_2 - x_1\| + \|x_1 - x_d^{01}(T_1)\| \\ &< \|x_2 - x_1\| + \varepsilon_1 \end{aligned} \tag{12}$$

and it is obvious that $\|x_2 - x_d^{01}(T_1)\|$ can be made smaller than $\delta_2$ by making $T_1$ sufficiently large (because $x_1$ is asymptotically stable). Like in the case of $x_1$ and $\delta_1$, it follows that $x_2$ can be selected such that $\|x_2 - x_1\| < \delta_2 - \varepsilon_1$. Next, the triangle inequality yields

$$\begin{aligned} &\|x_d^{12}(t) - x_{e12}\| \le \|x_d^{12}(t) - x_2\| + \|x_2 - x_{e12}\| \\ &< \varepsilon/2 + \max_{x_{e12} \in x_e^{12}} \|x_2 - x_{e12}\| < \varepsilon, \forall t \in [T_1 \quad T_2]. \end{aligned} \tag{13}$$

which proves that for $\forall \varepsilon > 0$, $(x_2, u_2)$ can be selected such that the distance between $x_d^{12}(t)$ and $x_e^{12}$ is smaller than $\varepsilon$. Figure 3 illustrates the construction process.

The process continues until the final equilibrium, $(x_f, u_f)$, is reached, after $N$ steps. Finally a *piecewise constant control* is built which guarantees that the deployment and equilibrium paths are within $\varepsilon$ distance, i.e. $E(t) < \varepsilon$, $t \in [0, \tau]$.

The constructive-proof presented in the above indicates two important facts. Firstly, each time-interval $[T_k \; T_{k+1}]$ must be *sufficiently long* such that $\|x_d^{k(k+1)}(T_{k+1}) - x_{k+1}\| < \varepsilon_{k+1}$ (i.e. the terminal point of the previous segment of the deployment path must be sufficiently close to $x_{k+1}$). Secondly, each space-interval (i.e. $\|x_k - x_{k+1}\|$) must be *sufficiently small* which, by continuity arguments, results in the conclusion that $\|u_k - u_{k+1}\|$ must be sufficiently small. These facts indicate that the closer the two paths are required to be (i.e. smaller $\varepsilon$ is desired) the longer the deployment time and the more refined the controls should be.

Even though this theorem gives a *sufficient* condition, the result is obviously more general than the result of Khalil [8].

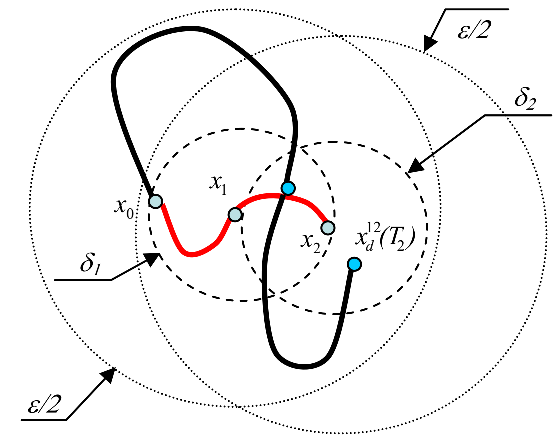


Fig. 3: The process of constructing $\varepsilon$-close deployment and equilibrium paths.

## IV. A CONDITION FOR QUASISTATIONARY DEPLOYMENT

In the following another important question, that of quasistationary deployment, will be addressed. The deployment is considered quasistationary if for a fixed $\eta > 0$, $\|\dot{x}_d(t)\| < \eta, \forall t \in [0, \tau]$. The question of interest is under what conditions there exists a control $u(t)$ such that $\|\dot{x}_d(t)\| < \eta, \forall t \in [0, \tau]$ (see [1] for motivations rooted deeply in practical applications). The following theorem provides *sufficient* conditions for this to happen.

**Theorem 2**: If the equilibrium path is asymptotically stable uniformly in $u$ for (1) and for any fixed $u$, $f(x, u)$ is Taylor series expandable in $x$, then for $\forall \eta > 0$ there exists a piecewise constant control, $u(t)$, *obtained using the equilibrium path* such that $\|\dot{x}_d(t)\| < \eta, \forall t \in [0, \tau]$.

**Proof:** The condition $\|\dot{x}_d(t)\| < \eta$ is equivalent to $\|f(x_d(t), u(t))\| < \eta$. Theorem 1 shows that for $\forall \varepsilon > 0$ a piecewise constant control can be applied to (1) such that the deployment and equilibrium path segments are within $\varepsilon$ distance from each other. Consider for example the first deployment path segment, $x_d^{01}(t)$. The Taylor series expansion of $f(x_d^{01}(t), u_1)$ around $x_1$ yields

$$f(x_d^{01}(t),u_1) = f(x_1,u_1) + J_1\varepsilon_1(t) + O\big(\varepsilon_1(t)\big), \forall t \in [0,T_1], \tag{14}$$

where $\varepsilon_1(t) = x_d^{01}(t) - x_1$, $J_1 = \dfrac{\partial f}{\partial x}(x_1,u_1)$; $O\big(\varepsilon_1(t)\big)$ represents terms of higher order in $\varepsilon_1(t)$. Since $f(x_1,u_1) = 0$,

$$\left\| f(x_d^{01}(t),u_1) \right\| \le \|J_1\| \|\varepsilon_1(t)\| + \left\| O\left( \|\varepsilon_1(t)\|^2 \right) \right\| < \bar{\sigma}\varepsilon + k\varepsilon^2. \tag{15}$$

Here $k$ is a positive constant and $\bar{\sigma}$ is selected such that $\bar{\sigma} > \max_{(x_e,u_e)} \sigma\left( \dfrac{\partial f}{\partial x}(x_e,u_e) \right)$, where $\sigma(*)$ is the maximum singular value of $*$. In the above $(x_e,u_e)$ varies along the entire equilibrium path, hence the maximum singular value of the Jacobian is selected over the entire equilibrium path. Since $\varepsilon$ can be made arbitrarily small by properly choosing $u_1$ (see Theorem 1 and its proof) the desired result is obtained: for any positive $\eta$, $u_1$ can be selected such that $\left\| f(x_d^{01}(t),u_1) \right\| < \eta$. The proof continues in a similar fashion for the remainder of the deployment path. Finally, a piecewise constant control $u(t)$ is constructed which guarantees that $\left\| f(x_d(t),u(t)) \right\| < \eta$.

The major implication of this result is that piecewise controls are sufficient to achieve quasistationary deployment. This shows that, for example in the case of a structure's deployment, the process can be conducted with *arbitrarily* small generalized velocities (hence vibrations) using only piecewise constant controls. This is a tremendous advantage since discrete controls are easily generated using digital technology.

With respect to the shortcomings of the two results presented herein, it is worth mentioning that they are *still* conservative, giving only sufficient conditions for arbitrarily close deployment and equilibrium paths, and quasistationary deployment, respectively. Hence one might expect that acceptable results for practical applications (i.e. *sufficiently* close deployment and equilibrium paths and *sufficiently* small $\|\dot{x}_d(t)\|$) are obtained under less restrictive conditions.

## V. Conclusions

A method for the control of nonlinear systems in which the key idea is to stay close to a preselected equilibrium path is investigated. Basic tools from topology and mathematical analysis are used to prove that if the equilibrium path is an asymptotically stable branch for the "frozen" system, piecewise constant control laws can be built which guarantee that the state space trajectory of the system (the deployment path) is arbitrarily close to an equilibrium path. A second result proved herein is that, if in addition, the vector function is Taylor series expandable in the state variables, the deployment can be conducted in a quasistationary manner with arbitrarily small generalized velocities. The piecewise constant controls are very advantageous in practical applications, because they are easy to generate.

The proofs give sufficient conditions, hence the results are (still) conservative. Future research should look into further relaxing the conditions presented in this paper.

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