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Strong Practical Stability and Stabilization of 2D Differential-Discrete Linear Systems

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***Abstract*— This paper considers two-dimensional (2D) differential linear systems recursive over the upper right quadrant described by well known state-space models. Included are differential linear repetitive processes which evolve over a subset of the upper right quadrant of the 2D plane. In particular, information propagation in one direction only occurs over a finite duration and is governed by a matrix differential linear equation. A stability theory exists for these processes but there has also been work which has led to the assertion that this is too strong in many cases of applications interest. This paper develops strong practical stability for differential linear repetitive processes as a possible alternative in such cases. Also stabilizing control law design algorithms are developed as the first step towards applying this new stability analysis to physical examples.**

I. INTRODUCTION

The unique characteristic of a repetitive, or multipass [1], process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

To introduce a formal definition, let $\alpha < +\infty$ denote the pass length (assumed constant). Then in a repetitive process the pass profile $y_k(t)$, $0 \le t \le \alpha$, generated on pass k acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile $y_{k+1}(t)$, $0 \le t \le \alpha$, $k \ge 0$.

Physical examples of these processes include long-wall coal cutting and metal rolling operations [1]. Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of such algorithmic applications include classes of iterative learning control schemes [2] and iterative algorithms for solving nonlinear dynamic optimal stabilization problems based on the maximum principle [3]. In this latter case, for example, use of the repetitive process setting provides the basis for the development of highly reliable and efficient iterative solution algorithms and in the former it provides a stability theory which, unlike many alternatives, provides information concerning an absolutely critical problem in this application area, i.e. the trade-off between convergence and the learnt dynamics.

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Attempts to control these processes using standard (or 1D) systems theory and algorithms fail (except in a few very restrictive special cases) precisely because such an approach ignores their inherent 2D systems structure, i.e. information propagation occurs from pass-to-pass (k direction) and along a given pass (t direction) and also the initial conditions are reset before the start of each new pass. To remove these deficiencies, a rigorous stability theory has been developed [1] based on an abstract model of the dynamics in a Banach space setting which includes a very large class of processes with linear dynamics and a constant pass length as special cases. Also the results of applying this theory to a range of sub-classes, including the differential linear repetitive processes considered here, have been reported [1].

Recognizing the unique control problem, the stability theory for linear repetitive processes is of the bounded input bounded output (BIBO) form, i.e. bounded inputs are required to produce bounded sequences of pass profiles (where boundedness is defined in terms of the norm on the underlying Banach space). Moreover, it consists of two concepts, one of which is defined over the finite pass length and the other is independent of this parameter. In particular, asymptotic stability guarantees this BIBO property over the finite and fixed pass length whereas stability along the pass is stronger since it requires this property uniformly (and hence it is not surprising that asymptotic stability is a necessary condition for stability along the pass).

If asymptotic stability holds for a differential linear repetitive process then any sequence of pass profiles it generates converges in the pass-to-pass direction to a limit profile which is described by a 1D differential linear systems state-space model. This fact has clear implications for the design of control schemes. Moreover, the condition for asymptotic stability is very easy to test whereas one of the extra for stability along the pass is much more involved and also its frequency domain interpretation raises the question of whether or not asymptotic stability alone would be sufficient for many practically relevant cases. The answer for differential processes is no but it may be acceptable to use strong practical stability which we develop here as an alternative to stability along the pass.

Throughout this paper, the null and identity matrices with the required dimensions are denoted by 0 and I respectively. Moreover, $M > 0$ (< 0) denotes a real symmetric positive (negative) definite matrix, $Sym\{M\}$ is used to denote $M +$

M^T, and $(\star)$ is used to denote block entries in the symmetric Linear Matrix Inequalities (LMIs) which are the means by which the necessary computations can be completed for a given numerical example.

II. Strong Practical Stability Analysis

The state-space model of a differential linear repetitive process [1] has the following form over $0 \le t \le \alpha,\ k \ge 0$

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + Bu_{k+1}(t) + B_0 y_k(t), \\ y_{k+1}(t) &= Cx_{k+1}(t) + Du_{k+1}(t) + D_0 y_k(t). \end{aligned} \tag{1}$$

Here on pass k, $x_k(t)$ is the $n \times 1$ state vector, $y_k(t)$ is the $m \times 1$ pass profile vector and $u_k(t)$ is the $l \times 1$ vector of control inputs, and $\alpha < \infty$ is termed the pass length. To complete the process description, it is necessary to specify the boundary conditions i.e. the state initial vector on each pass and the initial pass profile (i.e. on pass 0). For the purposes of this paper, no loss of generality occurs from assuming state initial vectors of the from $x_{k+1}(0) = d_{k+1}, k \ge 0$, where the $n \times 1$ vector d_{k+1} has known constant entries, and initial pass profile $y_0(t) = f(t)$, where $f(t)$ is a known vector over $0 \le t \le \alpha$.

The stability theory [1] for linear repetitive processes is based on an abstract model in a Banach space setting which includes a wide range of such processes as special cases, including those described by (1). In terms of their dynamics it is the pass-to-pass coupling (noting again their unique feature) which is critical. This is of the form $y_{k+1} = L_\alpha y_k$, where $y_k \in E_\alpha$ (E_α a Banach space with norm $||\cdot||$) and L_α is a bounded linear operator mapping E_α into itself. (In the case considered here L_α is a differential linear systems convolution operator.)

Asymptotic stability, i.e. BIBO stability over the fixed finite pass length $\alpha > 0$, requires the existence of finite real scalars $M_\alpha > 0$ and $\lambda_\alpha \in (0, 1)$ such that $||L_\alpha^k|| \le M_\alpha \lambda_\alpha^k,\ k \ge 0$, where $||\cdot||$ also denotes the induced operator norm. For processes described by (1) it has been shown elsewhere (see, for example, Chapter 3 of [1]) that this property holds if, and only if, all eigenvalues of the matrix D_0 have modulus strictly less than unity — written here as $r(D_0) < 1$ where $r(\cdot)$ denotes the spectral radius of its matrix argument.

Suppose that $r(D_0) < 1$ and the input sequence applied $\{u_{k+1}\}_k$ converges strongly as $k \to \infty$ (i.e. in the sense of the norm on the underlying function space) to u_∞. Then the strong limit $y_\infty := lim_{k\to\infty} y_k$ is termed the limit profile corresponding to this input sequence and its dynamics are described by

$$\begin{aligned} \dot{x}_\infty(t) &= (A + B_0(I - D_0)^{-1}C)x_\infty(t) \\ &+ (B + B_0(I - D_0)^{-1}D)u_\infty(t), \\ y_\infty(t) &= (I - D_0)^{-1}Cx_\infty(t) \\ &+ (I - D_0)^{-1}Du_\infty(t), \\ x_\infty(0) &= d_\infty, \end{aligned} \tag{2}$$

where (again a strong limit) $d_\infty := \lim_{k\to\infty} d_k$. In physical terms, this result states that under asymptotic stability the repetitive dynamics can, after a "sufficiently large" number of passes have elapsed, be replaced by those of a 1D differential linear system. This fact has clear implications in terms of the control of these processes — see [1] for a detailed treatment of this point.

Asymptotic stability does not guarantee that the limit profile has acceptable along the pass dynamics since it can be unstable in the 1D linear systems sense. A simple example here is the case when $A = -1$, $B = 1$, $B_0 = 1 + \beta$, $C = 1, D = D_0 = 0$, where $\beta > 0$ is a real scalar. Hence if $\beta > 0$ the limit profile for this proces is unstable.

If we wish to prevent cases such as the above example from arising, one route is to demand the BIBO property for any possible value of the pass length (mathematically this can be analyzed by letting $\alpha \to \infty$). This is the stability along the pass property which requires the existence of finite real scalars $M_\infty > 0$ and $\lambda_\infty \in (0, 1)$ such that $||L_\alpha^k|| \le M_\infty \lambda_\infty^k,\ k \ge 0$. For the processes considered here this requires that (i) $r(D_0) < 1$ (asymptotic stability), (ii) all eigenvalues of A have strictly negative real parts, and (iii) all eigenvalues of the transfer-function matrix $G(s) = C(sI - A)^{-1}B_0 + D_0$ must lie inside the unit circle in the complex plane for all $s = \imath\omega,\ \omega \ge 0$. In the case of the numerical example above it is this last condition which fails.

Stability along the pass for linear repetitive processes demands that the signals involved are uniformly bounded when both independent variables (k and t) are of unbounded duration. As noted above, this requires $r(G(\imath\omega)) < 1, \omega \ge 0$, which, as discussed in more detail at the end of this section, is a very strict condition. Strong practical stability relaxes this by removing the uniform boundedness requirement as both $k \to \infty$ and $\alpha \to \infty$ but still demands it when (i) both k and α are finite, (ii) the pass index $k \to \infty$ and the pass length α finite, and (iii) the pass index k is finite and the pass length $\alpha \to \infty$. Also cases (ii) and (iii) here have practical relevance which we discuss next in terms of a robotic system.

Consider the case of a gantry robot whose task is to collect an object from a location and place it on a moving conveyor belt after a finite time has elapsed, then return to the original location to pick up the next one and so on. Then this is an obvious application for iterative learning control [2], and hence repetitive process theory, in that the return journey can be used to update the control law using previous pass information to sequentially improve performance. Case (ii) here is a mathematical formulation of the desire to execute this operation a very large number of times without the need to stop and hence lose throughput. Case (iii) is the mathematical formulation where the process completes a finite number of passes but the pass length is 'very long' and there is a requirement to control the along the pass dynamics. Next we analyze these two cases in turn.

Consider the case of $t = 0$ with zero state initial vector sequence and zero control input vector. Then $y_k(0) = D_0^k y_0(0)$ and hence we require $r(D_0) < 1$. Under this condition, i.e. asymptotic stability, we achieve the limit profile (2) as $k \to \infty$ that is stable when the eigenvalues of $A + B_0(I - D_0)^{-1}C$ have strictly negative real parts. These are necessary conditions for strong practical stability.

Consider now any finite k. Then clearly (consider the case when there is no previous pass profile contribution) we require that all eigenvalues of the matrix A have strictly negative real parts. Also as $t \to \infty$

$$y_{k+1}(\infty) = \left(-CA^{-1}B_0 + D_0\right) y_k(\infty), \tag{3}$$

and hence we require $r(-CA^{-1}B_0 + D_0) < 1$. In summary, therefore, strong practical stability requires the following conditions to hold

[a] $r(D_0) < 1$,
[b] all eigenvalues of the matrix A have strictly negative real parts,
[c] all eigenvalues of the matrix $A + B_0(I - D_0)^{-1}C$ have strictly negative real parts, and
[d] $r(-CA^{-1}B_0 + D_0) < 1$.

To explain the basic difference between asymptotic stability, strong practical stability and stability along the pass, first note that $G(0) = -CA^{-1}B_0 + D_0$ and also $\lim_{|s|\to\infty} G(s) = D_0$. Consider also the case when there is zero control input and the state initial vector on each pass is zero. Then in Laplace transform terms $y_k(s) = G^k(s)y_0(s)$. Now set $s = \imath\omega$ and consider, for simplicity, the single-input single-output case and then $y_k(\imath\omega) = G^k(\imath\omega)y_0(\imath\omega)$, $k \geq 0$. Hence stability along the pass requires that each frequency component of the initial pass profile is attenuated from pass-to-pass, asymptotic stability (i.e. $r(D_0) < 1$) only requires this at high frequencies, and strong practical stability at both high and low frequencies together with conditions [b] and [c] above.

In terms of design to track a given reference vector, imposing the requirement for stability along the pass means that the control law must achieve the required level of attenuation over the complete frequency range and this, by comparison with the 1D systems case, is most likely to result in a very difficult design problem. In such cases, strong practical stability may lead to acceptable design, especially for applications where an unstable limit profile is not acceptable and/or some control is required over the along the pass dynamics. Note also that applications do exist where asymptotic stability is all that is required or can be achieved.

III. LMI based Stability Tests

The conditions for strong practical stability can, assuming no numerical problems with computing the eigenvalues of the matrices involved, be easily checked for a given example. Suppose, however, that the task is to ensure this property, by application of a control law (see also the next section) of the form

$$u_{k+1}(t) = K_1 x_{k+1}(t) + K_2 y_k(t), \tag{4}$$

which is a combination of current pass state feedback plus a feedforward term from the previous pass profile (in keeping with the fact that use of only current pass state or pass profile vector activated control laws cannot stabilize the process dynamics in all but a few restrictive special cases). Then when this control law is applied the process state-space model matrices A, B_0, C, D_0 are mapped to $A + BF, B_0 + FK_2, C + DK_1$, and $D_0 + DK_2$ respectively. Hence design to satisfy conditions [a] and [b] for the controlled process is simply the 1D pole placement problem for differential and discrete linear systems respectively. The case for conditions [c] and [d] is far from clear and hence as a preliminary step to overall control law design we make novel use of results from 1D singular differential and discrete linear systems theory for the state-space models

$$\begin{aligned} E\dot{x}(t) &= \widehat{A}x(t) + \widehat{B}u(t), \\ y(t) &= \widehat{C}x(t) + \widehat{D}u(t), \end{aligned} \tag{5}$$

and

$$\begin{aligned} Ex(h+1) &= \widehat{A}x(h) + \widehat{B}u(h), \\ y(h) &= \widehat{C}x(h) + \widehat{D}u(h), \end{aligned} \tag{6}$$

respectively, where E is a singular matrix. This will lead to stability tests in terms of LMIs which then (see the next section) lead to control law design algorithms.

The starting point in terms of condition [c] is to note that it is equivalent to stability of the 1D singular differential linear system with state-space model

$$E_1\dot{z}(t) = \begin{bmatrix} A & B_0 \\ C & D_0 - I \end{bmatrix} z(t) + \begin{bmatrix} B \\ D \end{bmatrix} u(h), \tag{7}$$

where

$$E_1 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \tag{8}$$

Similarly, condition [d] is equivalent to stability of the 1D singular discrete linear system

$$E_2 z(h+1) = \Phi z(h) + \begin{bmatrix} B \\ D \end{bmatrix} u(h), \tag{9}$$

where

$$E_2 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \quad \Phi = \begin{bmatrix} A & B_0 \\ C & D_0 \end{bmatrix}. \tag{10}$$

We also need the following definitions.

Definition 1: A 1D differential singular linear system of the form (5) is termed admissible [4] if it is stable, regular, impulse-free, i.e. $det(sE - \widehat{A})$ is not identically zero, and $deg\left(det(zE - \widehat{A})\right) = rank(E)$.

Definition 2: A 1D discrete singular linear system of the form (6) is termed admissible [4] if it is stable, regular, i.e. $det(zE - \widehat{A})$ is not identically zero, and $deg\left(det(zE - \widehat{A})\right) = rank(E)$.

It is clear that there exists (appropriately dimensioned) nonsingular matrices U and V such that

$$UEV = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \tag{11}$$

Also introduce

$$\begin{aligned} E^{\perp} &= V(I - UEV)U, \quad &(12)\\ E^{\ddagger} &= U^T(I - UEV)U^{-T}, \quad &(13) \end{aligned}$$

and

$$E^{\dagger} = U^{-1}(I - UEV)U. \quad (14)$$

Then we have the following result [5].

Lemma 1: A 1D differential singular linear system of the form (5) is admissible if, and only if, there exist appropriately dimensioned matrices X, Y and Z such that the following LMI is feasible

$$\begin{aligned} EXE^T + Sym\{E^{\dagger}Z\} &> 0, \quad (15)\\ Sym\{\widehat{A}XE^T\} + Sym\{\widehat{A}E^{\perp}Y\} &< 0. \quad (16)\end{aligned}$$

The corresponding result for the discrete case is as follows [6].

Lemma 2: A 1D discrete singular linear system of the form (6) is admissible if, and only if, there exist appropriately dimensioned matrices X, Y and G such that the following LMI is feasible for a given $\beta > 1$

$$\begin{aligned} &\begin{bmatrix} -EXE^T & (E^{\perp}YE^{\ddagger})^T \\ E^{\perp}YE^{\ddagger} & X \end{bmatrix} \\ &+ Sym\left\{\begin{bmatrix} \widehat{A}G \\ -G \end{bmatrix}\begin{bmatrix} U^{-T} & \beta V^T \end{bmatrix}\right\} < 0. \end{aligned} \quad (17)$$

Now we can establish the first major new result of this paper which gives computable necessary and sufficient conditions for strong practical stability.

Theorem 1: A differential linear repetitive process described by (1) is strongly practically stable if, and only if, there exist the appropriately dimensioned matrices $W_1 > 0$, $W_2 > 0$, $X_{11}^1 = (X_{11}^1)^T$, $X_{11}^2 = (X_{11}^2)^T$, $X_{22}^2 = (X_{22}^2)^T$, X_{21}^1, X_{21}^2, Y_{11}^2, Y_{22}^1, Y_{21}^1, Z_{22}^1, Z_{21}^1, $\widetilde{G}_2$. such that the following LMIs are feasible for a scalar $\beta > 1$

$$\begin{bmatrix} -W_1 & W_1^T D_0^T \\ D_0 W_1 & -W_1 \end{bmatrix} < 0, \quad (18)$$

$$A^T W_2 + W_2 A < 0, \quad (19)$$

$$\begin{bmatrix} X_{11}^1 & Z_{21}^T \\ Z_{21} & Z_{22}^T + Z_{22} \end{bmatrix} > 0, \quad (20)$$

$$Sym\left\{\begin{bmatrix} AX_{11}^1 + B_0X_{21}^1 + B_0Y_{21}^1 & B_0Y_{22}^1 \\ CX_{11}^1 + (D_0 - I)(X_{21}^1 + Y_{21}^1) & (D_0 - I)Y_{22}^1 \end{bmatrix}\right\} < 0, \quad (21)$$

$$\begin{aligned} &\begin{bmatrix} 0 & 0 & (Y_{11}^2)^T & 0 \\ 0 & -X_{22}^2 & 0 & 0 \\ Y_{11}^2 & 0 & X_{11}^2 & (X_{21}^2)^T \\ 0 & 0 & X_{21}^2 & X_{22}^2 \end{bmatrix} \\ &+Sym\left\{\begin{bmatrix} \Phi\widetilde{G}_2 \\ -\widetilde{G}_2 \end{bmatrix}\begin{bmatrix} U_2 & \beta U_2 \end{bmatrix}\right\} < 0, \end{aligned} \quad (22)$$

where

$$U_2 := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \quad (23)$$

Proof: In the case of conditions [a] and [b] respectively, simply note that the LMIs of (18) and (19) are commonly used in 1D linear systems stability theory. The fact that the LMIs (20) and (21) are equivalent to condition [c] follows immediately on interpreting Lemma 1 for this case. Finally, interpreting Lemma 2 for the case here shows that the LMI (22) is equivalent to condition [d]. ■

IV. CONTROL LAW DESIGN

If the control law (4) is applied then the controlled process state-space model is given by

$$\begin{aligned} \dot{x}_{k+1}(t) &= (A + BK_1)x_{k+1}(t) \\ &+ (B_0 + BK_2)y_k(t), \\ y_{k+1}(t) &= (C + DK_1)x_{k+1}(t) \\ &+ (D_0 + DK_2)y_k(t), \end{aligned} \quad (24)$$

and the following result gives the necessary and sufficient conditions for strong practical stability.

Theorem 2: The process of (24) is strongly practically stable in the strongly practically stable if, and only, if the following hold

[e] $r(D_0 + DK_2) < 1$,

[f] all eigenvalues of $A + BK_1$ have strictly negative real parts,

[g] all eigenvalues of $A + BK_1 + (B_0 + BK_2)(I - D_0 - DK_2)^{-1}(C + DK_1)$ have strictly negative real parts, and

[h] $r(D_0 + DK_2 - (C + DK_1)(A + BK_1)^{-1}(B_0 + BK_2)) < 1$.

Control law design based on Theorem 2 is somewhat complex since we only have the two control law matrices K_1 and K_2 to simultaneously satisfy the four conditions. Moreover, even though the first two are simply the pole placement problem for 1D differential and discrete linear systems respectively, the third and fourth clearly require further development and for that we require the following results from, for example, [5] and [6].

Lemma 3: Suppose that the control law

$$u(t) = Kx(t), \quad (25)$$

is applied to the 1D differential singular linear system described by (5). Then the resulting system is admissible if, and only if, there exist matrices X, Y, Z, R and G such that the following LMI is feasible

$$\begin{aligned} EXE^T + Sym\{E^{\dagger}Z\} &> 0, \quad (26)\\ Sym\{XE^T\} + Sym\{E^{\perp}Y\} &> 0, \quad (27)\\ \begin{bmatrix} 0 & (XE^T + E^{\perp}Y)^T \\ XE^T + E^{\perp}Y & 0 \end{bmatrix} &+ \\ +Sym\left\{\begin{bmatrix} \widehat{A}G + \widehat{B}R \\ -G \end{bmatrix}\begin{bmatrix} I & I \end{bmatrix}\right\} &< 0. \quad (28)\end{aligned}$$

If this condition holds, the stabilizing control law matrix is given by

$$K = RG^{-1}. \quad (29)$$

Lemma 4: Suppose that the control law

$$u(h) = Kx(h), \quad (30)$$

is applied to the 1D discrete singular linear system described by (6). Then the resulting system is admissible if, and only if, there exists matrices X, Y, G such that the following LMI is feasible

$$\begin{bmatrix} -EXE^T & (E^{\perp}YE^{\ddagger})^T \\ E^{\perp}YE^{\ddagger} & X \end{bmatrix} + Sym\left\{\begin{bmatrix} \widehat{A}G+\widehat{B}R \\ -G \end{bmatrix}\begin{bmatrix} U^{-T} & \beta V^T \end{bmatrix}\right\} < 0, \quad (31)$$

for a given $\beta > 1$. If this condition holds, the stabilizing control law matrix is given by

$$K = RG^{-1}. \quad (32)$$

The following is the second major new result in this paper and provides an effective solution to the control law design problem of this section.

Theorem 3: A controlled differential linear repetitive process described by (24) is strongly practically stable if, and only if, the following LMIs are feasible

$$\begin{bmatrix} W_2 - G_2 - G_2^T & (*) \\ D_0G_2 + DR_2 & -W_2 \end{bmatrix} < 0, \quad (33)$$

$$AW_1 + BR_1 + W_1A^T + R_1^TB < 0, \quad (34)$$

$$\begin{bmatrix} X_{11}^1 & (*) \\ Z_{21} & Z_{22}^T + Z_{22} \end{bmatrix} > 0, \quad (35)$$

$$Sym\left\{\begin{matrix} X_{11}^1 & 0 \\ X_{21}^1 + Y_{21}^1 & Y_{22}^1 \end{matrix}\right\} > 0, \quad (36)$$

$$\begin{bmatrix} 0 & (*) & (*) & (*) \\ 0 & 0 & (*) & (*) \\ X_{11}^1 & 0 & 0 & (*) \\ X_{21}^1 + Y_{21}^1 & Y_{22}^1 & 0 & 0 \end{bmatrix} + Sym\left\{\begin{matrix} AW_1 + BR_1 & B_0G_2 + BR_2 & AW_1 + BR_1 & B_0G_2 + BR_2 \\ CW_1 + DR_1 & D_0G_2 - G_2 + DR_2 & CW_1 + DR_1 & D_0G_2 - G_2 + DR_2 \\ -W_1 & 0 & -W_1 & 0 \\ 0 & -G_2 & 0 & -G_2 \end{matrix}\right\} < 0, \quad (37)$$

$$\begin{bmatrix} 0 & 0 & (Y_{11}^2)^T & 0 \\ 0 & -X_{22}^2 & 0 & 0 \\ Y_{11}^2 & 0 & X_{11}^2 & (X_{21}^2)^T \\ 0 & 0 & X_{21}^2 & X_{22}^2 \end{bmatrix} + Sym\left\{\begin{matrix} AW_1 + BR_1 & B_0G_2 + BR_2 & AW_1\beta + BR_1\beta & B_0G_2\beta + BR_2\beta \\ CW_1 + DR_1 & D_0G_2 + DR_2 & CW_1\beta + DR_1\beta & D_0G_2\beta + DR_2\beta \\ -W_1 & 0 & -W_1\beta & 0 \\ 0 & -G_2 & 0 & -G_2\beta \end{matrix}\right\} < 0, \quad (38)$$

for given $\beta > 1$ and $W_1 > 0$, $W_2 > 0$, $X_{11}^1 = (X_{11}^1)^T$, $X_{22}^1 = (X_{22}^1)^T$, $X_{11}^2 = (X_{11}^2)^T$, $X_{22}^2 = (X_{22}^2)^T$, X_{21}^1, X_{21}^2, Y_{11}^2, Y_{22}^1, Z_{21}, Z_{22}, G_2, R_1, R_2.

If these LMIs hold, stabilizing control law matrices are given by

$$\begin{aligned} K_1 &= R_1W_1^{-1}, \\ K_2 &= R_2G_2^{-1}. \end{aligned} \quad (39)$$

Proof: Conditions (33) and (34) are well known for nonsingular 1D discrete and differential linear systems respectively. The proofs for conditions [e] and [f] here follow the arguments of Lemmas 3 and 4 applied to the controlled process. However, the same blocks W_1 and G_2 have to ensure that [g] and [h] hold. This can be achieved by making the following choices in block structure of the matrices R and G in the application of Lemma 3 and 4 respectively to the controlled process.

- $R = \begin{bmatrix} R_1 & R_2 \end{bmatrix}$ and $G = \begin{bmatrix} W_1 & 0 \\ 0 & G_2 \end{bmatrix}$ for [(g)].
- $R = \begin{bmatrix} R_2 & R_1 \end{bmatrix}$ and $G = \begin{bmatrix} 0 & W_1 \\ G_2 & 0 \end{bmatrix}$ for [(h)].

Routine mathematical manipulations yield the required result. ■

Note here that the above result gives an arbitrary element of the set of all possible stabilizing control laws. We can limit this set by assuming, for example, block-diagonal decision matrices. Also by imposing additional constraints and use GEVP optimization procedures we can attempt to limit the values of the control signals required, which has obvious (potential) benefits in terms of applying this theory to physical examples.

V. A Numerical Example

Consider the case when

$$\left[\begin{array}{c|c|c} A & B & B_0 \\ \hline C & D & D_0 \end{array}\right] = \left[\begin{array}{cc|c|c} -0.746 & 1.5623 & -0.151 & 0.-0.005 \\ 1.156 & 0.110 & -1.090 & -0.546 \\ \hline -0.596 & -0.233 & 0.483 & -2.974 \end{array}\right],$$

with the boundary conditions

$$\begin{aligned} x_{k+1}(0) &= [1,1]^T, \quad k \geq 0, \\ y_0(t) &= -5, \quad 0 \leq t \leq 10, \end{aligned}$$

This process is asymptotically unstable and hence is neither stable along the pass or strongly practically stable. The LMIs of Theorem 3 are feasible for given $\beta > 1$ and we can attempt to use this last parameter to assist with along the pass performance of the resulting controlled process. Here we consider the cases of $\beta = 1.01$ and $\beta = 25.01$ respectively for which the corresponding control law matrices are

$$\begin{aligned} K_1 &= \begin{bmatrix} 1.2257 & 0.49659 \end{bmatrix}, \quad K_2 = 4.4262, \\ K_1 &= \begin{bmatrix} 1.2333 & 0.48497 \end{bmatrix}, \quad K_2 = 5.9488, \end{aligned}$$

Pass profiles dynamics generated by the controlled process for each value of β is given in Figs. 1 and 2 respectively. In the case of Fig. 1, and to a lesser extent Fig. 2, closer inspection shows that the along the pass dynamics contain oscillations which are eventually damped out but this could take a relatively long number of passes. For example, Fig. 3 shows the along the pass dynamics when $k = 69$ where this feature is still present (but much less prominent than on earlier passes and also the final value is lower). Along the pass oscillations are much less prominent in Fig. 2 (where the value of β is much larger). Clearly further research is required on how to chose this tunable parameter to best effect.

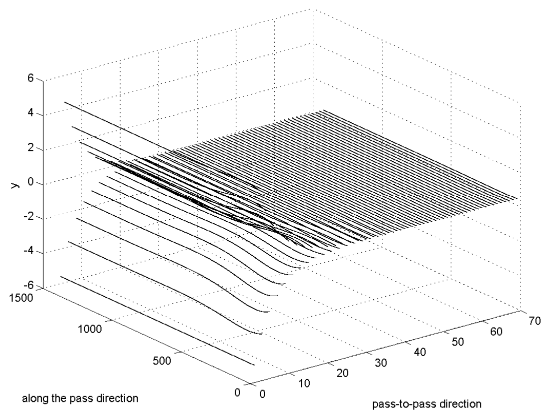

Fig. 1. Pass profile sequence generated by the controlled process with $\beta = 1.01$.

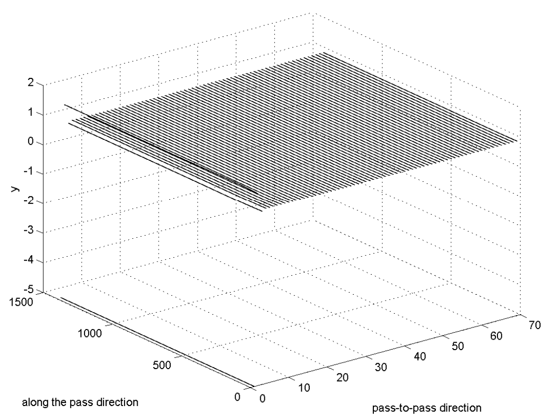

Fig. 2. Pass profile sequence generated by the controlled process with $\beta = 25.01$.

VI. Conclusions

Differential linear repetitive processes propagate information in two independent directions where for one of them the duration is finite and the dynamics are described by a matrix differential linear system. A stability theory for these processes exists which has clear physical motivation but more detailed studies have strongly suggested it is too strong for some cases. In this paper, we have developed strong practical stability as an alternative for such cases and characterized it in terms of necessary and sufficient LMI

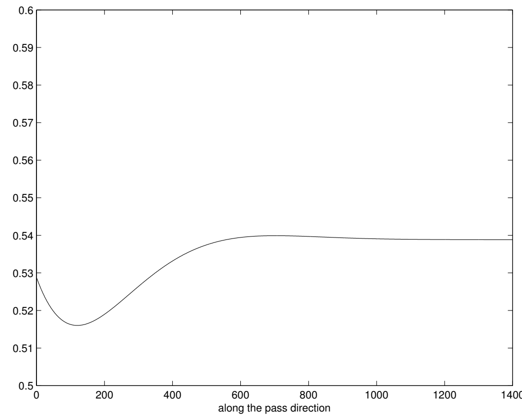

Fig. 3. Along the pass dynamics generated by the controlled process on pass $k = 69$ with $\beta = 1.01$.

conditions. This provides an immediate route to the design of stabilizing control laws. Moreover, a design parameter is available to tune the control law to meet performance specifications.

This last aspect is the subject of on-going work using differential repetitive process models obtained by modeling physical examples. Moreover, it should be possible to extend the analysis here to deal with cases where there is uncertainty associated with the process model and/or disturbances acting on both the current pass state and pass profile vectors, and hence the need for H_2 or H_∞ or mixed H_2/H_∞ analysis. Finally, the control law used here by no means exhausts the possibilities. For example, it should be possible to extend this control law to include terms related to the available information on pass $k+1$ and at point t. Options here include adding a term of the form $K_3 y_k(t-1)$ or $K_4 y_k(t+1)$, or replacing the current pass state vector by the current pass profile vector and hence avoid problems which could arise due to the fact that all elements in the current pass state vector may not be available for measurement.

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