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The Pontryagin Maximum Principle applied to Nonholonomic Mechanics

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***Abstract*— We introduce a method which allows one to recover the nonholonomic equations of motion of certain systems by instead finding a Hamiltonian via Pontryagin's Maximum Principle on an enlarged phase space, and then restricting the resulting canonical Hamilton equations to an appropriate invariant submanifold of the enlarged phase space. We illustrate the method through several examples, and discuss its relationship to the integrability of the system, and its quantization.**

I. INTRODUCTION

One of the hallmarks of continuous optimal control problems is that, under certain regularity assumptions, the optimal Hamiltonian can be found by applying the Pontryagin Maximum principle [4]. Moreover, in most cases of physical interest, the problem can be rephrased so as to be solved by using Lagrange multipliers [4]. Such a usage of the multiplier approach can also be applied to the mechanics of constrained physical systems with success in the case of holonomic (position dependent) constraints. On the other hand, many interesting mechanical systems are subject to additional velocity-dependent (i.e. nonholonomic) constraints. Typical engineering problems that involve nonholonomic constraints arise for example in robotics, where the wheels of a mobile robot are often required to roll without slipping, or where one is interested in guiding the motion of a cutting tool. In many cases the solution of these systems cannot be obtained analytically, and one can only analyze the systems by means of its qualitative and geometric features. Unfortunately, in the case of these nonholonomically constrained systems the Lagrange multiplier approach, also called the *vakonomic* approach by Arnold [2], generally leads to dynamics that do not reproduce the physical equations of motion (see [19] and references therein). Thus, the rich interplay between Pontryagin's Maximum Principle, the vakonomic approach, and the physical equations of motion of a constrained system breaks down when the constraints are nonholonomic. However as we showed in a previous paper [12], for certain systems and initial data the vakonomic approach and Lagrange-D'Alembert principle yield equivalent equations of motion.

The purpose of the current work is to close the gap left open in the relationship between Pontryagin's Maximum Principle and nonholonomically constrained systems. The main result is that for certain systems belonging to a class different from those considered in [12], we can recover the nonholonomic equations of motion by instead finding a Hamiltonian via Pontryagin's Maximum Principle on an enlarged phase space, and then restricting the resulting canonical Hamilton equations to an appropriate invariant submanifold of the enlarged phase space. In essence, the method we introduce here results in an unconstrained, variational system which when restricted to an appropriate submanifold reproduces the dynamics of the underlying nonholonomic system. The idea of enlarging the phase space of the system is not new, appearing rather naturally in the theory of singular Lagrangians with constraints and the resulting Dirac constraint theory [15], and also in some other works [10], [1]. However, unique to our method is the identification of the problem with one of optimal control, and an intriguing relationship to integrability and stability questions of the underlying nonholonomic system.

The paper is divided into four basic parts. We review the necessary background in nonholonomic mechanics in Section 1, introducing the dynamics on the constraint manifold $\mathcal{M}$. In Section 2 we introduce a class of systems for which our newly introduced method reproduces the nonholonomic dynamics. We then apply the main result of Section 2 to various examples in Section 3. In Section 4 we discuss the relationship of our method to questions of integrability of the momentum equations of the underlying nonholonomic system. Lastly, we outline some further avenues for research in the Conclusion.

II. NONHOLONOMIC MECHANICS

Nonholonomic mechanics takes place on a configuration space Q with a nonintegrable distribution $\mathcal{D}$ that describes the kinematic constraints of interest. These constraints are often given in terms of independent one-forms, whose vanishing in turn describes the distribution. Moreover, we typically choose a bundle and an Ehresmann connection A on that bundle such that $\mathcal{D}$ is given by the horizontal subbundle associated with A. One then derives the equations of motion using the Lagrange-d'Alembert principle, which takes into account the need for reaction forces that enforce the constraints throughout the motion of the system.

To this end, let $\{\omega^a\}$ be a set of m independent one-forms whose vanishing describes the constraints on the system. Locally, we can write

$$\omega^k(q) = ds^k + A^k_\alpha(r,s)dr^\alpha, \qquad k = 1,\ldots,m, \quad (1)$$

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where $q = (r, s) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$. In these coordinates we can choose the bundle to be that given by $(s, r) \mapsto r$, and the connection is, in this choice of bundle, defined by the constraints. Moreover, the distribution $\mathcal{D}$ is given by

$$\mathcal{D} = \text{span}\{\partial_{r^\alpha} - A^k_\alpha \partial_{s^k}\}. \tag{2}$$

One can then form the constrained Lagrangian

$$L_c(q, \dot{q}) = L(q, hor\ \dot{q}) \tag{3}$$

where $hor\ \dot{q}$ is the horizontal projection given in coordinates by $(\dot{r}^\alpha, \dot{s}^k) \mapsto (\dot{r}^\alpha, -A^k_\alpha(r, s)\dot{r}^\alpha)$. Given these data, one can write down the constrained Hamilton Equations on the constraint phase space $\mathcal{M} := \mathbb{F}L(\mathcal{D}) \subset T^*Q$ [4], [25]:

$$\begin{aligned} \dot{r}^\alpha &= \frac{\partial H_c}{\partial \tilde{p}_\alpha}, & (4)\\ \dot{s}^k &= -A^k_\beta \frac{\partial H_c}{\partial \tilde{p}_\beta}, & (5)\\ \dot{\tilde{p}}_\alpha &= -\frac{\partial H_c}{\partial r^\alpha} + A^k_\alpha \frac{\partial H_c}{\partial s^k} - K^\gamma_l \tilde{p}_\gamma B^l_{\alpha\beta} \frac{\partial H_c}{\partial \tilde{p}_\beta}, & (6) \end{aligned}$$

where K^γ_l is defined by $(g_{l\alpha} - g_{lk}A^k_\alpha)\partial H_c/\partial \tilde{p}_\alpha = K^\gamma_l \tilde{p}_\gamma$, with the g_{ij} the components of the kinetic energy metric of the unconstrained Lagrangian, $B^l_{\alpha\beta}$ is the curvature of the connection, $H_c = \tilde{p}_\alpha \dot{r}^\alpha - L_c$ is the constrained Hamiltonian, and $\tilde{p}_\alpha = p_\alpha - p_k A^k_\alpha$. These are a set of $2n-m$ equations on the submanifold $\mathcal{M}$ with induced coordinates $(r^\alpha, s^k, \tilde{p}_\alpha)$, and are manifestly non-Hamiltonian, a reflection of the fact that the presence of nonholonomic constraints induces additional forces that enforce those constraints.

III. The Maximum Principle and Nonholonomic Mechanics

In traditional investigations of the inverse problem of the calculus of variations [24], one is interested in whether or not a given system of differential equations is actually the Euler-Lagrange equations of some yet undetermined Lagrangian. Thus the objective is: given the equations of motion, attempt to construct an appropriate Lagrangian that reproduces them. For nonholonomic systems we know this is not possible, since these types of systems are not variational [4]. However, in [5] we explore a method that allows one to preserve much of the above reasoning.

In this paper, we will take a different approach: given the *solutions* to the equations of motion, attempt to construct a Hamiltonian H and a set $\mathcal{C}$ such that the canonical equations of H restricted to $\mathcal{C}$ reproduce the nonholonomic mechanics. Said differently, we are looking for a Hamiltonian vector field X and a subset $\mathcal{C}$ of its phase space such that $X|_\mathcal{C} = X_\mathcal{M}$, the nonholonomic vector field. Below we introduce a method which accomplishes this by identifying the problem with one of optimal control and applying the Pontryagin Maximum Principle. We then later apply the results to some well known nonholonomic systems considered in [1] and [4].

To begin, suppose that we have a nonholonomic system with configuration manifold Q. Consider the special class of systems whose equations of motion (4)-(6) have solutions given by

$$\begin{aligned} q_a &= u_a t + b_a, & a = 1, \ldots, J, && (7)\\ q_i &= u_i h_i(q_a) + b_i, & i = J+1, \ldots, m && (8)\\ q_\alpha &= c_\alpha(u_a, u_i) h_\alpha(q_a) + b_\alpha, & \alpha = 1, \ldots n-m && (9) \end{aligned}$$

where the u's and b's are constants dependent on the initial conditions. Differentiating the system (7)-(9) with respect to time yields:

$$\dot{q}_a = u_a, \quad \dot{q}_i = h_{ija} u_j u_a, \tag{10}$$

$$\dot{q}_\alpha = h_{\alpha\beta a} c_\beta u_a, \tag{11}$$

where we have adopted the notation $h_{ij} = h_i \delta_{ij}$, $h_{ija} = \partial h_{ij}/\partial q_a$ and employed Einstein's summation convention.

Now, consider the optimal control problem with cost function

$$G(q, u) = \frac{1}{2}(\delta_{ab} u_a u_b + h_{ija} u_i u_j u_a + h_{\alpha\beta a} u_\alpha u_\beta u_a), \tag{12}$$

subject to the constraints (10), and the following variation of (11):

$$\dot{q}_\alpha = h_{\alpha\beta a} u_\beta u_a. \tag{13}$$

Accordingly, we can form the Hamiltonian

$$\begin{aligned} H(q, p, u) = p_a u_a &+ p_i h_{ija} u_j u_a \\ &+ p_\alpha h_{\alpha\beta a} u_a u_\beta - G(q, u), \end{aligned} \tag{14}$$

and applying Pontryagin's Maximum Principle [4], the optimality conditions $\partial H/\partial u_a = 0$, $\partial H/\partial u_i = 0$, $\partial H/\partial u_\alpha = 0$ yield the optimal controls

$$\begin{aligned} u^*_a &= p_a + \frac{1}{2}(h_{ija} p_i p_j + h_{\alpha\beta a} p_\alpha p_\beta),\\ u^*_i &= p_i,\\ u^*_\alpha &= p_\alpha. \end{aligned} \tag{15}$$

Substituting this into (14) yields the Hamiltonian $H^*(q, p) = H(q, p, u^*)$, where

$$H^*(q, p) = \frac{1}{2}\sum_{a=1}^{J}\left[p_a + \frac{1}{2}\left(h_{ija} p_i p_j + h_{\alpha\beta a} p_\alpha p_\beta\right)\right]^2. \tag{16}$$

Now, by taking the resulting dynamics from the Hamilton equations corresponding to (16) and restricting to the submanifold of T^*Q defined by

$$p_\alpha = c_\alpha(u^*_a(q, p), p_i), \tag{17}$$

we recover the nonholonomic problem equations (10)-(11). The above derivation forms the basis for the main result.

Proposition 1: Suppose the solutions of a nonholonomic system with configuration manifold Q have the form (7)-(9). Then the canonical equations associated with the Hamiltonian (16), when restricted to the invariant submanifold $\mathcal{C} \subset T^*Q$ defined by (17), reproduce the nonholonomic solutions (7)-(9).

Proof: It suffices to show that the canonical equations when restricted to $\mathcal{C}$ reproduce (10)-(11). The canonical equations are given by:

$$\begin{aligned} \dot{q}_a &= p_a + \frac{1}{2}(h_{ija}p_ip_j + h_{\alpha\beta a}p_\alpha p_\beta), &(18)\\ \dot{q}_i &= h_{ija}p_j\dot{q}_a, \\ \dot{q}_\alpha &= h_{\alpha\beta a}p_\beta\dot{q}_a, \\ \dot{p}_a &= -\dot{q}_b\frac{\partial}{\partial q_b}\left[\frac{1}{2}(h_{ija}p_ip_j + h_{\alpha\beta a}p_\alpha p_\beta)\right], &(19)\\ \dot{p}_i &= \dot{p}_\alpha = 0. &(20) \end{aligned}$$

However, (19) can be re-written as

$$\frac{d}{dt}\left[p_a + \frac{1}{2}(h_{ija}p_ip_j + h_{\alpha\beta a}p_\alpha p_\beta)\right] = 0. \tag{21}$$

This verifies that the right-hand side of (18) is indeed constant, as are the p_i and p_α by (20). Lastly, imposing condition (17) in (18)-(20) gives back the kinematic equations (10)-(11), and by (20) we see at once that this submanifold is indeed invariant under the motion of the system. ■

IV. EXAMPLES

A. The Nonholonomic Free Particle

To illustrate the method, consider perhaps the simplest example: a nonholonomically constrained free particle (more details can be found in [4], [23]). In this example one has a free particle with Lagrangian and constraint given by

$$\begin{aligned} L = \tfrac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \\ \dot{z} + x\dot{y} = 0. \end{aligned} \tag{22}$$

We can form the constrained Lagrangian L_c by substituting the constraint into L, and proceed to compute the constrained equations (4)-(6), which take the form

$$\begin{aligned} \dot{x} &= \tilde{p}_x, & \dot{\tilde{p}}_x = 0, \\ \dot{y} &= \frac{\tilde{p}_y}{1+x^2}, & \dot{\tilde{p}}_y = \frac{x\tilde{p}_x\tilde{p}_y}{1+x^2}, \\ \dot{z} &= -\frac{x\tilde{p}_y}{1+x^2}. \end{aligned} \tag{23}$$

These equations have the solution

$$\begin{aligned} x &= u_1 t + x_0, &(24)\\ y &= u_2 \ln\left(x + \sqrt{1+x^2}\right) + y_0, &(25)\\ z &= -u_2\sqrt{1+x^2} + z_0, &(26) \end{aligned}$$

where $u_{1,2}$ and x_0, y_0, z_0 are constants dependent on the initial conditions. Now, differentiating (24)-(26) in time yields the system:

$$\begin{aligned} \dot{x} &= u_1 &(27)\\ \dot{y} &= \frac{u_1u_2}{\sqrt{1+x^2}}, &(28)\\ \dot{z} &= -\frac{xu_1u_2}{\sqrt{1+x^2}}. &(29) \end{aligned}$$

Considering the optimal control problem with cost function

$$L = \frac{1}{2}\left[u_1^2 + \frac{u_1}{\sqrt{1+x^2}}\left(u_2^2 + xu_3^2\right)\right] \tag{30}$$

and subject to (27)-(28) and

$$\dot{z} = \frac{xu_1u_3}{\sqrt{1+x^2}}. \tag{31}$$

we can apply the Maximum Principle to arrive at the Hamiltonian

$$H = \frac{1}{2}\left[p_x + \frac{1}{2\sqrt{1+x^2}}\left(p_y^2 + xp_z^2\right)\right]^2. \tag{32}$$

Moreover, an easy calculation shows that by restricting the dynamics from (32) to the submanifold

$$p_y + p_z = 0, \tag{33}$$

one indeed recovers the nonholonomic trajectories given by (24)-(26), taking into account the fact that since (32) is independent of y, z, then the restriction (33) is indeed constant in time.

B. The Knife Edge on the Plane

The knife edge on the plane corresponds physically to a blade moving in the xy plane at an angle ϕ to the x-axis (see [21]). The Lagrangian and constraints for the system are:

$$\begin{aligned} L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}J\dot{\phi}^2, \\ \dot{x}\sin\phi - \dot{y}\cos\phi = 0, \end{aligned} \tag{34}$$

where J is the moment of inertia of the blade about a vertical axis through the point of contact. Calculating the constrained equations yields:

$$\begin{aligned} \dot{\phi} &= \frac{1}{J}\tilde{p}_\phi, & \dot{\tilde{p}}_\phi = 0, \\ \dot{x} &= \frac{1}{m}\tilde{p}_x\cos^2\phi, & \dot{\tilde{p}}_x = \frac{1}{J}\tilde{p}_x\tilde{p}_\phi\tan\phi, \\ \dot{y} &= \frac{1}{m}\tan\phi\cos^2\phi\,\tilde{p}_x. \end{aligned} \tag{35}$$

Now, the solution to these equations is given by [21]

$$\begin{aligned} \phi &= u_1 t + \phi_0, \\ x &= u_2\sin\phi + x_0, \\ y &= -u_2\cos\phi + y_0, \end{aligned} \tag{36}$$

where $u_{1,2}$ and $\mathbf{q_0}$ depend on the initial conditions, and where we have absorbed m, J into the constants $u_{1,2}$. Proposition 1 again applies and yields the Hamiltonian and constraint:

$$H = \tfrac{1}{2}\left[p_\phi + \tfrac{1}{2}\left(p_x^2\cos\phi - p_y^2\sin\phi\right)\right]^2, \qquad p_x + p_y = 0. \tag{37}$$

C. The Vertical Rolling Disk

The vertical rolling disk is a homogeneous disk rolling without slipping on a horizontal plane (see [4]). The system has the Lagrangian and constraints given by

$$\begin{aligned} L &= \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) + \tfrac{1}{2}I\dot{\theta}^2 + \tfrac{1}{2}J\dot{\varphi}^2, \\ \dot{x} &= R(\cos\varphi)\dot{\theta}, \\ \dot{y} &= R(\sin\varphi)\dot{\theta}, \end{aligned} \tag{38}$$

and the constrained equations of motion are:

$$\begin{aligned} \dot{\theta} &= \frac{\tilde{p}_\theta}{\tilde{I}}, & \dot{\tilde{p}}_\theta &= 0 \\ \dot{\varphi} &= \frac{\tilde{p}_\varphi}{J}, & \dot{\tilde{p}}_\varphi &= 0 \\ \dot{x} &= \frac{R}{\tilde{I}}\cos\varphi\tilde{p}_\theta, \\ \dot{y} &= \frac{R}{\tilde{I}}\sin\varphi\tilde{p}_\theta, \end{aligned} \tag{39}$$

where $\tilde{I} = I + mR^2$. These equations have the immediate solutions

$$\begin{aligned} \theta &= u_1 t + \theta_0, \\ \varphi &= u_2 t + \varphi_0, \\ x &= \left(\frac{u_1}{u_2}\right)\sin\varphi + x_0, \\ y &= \left(-\frac{u_1}{u_2}\right)\cos\varphi + y_0, \end{aligned} \tag{40}$$

where again we have absorbed the system's parameters into $u_{1,2}$. The system has the form of Proposition 1, and in this case we have $u_\alpha = u_{3,4}$, where $u_3 = u_1/u_2$, and $u_4 = -u_1/u_2$, or equivalently, $u_3 + u_4 = 0$. Now, by applying Proposition 1 the optimal controls (15) turn out to be:

$$u_1^* = p_\theta, \quad u_3^* = p_x, \quad u_4^* = p_y, \tag{41}$$

$$u_2^* = p_\varphi + \tfrac{1}{2}\left(p_x^2\cos\varphi - p_y^2\sin\varphi\right). \tag{42}$$

Thus, the resulting H and $\mathcal{C}$ are:

$$\begin{aligned} H &= \tfrac{1}{2}\left[p_\theta^2 + \left\{p_\varphi + \tfrac{1}{2}\left(p_x^2\cos\varphi - p_y^2\sin\varphi\right)\right\}^2\right], \\ & p_x + p_y = 0, \\ & p_x\left\{p_\varphi + \tfrac{1}{2}\left(p_x^2\cos\varphi - p_y^2\sin\varphi\right)\right\} - p_\theta = 0. \end{aligned} \tag{43}$$

V. Related Research Directions and Conclusions

Proposition 1 represents a new link between the fields of optimal control, where equations are derived from a Hamiltonian, and nonholonomic mechanics, where equations are derived from a Hamiltonian and constraint reaction forces. The Proposition shows that by combining elements of both derivations, for certain systems one can formulate the mechanics in a form analogous to the treatment of constraints arising from singular Lagrangians that leads to the Dirac theory of constraints [15]. Moreover, two important bi-products of our method have emerged which we are currently researching: applications to the quantization of nonholonomic systems [13], and applications to integrability [14].

A. Quantization of Nonholonomic Systems

Dirac's theory allows for the quantization of constrained systems wherein the constraints typically arise from a singular Lagrangian (see [18] and references therein), and central to the method is the modification of the Hamiltonian to incorporate so-called first and second class constraints. The method proposed in this paper provides an analogue to Dirac's theory and allows for the quantization of certain nonholonomic systems by similarly modifying the usual Hamiltonian.

Although there have already been some attempts to quantize nonholonomic systems [6], [8], [1], [17], [22], the results have been mixed, mainly due to the inherent difficulties arising in the quantization procedure. However, taking the knife edge on the plane as an example, in view of (37) the quantum Hamiltonian $\hat{H}$ has the form

$$\hat{H} = -\frac{\hbar^2}{2}\left[\frac{\partial}{\partial\phi} - i\frac{\hbar}{2}\left(\cos\phi\frac{\partial^2}{\partial x^2} - \sin\phi\frac{\partial^2}{\partial y^2}\right)\right]^2, \tag{44}$$

which is a Hermitian operator. We also see at once that since $\hat{H}$ is independent of x and y, it commutes with $\hat{p}_x$ and $\hat{p}_y$, making the task of finding the eigenfunctions easier [13]. Moreover, one can show [13] that ϕ is quantized due to its periodicity, which makes finding the eigenenergies easier.

Treating the constraints is the difficult part. There have in the literature been essentially two different ways to impose the quantum version of (37) in this case: *strongly* and *weakly*. One may require that the quantum version of (37) hold *strongly* by restricting the set of possible eigenstates of (44) to those which satisfy the quantum version of (37). On the other hand, one may only require that the eigenstates satisfy the quantum version of (44) on average, a weaker condition but arguably a more physically relevant viewpoint also advocated in [17], [1]. Details will be given in [13].

B. Nonholonomic Phase Space Transformations

Although recent authors [16], [3] have been developing Hamilton-Jacobi theory for nonholonomic systems with great success, many fundamental questions remain. For example, it is well known that the flow of a general nonholonomic system does not consist of canonical transformations [4], and as such the task of transforming coordinates in phase space to achieve a certain transformed Hamiltonian, as is usually done to arrive at the Hamilton-Jacobi equation in the unconstrained case, is made more difficult. However, common to all the examples in Section 3 is the fact that the momentum equations are integrable. In the most obvious case, that of the momentum equations of the vertical disk (39), the momenta are in fact constant in time.

Although we have yet to focus on this point, let us recall the origin of the invariant submanifold $\mathcal{C}$, equations (17). Within the context of the assumed solutions (4)-(6), the substitution (17) amounts to the claim that the newly defined momenta p_i are constant in time, as are then the p_α, being defined as they are in (17). Thus emerges another avenue for constructing the Hamiltonian (16): if one can find a coordinate transformation taking $\tilde{p} \to p$ such that $\dot{p}_I = 0\,\forall I$, then the constrained equations of motion (4)-(6) will be transformed into a *kinematic set*, that is, a set of n equations for which the conjugate momenta are all constant in time. In these cases the problem is then reduced to a set of equations of the form (10)-(11) [14]. From there we can essentially apply the same procedure leading up to Proposition 1 to obtain H and $\mathcal{M}$, with minor modifications.

The advantage of such an alternative is that one would no longer require the solutions to the nonholonomic system to find H and $\mathcal{C}$. One case in which this direction seems to work rather well is when the momentum equations are integrable [4], [27]. To illustrate this, consider again the nonholonomic free particle with constrained equations of motion (23). One can easily show that the momentum equations are integrable, with $\tilde{p}_x = u_1$, and $\tilde{p}_y = k\sqrt{1+x^2}$, where u_1, k are constants. Consider now the transformation $\tilde{p} \to p$ given by:

$$\begin{aligned} \tilde{p}_x &= u_1, \\ \tilde{p}_y &= u_1 p_y \sqrt{1+x^2}. \end{aligned} \tag{45}$$

Under this transformation, equations (23) become:

$$\dot{x} = u_1, \qquad \dot{u}_1 = 0, \tag{46}$$

$$\dot{y} = \frac{u_1 p_y}{\sqrt{1+x^2}}, \qquad \dot{p}_y = 0, \tag{47}$$

$$\dot{z} = -\frac{u_1 x p_y}{\sqrt{1+x^2}}. \tag{48}$$

From here, we can define $p_z = -p_y$ as in (17) and repeat the earlier procedure from Proposition 1 to arrive at the Hamiltonian and constraint (32)-(33) as before.

Lastly, we expect the existence of such transformations to relate to the existence of an invariant measure for the constrained dynamics [14]. One can show [7] that the nonholonomic free particle's constrained dynamics (23) have an invariant measure with density $k = \left(1+x^2\right)^{1/2}$, which is the factor used in (45) to effect the phase space transformation. One can also show [14] that there exists a similar relationship between the constrained system's invariant measure density and the corresponding phase space transformation.

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References

[1] Abud, M., et al, The Quantization of Classical Non-holonomic Systems, *Revista Brasileira de Fisica*, 13 (1983), no. 2.

[2] Arnold, V.I., et al, *Dynamical Systems III*, Springer-Verlag (1987).

[3] Bates, L., R. Cushman. What is a completely integrable nonholonomic dynamical system? *Rep. Math. Phys.*, 44 (1999), no. 1-2, 29-35.

[4] Bloch, A.M., *Nonholonomic Mechanics and Control*, Springer NY (2003).

[5] Bloch, A.M., Fernandez, O.E., and Mestdag, T., Hamiltonization of nonholonomic systems and the inverse problem of the calculus of variations, preprint (2008).

[6] Bloch, A.M., Rojo, A., Quantization of a nonholonomic system, *preprint*.

[7] Cortes, J., *Geometric, control and numerical aspects of nonholonomic systems*, Springer (2002).

[8] Eden, R.J., *Proc. Royal Soc. London*, 205 (1951), no. 1083, 583-595.

[9] Ehlers, K., et al, Nonholonomic systems via moving frames: Cartan equivalence and Chaplygin Hamiltonization, *arXiv math-ph/0408005* (2004).

[10] Espindola. O., et al, Hamiltonisation of classical non-holonomic systems, *J. Phys. A: Math. Gen.*, 20 (1987), 1713-1717.

[11] Federov, Y.N., Jovanovic, B., Quasi-Chaplygin Systems and Nonholonomic Rigid Body Dynamics, *Lett. Math. Phys.*, 76 (2006), no. 2-3, 215-230.

[12] Fernandez, O.E., Bloch, A.M., Equivalence of the Dynamics of Nonholonomic and Variational Nonholonomic Systems for certain Initial Data, *to appear in J. Phys. A: Math. Theor., 41 (2008)*.

[13] Fernandez, O.E., Bloch, A.M., Uribe, A., Quantum Mechanics of Nonholonomic Systems, *in preparation*.

[14] Fernandez, O.E., Bloch, A.M., Uribe., A., Mestdag, T., Nonholonomic Phase Space Transformations, *in preparation*.

[15] Henneaux, M., Teitelboim, C., *Quantization of Gauge Systems*, Princeton Univ. Press (1994).

[16] Iglesias-Ponte, D., et al, Towards a Hamilton-Jacobi theory for nonholonomic mechanical systems, *J. Phys. A: Math. Theor.*, 41 (2008), 015205.

[17] Kaplan, K., et al, Quantizing constrained systems, *Phys. Rev. A*, 56 (1997), no. 4, 2592-2599.

[18] Klauder, J., *Beyond Conventional Quantization*, Cam. Univ. Press (1999).

[19] Lewis, A., Murray, R., Variational principles for constrained systems: theory and experiment, *The Intl. J. Nonlinear Mechanics*, 30(6) (1995), 793-815.

[20] Marsden, J.E., Scheurle, J. The reduced Euler-Lagrange equations, *Fields Inst. Com.*, 1 (1993), 139-164.

[21] Neimark, J.I., Fufaev, N.A., *Dynamics of Nonholonomic Systems*, Amer. Math. Soc. (1972).

[22] Pitanga, P., Quantization of a non-holonomic system with symmetry, *Il Nuovo Cimento B*, 109 (1994), no. 6, 583-594.

[23] Rosenberg, R.M., *Analytical Dynamics of Discrete Systems*, Plenum Press, NY (1977).

[24] Santilli, R. M., *Foundations of Theoretical Mechanics: The Inverse Problem in Newtonian Mechanics*, Spinger-Verlag (1978).

[25] van der Schaft, A.J., Maschke, B.M., On the Hamiltonian formulation of nonholonomic mechanical systems, *Rep. on Math. Phys.* 34, 225-233.

[26] Veselov, A.V., Veselova, L.E., Integrable nonholonomic systems on Lie groups, *Math. Notes*, 44 (1988), no. 5, 810-819.

[27] Zenkov, D. Bloch, A.M. and Marsden, J.E. The energy-momentum method fo the stability of nonholonomic systems, *Dyn. Stab. Syst*, 13 (1998), 123-166.