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Lossless scalar functions: boundary interpolation, Schur algorithm and Ober's canonical form.

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***Abstract*— In [1] a balanced canonical form for continuous-time lossless systems was presented. This form has a tridiagonal dynamical matrix A and the useful property that the corresponding controllability matrix K is upper triangular. In [2], this structure is also derived from a LC ladder. In this paper, a connection is established between Ober's canonical form and a Schur algorithm built from angular derivative interpolation conditions. It provides a new interpretation of the parameters in Ober's form as interpolation values at infinity and a recursive construction of the balanced realization.**

I. Introduction

There is a rich literature on the construction of canonical forms for various classes of linear systems. It may be noticed that for balanced state-space canonical forms, the class of *lossless systems* lies at the heart of many such constructions ([3], [4]). Lossless systems are important in their own right too because of their usefulness for several other purposes: (i) to study H_2-model approximation problems; (ii) to perform system identification by the method of separable least squares (see [5]) using output normal forms; (iii) to generate polyphase representations of orthogonal filter banks when implementing orthogonal wavelet decomposition schemes.

In the *continuous-time* case, a balanced canonical form for lossless systems was first presented in the work of Ober ([6], [1]). This form has a tridiagonal dynamical matrix A and the useful property that the corresponding controllability matrix K is upper triangular. To deal with the *discrete-time* case one may apply the bilinear transform, which preserves balancedness but destroys upper triangularity of K. In [7] another balanced canonical form for discrete-time lossless systems was developed directly starting from upper triangularity of K. This canonical form is conveniently parameterized with Schur parameters. In [8] this result is generalized to the multivariable case by exploiting the connection with the tangential Schur algorithm. In the discrete-time setting, the construction of such balanced canonical forms is achieved in a recursive way with unitary matrix multiplications. Each step of the recursion involves an interpolation condition at a point located either inside or outside the unit circle.

To transfer these discrete-time results back to continuous-time, the bilinear transform can again be used. As a result, unitary matrix multiplications are replaced by more general linear fractional transformations. However, it can be established that the original balanced canonical form of Ober is *not* recovered in this way.

In the present paper we note that the canonical form of Ober involves interpolation conditions *at infinity*. Using the results of [9] on boundary interpolation problems we give a parametrization of continuous-time lossless functions. By making appropriate choices for the interpolation conditions, we show how the balanced state-space canonical form of Ober can now be recovered.

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II. Preliminaries.

We denote by $\Pi^+ := \{s \in \mathbb{C};\ \mathrm{Re}\ s > 0\}$ and $\Pi^- := \{s \in \mathbb{C};\ \mathrm{Re}\ s < 0\}$ the right and left half-planes, by $\mathbb{D}$ the open unit disk and by $\mathbb{T}$ the unit circle. For any matrix function $R(s)$, we define the matrix functions $R^*(s)$ and $R^\sharp(s)$ by

$$R^*(s) := R(\bar{s})^*, \quad \text{and} \quad R^\sharp(s) := R^*(-s). \tag{1}$$

Note that if s lies on the imaginary axis, then $R^\sharp(s) = R(s)^*$.

Let

$$J = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}.$$

A $2p \times 2p$ rational matrix function $\Theta(s)$ is called J-lossless if, at every point of analyticity s of Θ it satisfies

$$\begin{aligned} \Theta(s)^* J \Theta(s) &\leq J, \quad \mathrm{Re}\ s > 0, & (2)\\ \Theta(s)^* J \Theta(s) &= J, \quad \mathrm{Re}\ s = 0, & (3)\\ \Theta(s)^* J \Theta(s) &\geq J, \quad \mathrm{Re}\ s < 0. & (4) \end{aligned}$$

It can be shown that this definition contains redundancy in the sense that either one of these three defining properties (2)-(4) is implied by the other two (cf., e.g., [10], [11]).

A $p \times p$ rational matrix function $F(s)$ is called lossless, if and only if

$$F(s)F(s)^* \leq I_p, \quad \mathrm{Re}\ s > 0, \tag{5}$$

with equality on the imaginary axis.

By analytic continuation, the identity on the imaginary axis extends almost everywhere, so that any rational lossless function $F(s)$ is invertible, its inverse being $(-I)$-lossless and given by $F(s)^{-1} = F^\sharp(s)$.

Along with a $2p \times 2p$ rational function $\Theta(s)$ block-partitioned as follows

$$\Theta(s) = \begin{bmatrix} \Theta_{11}(s) & \Theta_{12}(s) \\ \Theta_{21}(s) & \Theta_{22}(s) \end{bmatrix}, \tag{6}$$

with each block of size $p \times p$, we associate the linear fractional transformation T_Θ which acts on $p \times p$ rational functions $F(s)$:

$$T_\Theta(F) = [\Theta_{11} F + \Theta_{12}][\Theta_{21} F + \Theta_{22}]^{-1}. \qquad (7)$$

Linear fractional transformations occur extensively in representation formulas for the solution of various interpolation problems ([9]). If $\Theta(s)$ is a J-lossless matrix function, then the map T_Θ sends every lossless function onto a lossless function.

If (A, B, C, D) is a (minimal) *balanced realization* of a lossless function $G(s)$, then it holds that

$$D \text{ is unitary (of size } p \times p), \qquad (8)$$
$$C = -DB^* \text{ (of size } p \times n), \qquad (9)$$
$$A + A^* = -BB^* \ . \qquad (10)$$

Conversely, if (A, B, C, D) is a matrix quadruple which satisfies these three conditions, then the associated transfer function $G(s) = D+C(sI_n-A)^{-1}B$ is lossless of McMillan degree $\leq n$. In that case, the McMillan degree is equal to n if and only if A is (continuous-time) asymptotically stable (i.e., al its eigenvalues are in the open left half plane), in which case (A, B, C, D) is minimal and balanced.

In the sequel, since we deal with scalar functions, only the case $p = 1$ is needed. The natural framework for interpolation theory is that of complex functions. However, systems are often real-valued and their transfer function real, that is, they satisfy the relation $\overline{F(s)} = F(\bar{s})$. In this paper, we are mainly interested in this case.

III. The canonical form of Ober

A balanced canonical form for real lossless systems was first presented in the work of Ober ([6], [1]): any lossless function $G(s)$ of McMillan degree n has a unique balanced realization $G(s) = D + C(sI_n - A)^{-1}B$ parametrized by $s_1 = \pm 1$, and $b_1, \alpha_1, \alpha_2, \dots, \alpha_{n-1}$ positive real numbers such that

$$\begin{array}{rcl} B^T & = & \begin{bmatrix} b_1 & 0 & \cdots & 0 \end{bmatrix}, \\ C^T & = & s_1 B, \\ A & = & \begin{bmatrix} -\frac{b_1^2}{2} & \alpha_1 & & & & \\ -\alpha_1 & 0 & \alpha_2 & & 0 & \\ & -\alpha_2 & 0 & & & \\ & & & \ddots & & \\ & 0 & & & 0 & \alpha_{n-1} \\ & & & & -\alpha_{n-1} & 0 \end{bmatrix}, \\ D & = & -s_1. \end{array} \qquad (11)$$

The connection between the matrix $G(s)$ and the parameters in the canonical form is better understood with the help of the following family of polynomials (see [1]): for $k = 0, \dots n-2$, let

$$\Delta_{n-k}(s) = \det(sI_{n-k} - A_{n-k}), \qquad (12)$$

$$A_{n-k} = \begin{bmatrix} 0 & \alpha_{k+1} & & 0 & \\ -\alpha_{k+1} & 0 & \alpha_{k+2} & & \\ & -\alpha_{k+2} & & & \\ & & \ddots & \ddots & \\ & 0 & & 0 & \alpha_{n-1} \\ & & & -\alpha_{n-1} & 0 \end{bmatrix}.$$

These polynomials satisfy the recursion

$$\Delta_{n-k}(s) = s\Delta_{n-k-1}(s) + \alpha_{k+1}^2 \Delta_{n-k-2}(s), \qquad (13)$$

Δ_j is monic and is an even (resp. odd) function for j even (resp. odd). We have that

$$G(s) = -s_1 \frac{\Delta_n(s) - \frac{b_1^2}{2}\Delta_{n-1}(s)}{\Delta_n(s) + \frac{b_1^2}{2}\Delta_{n-1}(s)}. \qquad (14)$$

From a coprime fraction representation of $G(s)$, with monic denominator, the parameters $b_1, \alpha_1, \dots, \alpha_{n-1}$ can be easily recovered by separating the odd and even part of the numerator and using the recursion (13).

If $g(s) = G(1/s)$, it is easily verified that the parameter b_1 satisfies the interpolation condition

$$g'(0) = s_1 b_1^2.$$

IV. Boundary interpolation and parametrization of lossless functions

In this section we adress the complex case again. Interpolation problems in which an interpolation condition is prescribed at the boundary of the stability domain have been studied in [9, chap.21].

Let $f(s)$ be a lossless function; let $\sigma \in i\mathbb{R}$, and consider the boundary interpolation condition $f(\sigma) = \xi$ with $|\xi| = 1$. For the interpolation problem to be well-posed, some extra condition must be prescribed on the angular derivative $f'(\sigma)$ of f at σ. Since $|f(\sigma)| = 1$, it can be proved that the quantity

$$\rho = -\overline{f(\sigma)} f'(\sigma)$$

is a positive real number. This suggests the following angular derivative interpolation problem:

(ADI) Given $\sigma \in i\mathbb{R}$, $\xi \in \mathbb{T}$ and $\rho > 0$, find all the lossless functions f such that

$$\left\{ \begin{array}{rcl} f(\sigma) & = & \xi \\ f'(\sigma) & = & -\xi\rho \end{array} \right. \qquad (15)$$

With these interpolation data, we associate the J-lossless function

$$\theta_{\sigma,\rho,\xi}(s) = J - \frac{1}{(s-\sigma)\rho} \begin{bmatrix} \xi \\ 1 \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix}^*. \qquad (16)$$

Any J-lossless function with a pole at σ can be written in this form up to a right constant J-unitary factor H. The matrix $\theta_{\sigma,\rho,\xi}$ satisfies $\theta_{\sigma,\rho,\xi}(\infty) = J$. The reason for this particular choice will become clear in section V.

Theorem 1: The problem (ADI) always has a solution. The set of all solutions is given by

$$f = T_{\theta_{\sigma,\rho,\xi}}(g), \qquad (17)$$

where $\theta_{\sigma,\rho,\xi}(s)$ is given by (16) and $g(s)$ is an arbitrary lossless function such that $g(\sigma) \neq -\xi$. Moreover, we have that

$$\deg\ f = \deg\ g + 1.$$

Proof. This result is a particular case of Theorem 21.1.5. in [9], which more generally describes all the Schur functions which satisfy a set of interpolation conditions of the form (15). Here the choice of the J-lossless matrix $\theta_{\sigma,\rho,\xi}$ is also different. We give the main lines of the proof. The linear fractional transformation (17) can be computed as

$$f(s) = \frac{\xi + (1 - \rho(s - \sigma))g(s)}{\bar{\xi}g(s) + 1 + \rho(s - \sigma)}.$$

Assume that $g(s)$ is a lossless function and that it satisfies $g(\sigma) \neq -\xi$. Then $\theta_{\sigma,\rho,\xi}$ being J-lossless, $f(s)$ is lossless too and we have

$$f(\sigma) = \xi\frac{\xi + g(\sigma)}{\xi + g(\sigma)}, \quad f'(\sigma) = -\xi\rho\frac{(\xi + g(\sigma))^2}{(\xi + g(\sigma))^2},$$

so that, with the assumption $g(\sigma) \neq -\xi$, the interpolation conditions (15) are satisfied. The assumption $g(\sigma) \neq -\xi$ also ensures that $\deg f = \deg g + 1$ since then, no pole-zero cancellation can occur.

Conversely assume that $f(s)$ is a lossless function of McMillan degree n satisfying (15), and define

$$g(s) = \frac{\xi - (1 + \rho(s - \sigma))f(s)}{\bar{\xi}f(s) - 1 + \rho(s - \sigma)}.$$

The function g is again lossless and has degree at most $n+1$. However, the interpolations conditions ensure that a double pole-zero cancellation occurs, so that the degree of g is in fact $n - 1$. Applying l'Hôpital's rule, we can compute the value of g at σ upon differentiating twice the numerator and denominator expressions, and then evaluating at σ. We get

$$g(\sigma) = -\xi - 2\rho\xi\frac{f'(\sigma)}{f''(\sigma)},$$

which proves that $g(\sigma) \neq -\xi$. □

Let $f = f_n$ be a lossless function of McMillan degree n, then a *Schur algorithm* can be performed using angular derivative interpolation conditions. It consists of the construction of a sequence of lossless functions of decreasing degree $f = f_n, f_{n-1}, \ldots, f_j \ldots f_0$. Assume that f_j has been constructed and let

$$f_j(\sigma_j) = \xi_j \quad (18)$$
$$f'_j(\sigma_j) = -\xi_j\rho_j, \quad (19)$$

then the function f_{j-1} is given by

$$f_{j-1} = T^{-1}_{\theta_{\sigma_j,\rho_j,\xi_j}}(f_j),$$

where $\theta_{\sigma_j,\rho_j,\xi_j}$ is given by (16). It has degree $j - 1$ and satisfies the condition $f_{j-1}(\sigma_j) \neq -\xi_j$. The function f_0 is a constant ξ_0 of modulus 1.

The interpolation points σ_j being fixed, the interpolation values

$$(\xi_0, \xi_1, \ldots, \xi_n, \rho_1, \rho_2, \rho_n)$$

can be taken as parameters as in the classical Schur algorithm. We have the right number of real parameters: $2n + 1$ in the complex case and n in the real case. Using the reverse Schur algorithm, we define a map

$$(\xi_0, \xi_1, \ldots, \xi_n, \rho_1, \rho_2, \rho_n) \to f,$$

where f is a lossless function of degree n, which is clearly one-to-one on its domain of definition: the parameters $(\xi_0, \xi_1, \ldots, \xi_n, \rho_1, \rho_2, \rho_n)$ such that, for $j = 1, \ldots, n$, $\rho_j > 0$ and ξ_j is a unit complex number such that $f_{j-1}(\sigma_j) \neq -\xi_j$, where the f_j are built recursively, so that f_j only depends on the first $2j + 1$ parameters $(\xi_0, \xi_1, \ldots, \xi_j, \rho_1, \rho_2, \rho_j)$. When all interpolation points are equal, say $\sigma_j = \sigma$, $j = 1 \ldots, n$, the condition $f_{j-1}(\sigma_j) \neq -\xi_j$ simplifies to

$$\xi_j \neq -\xi_{j-1}, \quad = 1 \ldots, n, \quad (20)$$

and the domain of definition becomes simpler.

V. Connection with Ober's canonical form

We shall study the real case with all the interpolation points at zero $\sigma_j = 0$, $|\xi_j| = 1$, $j = 1, \ldots n$. Condition (20) implies that $\xi_0 = \xi_1 = \ldots = \xi_n = \pm 1$, so that the parameters are now $\xi_0 = \pm 1$ and the ρ_j, $\rho_j > 0$, $j = 1, \ldots n$.

Remark. The choice of the matrix $\theta_{\sigma,\rho,\xi}$ in (16) satisfying $\theta_{\sigma,\rho,\xi}(\infty) = J$ allows to take all the interpolations values ξ_j equal. The more usual choice $\theta_{\sigma,\rho,\xi}(\infty) = I$ will correspond to a sequence $\xi_0, \xi_1, \ldots, \xi_n$ alternating between the values 1 and -1.

To handle Ober's form which requires interpolation conditions at infinity, we shall consider the functions

$$F_j(s) = f_j(1/s), \quad j = 1, \ldots, n.$$

From $F_0 = \xi_0$, for $j = 1, \ldots, n$, the Schur recursion gives

$$F_j(s) = \frac{(s - \rho_j)F_{j-1}(s) + \xi_0\, s}{(s + \rho_j) + \xi_0\, sF_{j-1}(s)},$$

that is $F_j(s) = T_{\theta_j(1/s)}(F_{j-1}(s))$, where θ_j is the J-lossless matrix

$$\theta_j(s) = J - \frac{1}{\rho_j s} \begin{bmatrix} \xi_0 \\ 1 \end{bmatrix} \begin{bmatrix} \xi_0 \\ 1 \end{bmatrix}^T. \quad (21)$$

Putting $a_j = \rho_j/2$ and $\xi_0 = 1$, we get:

$$\begin{aligned} F_0(s) &= 1 \\ F_1(s) &= \frac{s - a_1}{s + a_1} \\ F_2(s) &= \frac{s^2 - a_2 s + a_1 a_2}{s^2 + a_2 s + a_1 a_2} \\ F_3(s) &= \frac{s^3 - a_3 s^2 + (a_1 a_2 + a_2 a_3)s - a_1 a_2 a_3}{s^3 + a_3 s^2 + (a_1 a_2 + a_2 a_3)s + a_1 a_2 a_3} \end{aligned}$$

If $\xi_0 = -1$, then each F_j is just multiplied by -1.

Theorem 2: Let $\rho_1, \rho_2, \ldots, \rho_n$ be strictly positive real numbers; let $F_0 = \xi_0$ and for $j = 1, \ldots, n$

$$F_j(s) = T_{\theta_j(1/s)}(F_{j-1}(s)),$$

where the θ_j are given by (21). Then, $F_n = F$ is a lossless function of McMillan degree n, and the parameters $b_1, \alpha_1, \ldots, \alpha_n$ in the canonical form (11) of F are related to the $a_j = \rho_j/2$, $j = 1, \ldots, n$ as follows:

$$\begin{array}{rcl} s_1 & = & -\xi_0 \\ b_1^2 & = & \rho_n \\ \alpha_1^2 & = & a_n a_{n-1} \\ \alpha_2^2 & = & a_{n-1} a_{n-2} \\ & \vdots & \\ \alpha_{n-1}^2 & = & a_2 a_1 \end{array} \tag{22}$$

Moreover,

$$F_j(s) = \xi_0 \, \frac{\Delta_j(s) - a_j \Delta_{j-1}(s)}{\Delta_j(s) + a_j \Delta_{j-1}(s)},$$

where Δ_j is the polynomial defined by (12).

Proof. For simplicity, we assume in the proof that $\xi_0 = 1$. We first prove that for $j = 1, \ldots, n$

$$F_j(s) = \frac{P_j(s) - a_j Q_j(s)}{P_j(s) + a_j Q_j(s)},$$

where P_j and Q_j are polynomials, P_j is monic and has same parity as j while Q_j has the opposite one. It holds for $j = 1$. We have that $P_1(s) = s$ and $Q_1(s) = 1$. Assume it holds for j, then

$$\begin{array}{rcl} F_{j+1} & = & \dfrac{(s - \rho_{j+1})(P_j - a_j Q_j) + s(P_j + a_j Q_j)}{(s + \rho_{j+1})(P_j + a_j Q_j) + s(P_j - a_j Q_j)} \\ & = & \dfrac{sP_j - a_{j+1} P_j + a_j a_{j+1} Q_j}{sP_j + a_{j+1} P_j + a_j a_{j+1} Q_j}, \end{array}$$

so that the relation is also true at $j + 1$ with

$$\left\{ \begin{array}{rcl} P_{j+1} & = & sP_j + a_j a_{j+1} Q_j \\ Q_{j+1} & = & P_j. \end{array} \right.$$

Now it is clear from (14) that $\Delta_n = P_n$, $\Delta_{n-1} = Q_n = P_{n-1}$ and $\rho_n = -b_1^2$. Using (13), we have that

$$\alpha_{j+1}^2 \Delta_{n-j-2} = \Delta_{n-j} - s\Delta_{n-j-1} = P_{n-j} - sP_{n-j-1},$$

by the induction hypothesis, and therefore

$$\alpha_{j+1}^2 \Delta_{n-j-2} = a_{n-j} a_{n-j-1} P_{n-j-2},$$

so that for $j = 0, \ldots n-2$,

$$P_{n-j-2} = \Delta_{n-j-2}, \quad \alpha_{j+1}^2 = a_{n-j} a_{n-j-1}.$$

□

For $j = 1, \ldots, n$, the lossless functions $F_j(s)$ satisfy the interpolation conditions

$$\begin{array}{rcl} F_j(\infty) & = & \xi_0 \\ \lim_{s\to\infty} s^2 F_j'(s) & = & \xi_0 \rho_j. \end{array}$$

It is interesting to notice that the parameters a_i above have a direct electrical interpretation. In [2], the state-space structure (11) is derived from a singly-terminated LC ladder, the parameters a_i in (22) being the reciprocal values of capacitor and inductor values, alternatively.

VI. Balanced realizations

In this section, a recursive construction of balanced realizations is associated with the Schur algorithm described in section IV. We study the general case of complex transfer functions, but focus on the interpolation points all at infinity, which gives a nice structure. In the real case, it leads to Ober's canonical form. This construction is based on the results of [12] in which linear fractional transformations on transfer functions are represented by corresponding linear fractional transformations on state-space realization matrices. Namely, there we have the following result.

Theorem 3: Suppose that $\theta(s)$ is proper, has McMillan degree 1 and admits a state-space realization

$$\theta(s) = \begin{bmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{D}_{21} & \mathcal{D}_{22} \end{bmatrix} + \begin{bmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{bmatrix} (s - \sigma)^{-1} \begin{bmatrix} \mathcal{B}_1 & \mathcal{B}_2 \end{bmatrix}.$$

Let $G(s) = D + C(sI_n - A)^{-1}B$ be a realization of $G(s)$, then a realization of $F(s) = T_{\theta(1/s)}(G(s)) = \tilde{D} + \tilde{C}(sI_{n+1} - \tilde{A})^{-1}\tilde{B}$ is given by the linear fractional transformation:

$$\begin{bmatrix} \tilde{D} & \tilde{C} \\ \tilde{B} & \tilde{A} \end{bmatrix} = T_\Phi \left(\begin{bmatrix} D & 0 & C \\ 0 & 1 & 0 \\ B & 0 & A \end{bmatrix} \right)$$

where

$$\Phi = \left[\begin{array}{ccc|ccc} \mathcal{D}_{11} & 0 & 0 & \mathcal{D}_{12} & \mathcal{C}_1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 \\ \hline \mathcal{D}_{21} & 0 & 0 & \mathcal{D}_{22} & \mathcal{C}_2 & 0 \\ \mathcal{B}_1 & 0 & 0 & \mathcal{B}_2 & \sigma & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n \end{array} \right].$$

We now apply this result to the J-inner matrix

$$\theta(s) = J - \frac{1}{\rho s} \begin{bmatrix} \xi \\ 1 \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix}^*,$$

considered in section V, and consider the linear fractional transformation

$$F(s) = T_{\theta(1/s)}(G(s)),$$

which provides the solution to an interpolation problem at ∞, namely

$$\begin{array}{rcl} F(\infty) & = & \xi \\ \lim_{s\to\infty} s^2 F'(s) & = & \xi\rho. \end{array}$$

The matrix Φ in theorem 3 is then

$$\Phi = \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 0 & \xi/\sqrt{\rho} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1/\sqrt{\rho} & 0 \\ \bar{\xi}/\sqrt{\rho} & 0 & 0 & 1/\sqrt{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n \end{array} \right],$$

and the transformation on balanced realizations becomes

$$\begin{bmatrix} \tilde{D} & \tilde{C} \\ \tilde{B} & \tilde{A} \end{bmatrix} = \begin{bmatrix} \bar{\xi} & -\sqrt{\rho} & 0 \\ \bar{\xi}\sqrt{\rho} & -\frac{\rho}{1+\xi D} & \frac{\sqrt{\rho}}{1+\xi D} C \\ 0 & \xi \frac{\sqrt{\rho}}{1+\xi D} B & A - \frac{\xi}{1+\xi D} BC \end{bmatrix}.$$

In the real case, with $\xi_0 = \xi_1 = \ldots = \xi_n = 1$, the linear fractional fractional transformation

$$F_j(s) = T_{\theta_j(1/s)}(F_{j-1}(s)),$$

corresponds to the following recursion on state space realizations:

$$\begin{aligned} A_j &= \begin{bmatrix} -\rho_j/2 & \frac{\sqrt{\rho_j}}{2} C_{j-1} \\ \frac{\sqrt{\rho_j}}{2} B_{j-1} & A_{j-1} - \frac{B_{j-1}C_{j-1}}{2} \end{bmatrix}, \\ B_j &= \begin{bmatrix} \sqrt{\rho_j} \\ 0 \end{bmatrix}, \\ C_j &= \begin{bmatrix} -\sqrt{\rho_j} & 0 \end{bmatrix}, \\ D_j &= 1, \end{aligned}$$

which leads to the balanced canonical form of Ober, with $s_1 = -1$. Note that subdiagonal entries are all positive while they are all negative in Ober's canonical form.

An example: consider the ladder filter in [2]: $F(s) = \frac{(-1)^n q(-s)}{q(s)}$ where $q(s) = s^5 + .9287s^4 + 1.7726s^3 + 1.0557s^2 + .6917s + .1739$.
To compute its first parameter $a_5 = \rho_5/2$, we separate the odd and even part of $q(s)$, which gives

$$\begin{aligned} \Delta_5 &= s^5 + 1.7726s^3 + .6917s \\ \Delta_4 &= s^4 + .18725 + 1.1367s^2 \\ a_5 &= .9287 \end{aligned}$$

Then we use the recursion formula (13) and (22) to compute the $a_i = \rho_i/2$. We get

$$\begin{aligned} \Delta_3 &= s^3 + .7933s & a_4 &= .6847 \\ \Delta_2 &= s^2 + .5452 & a_3 &= .5016 \\ \Delta_1 &= s & a_2 &= .4946 \\ \Delta_0 &= 1 & a_1 &= 1.1025 \end{aligned}$$

Then, a balanced realization can be computed by the recursions below:

$$A = \begin{bmatrix} -.9287 & -.7974 & 0 & 0 & 0 \\ .7974 & 0 & -.586 & 0 & 0 \\ 0 & .586 & 0 & -.498 & 0 \\ 0 & 0 & .498 & 0 & -.73843 \\ 0 & 0 & 0 & .73843 & 0 \end{bmatrix}$$

$$B^T = -C = \begin{bmatrix} 1.3629 & 0 & 0 & 0 & 0 \end{bmatrix}$$

VII. Conclusion

In this paper, a connection has been established between Ober's canonical form for continuous-time lossless systems and a Schur algorithm built from angular derivative interpolation conditions. It provides a new interpretation of the parameters in Ober's form in terms of boundary interpolation values and a recursive construction of the balanced realization. We thus have an analog in continuous-time of the connection between Hessenberg canonical form and Schur parameters in discrete-time ([7]).

Schur parameters present interesting properties from an optimization view point. Stability and the order of the system are a built-in property, computations can be performed in state-space using balanced realizations and often exhibit a nice behavior. In discrete-time, Schur parameters have been successfully used as optimization parameters, mainly in rational L^2 approximation ([13]). Algorithm in discrete-time can be used for continuous-time identification problems. However, the approach can now be developed directly in continuous-time for a better adaptation.

Applications to the design of hyperfrequency filters are expected but they would require the multivariable case since these filters are 2×2 systems. In this problem, as often, the physical parameters are not suitable for optimization. It is therefore natural to split the design into two steps: first, solve a rational approximation problem using dedicated parameters and then compute all possible physical realizations of the filter. These physical realizations have a particular structure connected with the geometry of the filter, made of coupled resonant cavities. They are balanced realizations of the form (A, C^T, C, I) where

$$\begin{aligned} A &= -\frac{1}{2} C^T C - j\, M \\ C &= \begin{bmatrix} j\sqrt{2R_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & j\sqrt{2R_2} \end{bmatrix}, \quad R_1, R_2 > 0. \end{aligned}$$

The coupling matrix M is symmetric and real, it indicates the way resonators are coupled to one an other. An approach based on computer algebra is presently used to find all the possible physical realizations corresponding to the rational approximation computed at the first step. The use of optimization parameters coming from the MIMO Ober's form would provide an approximant with a realization closer to the desired structure and would facilitate the second step. But we now expect that boundary interpolation problems for MIMO transfer functions would provide such parameters. This generalization is under study.

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