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Periodic Motion Planning and Analytical Computation of Transverse Linearizations for Hybrid Mechanical Systems

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***Abstract*— This paper introduces a method for analytical planning of feasible hybrid periodic trajectories in non-feedback-linearizable impulsive mechanical systems with control inputs. For a planned motion, a procedure for computation of a transverse linearization for a class of hybrid controlled mechanical systems with underactuation one is presented. The resulting linear comparison system can be used for stability analysis and for design of orbitally stabilizing controllers.**

***Index Terms*— Moving Poincaré section; Transverse linearization; Orbital stability; Hybrid mechanical systems; Virtual holonomic constraints; Walking robots**

I. Introduction

Controlling mechanical systems with limited numbers of actuators is a challenging task. The challenges become even greater, when besides the Euler-Lagrange equations, the description of the system dynamics includes a discontinuous updating law acting from time to time on the solutions, e.g. due to impact forces exerted on a walking/running robot when its feet hit the ground. Augmenting Euler-Lagrange equations, representing the *continuous part* of the dynamics, with an updating law, representing the *discrete part* of the dynamics, makes the control system *hybrid*. When the target behavior is more complicated than a simple equilibrium, e.g. a periodic trajectory, even establishing existence of such motions in a nonlinear system is often difficult [18], [27]. These difficulties are clearly inherited for hybrid nonlinear systems. In this work, we focus on periodic motion planning and orbital stability/stabilization of periodic motions for hybrid mechanical systems that have one less independent control input than the number of degrees of freedom.

Even with this restricted focus, the problems are sufficiently challenging, and the class of motivating examples is sufficiently rich, to warrant a detailed study. This class includes, among others, challenging robotics applications such as humanoid robots and various juggling devices [7], [3], [4], [14], [5], [8], [13], [23], [26].

The main contributions of this paper are as follows:

- A new approach for analytical planning periodic motions for mechanical systems with impulsive update laws;
- For a given cycle of the hybrid mechanical system, a hybrid linear comparison system, which is a linearization of the systems's dynamics transverse to the orbit, is derived;
- It is shown that stabilization of the origin of this hybrid linear comparison system allows a straightforward design of a feedback controller for orbital exponential stabilization of the target cycle for the hybrid nonlinear system.

A. Preliminaries: Transverse Linearization

As known, the linearization of an autonomous nonlinear system around its periodic solution cannot be asymptotically stable. Hence, one should consider not stabilization to a solution but to a desired orbit. The geometrical interpretation of this fact is that one needs to find a change of variables such that the system's states are decomposed into:

1) a scalar variable representing position *along the cycle*, which can be safely disregarded,
2) the remaining coordinates, representing the dynamics *transverse to the cycle*; in other words, coordinates that define a moving Poincaré section [12], to be stabilized.

A diffeomorphism to a new set of coordinates with this form always exists in a vicinity of a periodic orbit [24], [9], [18], [10], [15]. Finding such a change of coordinates in an explicit form is difficult. However, for analysis and orbital stabilization of the dynamics near the target orbit, construction of a *transverse linearization* about the orbit is sufficient, and explicit formulae for the full change of variables are not required. Such an analytic construction is the main contribution of this paper. The concept of transverse linearization has been used for feedback control of various classes of systems, see e.g. [21], [11], [1], [2].

The result of computations is a system of N-linear control systems combined with N linear switching laws, which all together constitute the *hybrid transverse linearization* for the nonlinear system around the cycle. Having computed this linear system, one has a tool for analysis of the nonlinear system around the cycle for the case when controller is already fixed. But mostly important, it provides a tool for designing orbitally stabilizing feedback controllers. Even the procedure below will be elaborated for planning and stabilizing the hybrid cycle with one jump, it is generic and can be readily extended for cycles with many jumps.

B. Re-parameterizing a Continuous-Time-Part of the Cycle

Continuous-time-parts of a periodic solution of a hybrid system are solutions of Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}(q,\dot q)}{\partial \dot q}\right) - \frac{\partial \mathcal{L}(q,\dot q)}{\partial q} = B(q)\,u, \qquad (1)$$

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defined on one or several finite-time intervals dependent on the number of switches along the orbit.Here q is a vector of generalized coordinates, u is a vector of control inputs,

$$\mathcal{L}(q,\dot{q}) = \tfrac{1}{2}\,\dot{q}^T M(q)\dot{q} - V(q) \tag{2}$$

is the Lagrangian of the system (1), $M(q)$ is a matrix of inertia, $V(q)$ is a potential energy, and $B(q)$ is a full-rank matrix function of appropriate dimensions. For future references we note that the system (1) can be rewritten [16] as

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = B(q)\,u, \tag{3}$$

where $G(q) = \left[\frac{\partial V(q)}{\partial q_1}, \ldots, \frac{\partial V(q)}{\partial q_n}\right]^T$ and $C(q,\dot{q})$ is a matrix of Coriolis and centrifugal forces.

If one wants to describe a solution of this system – either in an open-loop or under a feedback action – there is a number of possible formats. The obvious is defining an evolution of the generalized coordinates as functions of time:

$$q_1 = q_{1\star}(t), \quad \ldots, \quad q_n = q_{n\star}(t), \quad t \in [T_b, T_e] \tag{4}$$

However, the concept of orbital stability and an eventual goal to design state feedback control laws motivate an orbit description without an explicit reference to time. One can introduce a geometric description of the same orbit:

$$q_1 = \phi_1(\theta_\star), \quad \ldots, \quad q_n = \phi_n(\theta_\star), \quad \theta_\star \in [\Theta_b, \Theta_e], \tag{5}$$

where $\theta_\star$ could be some geometrical parameter or in many cases one can choose $\theta_\star$ to be one of the q-coordinates.

Geometrical specifications of coordinate motions, as in (5), are known as *virtual holonomic constraints* [7], [25], [19], since they express restrictions to the generalized coordinates $q_1, \ldots, q_n$ but imposed by control action. It is clear that relations (5) can always be found for a feasible motion.

If the system is fully actuated, typically the dynamics along the orbit can be controlled. However, for underactuated systems, once the shape of an orbit is specified, the dynamics along the orbit are also thereby fixed, and can be explicitly calculated.

If the system has underactuation one, i.e.

$$\dim u = \text{rank}\,\{B(q)\} = n - 1. \tag{6}$$

and the relations (5) are invariant under control action, then $\theta_\star$ is the solution of the system

$$\alpha(\theta_\star)\ddot{\theta}_\star + \beta(\theta_\star)\dot{\theta}_\star^2 + \gamma(\theta_\star) = 0, \tag{7}$$

where $\alpha(\theta_\star)$, $\beta(\theta_\star)$, and $\gamma(\theta_\star)$ are scalar functions. The differential equation has a general integral of motion [17], which keeps its value along the solution $\theta_\star = \theta(t)$ and is given by

$$\begin{aligned} I\Big(\theta(t), \dot{\theta}(t), \theta(0), \dot{\theta}(0)\Big) &= \dot{\theta}^2(t) - e^{\left\{-\int\limits_{\theta(0)}^{\theta(t)} \frac{2\,\beta(\tau)}{\alpha(\tau)}d\tau\right\}}\dot{\theta}^2(0) + \\ &+ \int_{\theta(0)}^{\theta(t)} e^{\left\{\int\limits_{\theta(t)}^{s} \frac{2\,\beta(\tau)}{\alpha(\tau)}d\tau\right\}}\,\frac{2\,\gamma(s)}{\alpha(s)}ds \end{aligned} \tag{8}$$

C. Defining Updating Laws

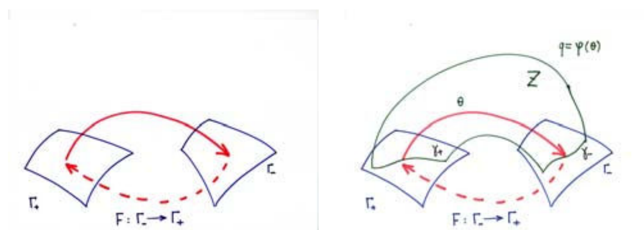
Discrete part of the system dynamics is defined by a collection of hyper-surfaces $\{\Gamma_-^{(i)}, \Gamma_+^{(i)}\}$ in the state space of the mechanical system (1) and a family of mappings $F^{(i)}(\cdot)$ between pairs of these hyper-surfaces $F^{(i)} : \Gamma_-^{(i)} \to \Gamma_+^{(i)}$. Here i is an index for labeling the triples $\{\Gamma_-^{(i)}, \Gamma_+^{(i)}, F^{(i)}(\cdot)\}$ running through $i \in \{1, \ldots, N_d\}$ where N_d can be arbitrarily large but finite. It is assumed that the hyper-surfaces $\Gamma_-^{(i)}$, $\Gamma_+^{(i)}$ and the mappings $F^{(i)}(\cdot)$ are C^1-smooth. The system (1) augmented with the discrete dynamics, i.e. triples

$$\{\Gamma_-^{(i)}, \Gamma_+^{(i)}, F^{(i)}(\cdot)\}, \quad i = 1, \ldots, N_d \tag{9}$$

constitutes the *hybrid* (impulsive) controlled system, whose periodic solutions are in the focus of this investigation.

D. Periodic Motions of (1), (9) with One Jump

Planning periodic solutions for hybrid mechanical system (1), (9) is a challenging task. Prior to solving this problem, let us make some observations about a system admitting a nontrivial periodic solution $q_\star(t) = q_\star(t + T_h)$ for an appropriate choice of the control input $u = u_\star(t)$ with the discrete dynamics acting only once along the cycle, i.e. $N_d = 1$. This situation is depicted on Fig. 1(a).



(a) (b)

Fig. 1. (a) A hybrid cycle of (1), (9). Here the solid arrow denotes the continuous part of the cycle $q_\star(t)$, the dashed arrow shows the result of a discrete mapping $F(\cdot)$ between the switching surfaces Γ_-, Γ_+; (b) if one introduces a control signal that guarantees invariance of virtual holonomic constraints (5), then the system evolves on a 2-dimensional surface Z. The curves γ_+ and γ_- are the intersections of Z with Γ_+ and Γ_-.

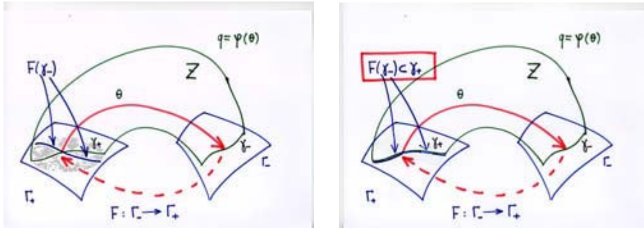
As argued above, the continuous-in-time part (4) of this hybrid solution can be always re-parameterized as in (5). This defines the functions $\phi_1(\cdot), \ldots, \phi_n(\cdot)$ of (5). Suppose that one introduces an *artificial control input* $u_a = u_a(q,\dot{q})$, which keeps of the relations

$$q_1 = \phi_1(\theta), \quad q_2 = \phi_2(\theta), \quad \ldots, \quad q_n = \phi_n(\theta), \tag{10}$$

invariant for (1). Then evolution of θ is not arbitrary, it is defined by (7), which state lives on a 2-dimensional surface Z. The trajectory $q_\star(t)$ corresponds to only one solution $\theta_\star(t)$ out of many possible solutions of (7). Defined in such way, the 2-dimensional manifold Z intersects both hyper-surfaces Γ_+ and Γ_- resulting in curves denoted as γ_+ and γ_- respectively, see Fig. 1 (b).

The discrete part of the dynamics maps the line $\gamma_- \subset \Gamma_-$ into $\Gamma_+ = F(\Gamma_-)$. In general, the invariance of Z may be broken by the discrete dynamics, see Fig. 2 (a), so that

$$F(\gamma_-) \not\subset \gamma_+ \tag{11}$$



(a) (b)

Fig. 2. (a) The discrete dynamics of (1), (9) do not necessary keep the 2-dimensional manifold Z invariant. In general, $F(\gamma_-) \not\subset \gamma_+$; (b) For some class of mechanical systems the conditions, when this inclusion $F(\gamma_-) \subset \gamma_+$ holds. These conditions are properties of the cycle and the system, they are independent on a controller designed for stabilizing the target cycle.

Conditions to ensure existence of a hybrid cycle for which Z is invariant over the discrete part of the dynamics $F(\cdot)$, see Fig. 2 (b), are reported in [25], [26] for a class of models of planar bipeds. Such invariance allowed the authors to construct orbitally stabilizing controllers for bipeds by finding virtual constraints (10) for which dynamics *within* Z are stable, and then enforcing the virtual constraints with high-gain or finite-time controllers.

II. Planning Periodic Solutions for Hybrid Mechanical Systems with a Single Jump

Theorem 1: *Given the hybrid mechanical system, where the continuous-time dynamics (1) have n-degrees of freedom and are of underactuation one, and where the hyper-surfaces Γ_- and Γ_+ and the map $F : \Gamma_- \to \Gamma_+$ define the discrete dynamics. Suppose for a control signal $u_\star(t) = u_\star(t+T_h)$, the hybrid mechanical system has a periodic solution*

$$q_\star(t) = q_\star(t+T_h), \quad \forall t, \quad T_h > 0 \tag{12}$$

with only one jump, i.e. $[q_\star(0), \dot q_\star(0)] \in \Gamma_+$, $[q_\star(T_h-), \dot q_\star(T_h-)] \in \Gamma_-$, $[q_\star(t), \dot q_\star(t)] \notin \Gamma_+ \cup \Gamma_-$ for any $t \in (0, T_h)$ and $[q_\star(T_h+), \dot q_\star(T_h+)] = F([q_\star(T_h-), \dot q_\star(T_h-)]) = [q_\star(0), \dot q_\star(0)]$. Introduce a variable $\theta_\star(t)$ and find C^2-functions $\phi_1(\cdot), \ldots, \phi_n(\cdot)$ such that the continuous-in-time arc of this motion is represented as $q_\star(t) = \phi(\theta_\star(t))$, i.e. in coordinates

$$q_{1\star}(t) = \phi_1(\theta_\star(t)), \;\ldots,\; q_{n\star}(t) = \phi_n(\theta_\star(t)), \quad t \in [0, T_h]$$

Compute the dynamics of (1), when these relations are kept invariant, i.e. the coefficients of the second order system (7). Then, by necessity, the algebraic equations

$$I\Big(\theta_\star(0), \dot\theta_\star(0), \theta_\star(T_h), \dot\theta_\star(T_h)\Big) = 0 \tag{13}$$

$$F\Big([q_-, \dot q_-]\Big)\Big|_{\substack{q_- = \phi(\theta_\star(T_h)) \\ \dot q_- = \phi'(\theta_\star(T_h))\,\dot\theta_\star(T_h)}} = [q_+, \dot q_+]\Big|_{\substack{q_+ = \phi(\theta_\star(0)) \\ \dot q_+ = \phi'(\theta_\star(0))\,\dot\theta_\star(0)}} \tag{14}$$

hold. Here the function $I(\cdot)$ is defined in (8). ∎

The relations (13), (14) can be used for planning cycles of (1), (9). To this end one can follow the next steps:

1) Choose a parametric set of C^2-smooth functions

$$\phi(\theta, P) = \{\phi_1(\theta,P), \phi_2(\theta,P), \ldots, \phi_n(\theta,P)\},$$

where $P = (p_1, \ldots, p_k)$ is a vector of parameters.

2) Compute the dynamics of (1) provided the relations

$$q_1 = \phi_1(\theta, P), \; q_2 = \phi_2(\theta, P), \; \ldots, \; q_n = \phi_n(\theta, P)$$

are kept invariant. This computation results in the parametric family of 2-dimensional manifolds $Z(P)$ of the state space of (1) and systems

$$\alpha(\theta,P)\ddot\theta + \beta(\theta,P)\dot\theta^2 + \gamma(\theta,P) = 0 \tag{15}$$

solutions of which live on $Z(P)$. For any choice of P, the system (15) is integrable So that for any solution $\theta(t,P)$ of this system, the function $I(\cdot)$ computed from the functions $\alpha(\cdot,P)$, $\beta(\cdot,P)$, $\gamma(\cdot,P)$, see (8), satisfies

$$I(\theta(\tau_1,P), \dot\theta(\tau_1,P), \theta(0,P), \dot\theta(0,P)) = 0$$

provided the solution $\theta(t,P)$ is defined for $t = \tau_1$.

3) Compute the projection of the discrete dynamics of the hybrid system on the relations of 2), i.e. define the lines and the mapping $\mathcal{F} : \gamma_- \to \Gamma_+$ (see Fig. 1):

$$\mathcal{F}\Big([\theta, \dot\theta]\Big)\Big|_{[\theta,\dot\theta]\in\gamma_-} = F\Big([q, \dot q]\Big)\Big|_{\substack{q=\phi(\theta,P) \\ \dot q = \phi'(\theta,P)\,\dot\theta}};$$

4) Search for $P = P_\star$ so that the next *algebraic* equations have a solution: $[a,b] \in \gamma_+$, $[x,y] \in \gamma_-$,

$$I(a,b,x,y) = 0, \quad \mathcal{F}([x,y]) = [a,b].$$

If such search is successful, $\theta(t, P_\star)$ is the solution of (15) with $P = P_\star$ initiated at $\theta_\star(0,P_\star) = a$, $\dot\theta_\star(0,P_\star) = b$, then (1), (9) has the hybrid cycle

$$q_1(t) = \phi_1(\theta(t,P_\star), P_\star), \ldots, q_n(t) = \phi_n(\theta(t,P_\star), P_\star).$$

III. Transverse Linearizations of (1), (9) Around Its Cycle with One Jump

Given a hybrid cycle with one jump of the system (1), (9), the procedure for computing coefficients of *transverse linearization* – a hybrid linear control system of dimension $(2n-1)$ – consists of three steps[1]:

Step 1: Linearizing the update law $\{\Gamma_+, \Gamma_-, F(\cdot)\}$ around the point $[q_\star(T_h-), \dot q_\star(T_h-)] \in \Gamma_-$;

Step 2: Computing a linearization of the transverse dynamics of the controlled Euler-Lagrange system (1) along the continuous-in-time sub-arc of $q_\star(t)$;

Step 3: Merging the linearizations of the continuous and discrete dynamics computed on the previous steps.

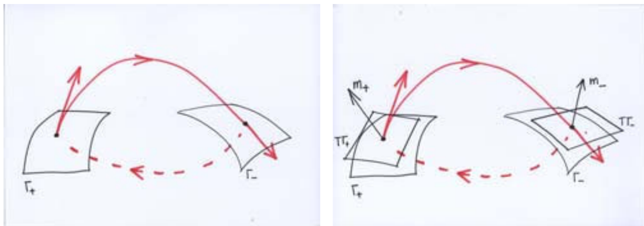
A. Step 1: Linearizing the Discrete-in-Time Dynamics

Linearizing the update law $\{\Gamma_+, \Gamma_-, F(\cdot)\}$ around the hybrid cycle $q_\star(t)$, is straightforward. Indeed, it is the differential of $F(\cdot)$ calculated at $[q_\star(T_h-), \dot q_\star(T_h-)] \in \Gamma_-$:

$$dF\Big|_{\substack{q=q_\star(T_h-) \\ \dot q = \dot q_\star(T_h-)}} : T\Gamma_-\big|_{\substack{q=q_\star(T_h-) \\ \dot q = \dot q_\star(T_h-)}} \to T\Gamma_+\big|_{\substack{q=q_\star(0+) \\ \dot q = \dot q_\star(0+)}} \tag{16}$$

where $T\Gamma_-$, $T\Gamma_+$ are tangent planes to the C^1-smooth manifolds Γ_-, Γ_+ at two points where the cycle $q_\star(t)$ hits and originates from the switching surfaces, see Fig. 3.

[1] Only the second and third steps are nontrivial.



(a) (b)

Fig. 3. (a) A hybrid cycle with one jump; (b) Tangent planes $T\Gamma_-$ and $T\Gamma_+$ of the switching surfaces Γ_- and Γ_+ at two points, where the periodic trajectory $q_\star(t)$ hits and originates from the switching surfaces. The linearization of $F(\cdot)$ in a vicinity of the hybrid cycle is the linear mapping $dF : T\Gamma_- \to T\Gamma_+$, see (16). The hyper-surfaces $T\Gamma_-$ and $T\Gamma_+$ can be also defined with the use of normal vectors m_- and m_+ respectively.

B. Step 2: Linearizing the Continuous-in-Time Dynamics

Computing a linearization of the transverse dynamics of the system (1) along its solution is based on concepts of moving Poincaré section [12] and transverse dynamics [24].

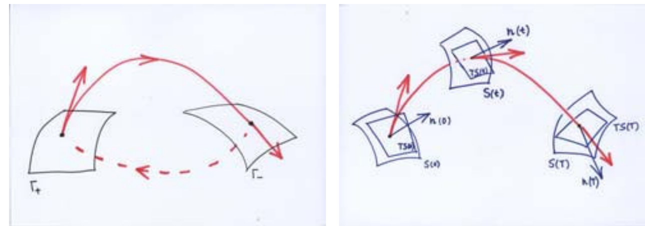
Definition 1: Let $q_s(t)$, $t \in [0, T_h]$, be a solution of the n-degree-of-freedom Euler-Lagrange system (1) with the initial conditions at $q_s(0) = q_0$, $\dot q_s(0) = \dot q_0$, driven by the control signal $u_s(t) \in C^1([0, T_h])$ such that $\left(|\dot q_s(t)|^2 + |\ddot q_s(t)|^2\right) > 0$ for all $t \in [0, T_h]$. Then
1) A family of $(2n-1)$-dimensional C^1-smooth surfaces $\{S(t),\, t \in [0, T_h]\}$ is called a *moving Poincaré section* associated with the solution $q_s(t)$, $t \in [0, T_h]$, if

- surfaces $S(t)$ are disjoint, i.e. $S(\tau_1) \cap S(\tau_2) = \emptyset$, $\forall\, \tau_1$, $\tau_2 \in [0, T_h)$, $\tau_1 \neq \tau_2$;
- $\exists \varepsilon > 0$: $S(\tau) \cap \{[q_s(t), \dot q_s(t)],\ |t - \tau| < \varepsilon\} = \{[q_s(\tau), \dot q_s(\tau)]\}$ for each $\tau \in [0, T_h]$;
- surfaces $S(t)$ are smoothly parametrized by t: $\exists\, f \in C^1(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R})$: $S(t) = \{[q, \dot q] : f_s(q, \dot q, t) = 0\}$;
- surfaces $S(t)$ are transversal to the solution $(q_s(t), \dot q_s(t))$, i.e. $\forall\, t \in [0, T_h]$

$$\frac{\partial}{\partial q} f_s(q, \dot q, t)\Big|_{\substack{q = q_s(t)\\ \dot q = \dot q_s(t)}} \cdot \dot q_s(t) + \frac{\partial}{\partial \dot q} f_s(q, \dot q, t)\Big|_{\substack{q = q_s(t)\\ \dot q = \dot q_s(t)}} \cdot \ddot q_s(t) \neq 0$$

2) Given a moving Poincaré section $\{S(t),\, t \in [0, T_h]\}$, in some vicinity of the trajectory $[q_s(t), \dot q_s(t)]$, $t \in [0, T_h]$, the states $[q, \dot q]$ of (1) can be changed into: the scalar variable $\psi(t)$ that parameterizes a position along the trajectory and the $(2n-1)$-dimensional vector $x_\perp(t)$ that defines location on the surface $S(t)$. $x_\perp(\cdot)$ is known as a vector of *transverse coordinates*, while $x_\perp$-dynamics are called *transverse*.
3) The dynamics of (1) rewritten in $[\psi, x_\perp]$-coordinates and linearized along the solution $q_s(t)$, $t \in [0, T_h]$ give rise to the linear time-varying control system of dimension $2n$ defined on $t \in [0, T_h]$. The subsystem that corresponds to linearization of the dynamics of transverse coordinates $x_\perp$ is called a *transverse linearization*. ■

As seen the transverse linearization is not unique and depends on the choice of a moving Poincaré section $\{S(t),\, t \in [0, T_h]\}$, which defines the transverse coordinates $x_\perp$. Fig. 4 illustrates the concept of a moving Poincaré section for the continuous-in-time sub-arc of hybrid cycle $q_\star(t)$. Note that $S(0)$, $S(T)$ won't necessarily coincide with switching surfaces Γ_+ and Γ_-.



(a) (b)

Fig. 4. (a) A hybrid cycle with one jump; (b) The moving Poincaré section – a family of $(2n-1)$-dimensional surfaces $S(t)$ transversal to the continuous-in-time sub-arc of the hybrid cycle. The linearization of transverse dynamics, *transverse linearization*, is linear control system defined on tangent planes $TS(t)$, $n(t)$ are vectors normal to $TS(t)$

It is a remarkable fact that for any motion of the system (1), there is a generic choice of transverse coordinates. With such transverse coordinates the coefficients of $(2n-1)$-dimensional linear control system – *transverse linearization* – can be found analytically. It has the form

$$\dot\zeta(t) = \underbrace{\begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} & I_{(n-1)\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} & 0_{(n-1)\times(n-1)} \end{bmatrix}}_{=A(t)} \zeta(t) + \underbrace{\begin{bmatrix} b_1(t) \\ 0_{(n-1)\times(n-1)} \\ I_{(n-1)\times(n-1)} \end{bmatrix}}_{=B(t)} w(t) \quad (17)$$

Here $\zeta \in \mathbb{R}^{2n-1}$, $w \in \mathbb{R}^{n-1}$; $a_{11}(\cdot)$, $a_{12}(\cdot)$, $a_{13}(\cdot)$, and $b_1(\cdot)$ are functions on $[0, T_h]$ of appropriate dimensions. They are computed according to the following steps[2]:
1) Introduce new generalized coordinates for (1) in a vicinity of the motion: Given the functions $\phi_1(\cdot)$, …, $\phi_n(\cdot)$, see (5), consider quantities

$$\theta,\ y_1 = q_1 - \phi_1(\theta),\ \ldots,\ y_n = q_n - \phi_n(\theta) \qquad (18)$$

They are excessive coordinates for the n-degrees of freedom Euler-Lagrange system (1), therefore one of them can be expressed as a function of others. Assume this is the case for y_n, so new generalized coordinates are

$$y = (y_1, \ldots, y_{n-1})^T \quad \text{and} \quad \theta, \qquad (19)$$

and that the last relation – $y_n = q_n - \phi_n(\theta)$ – becomes

$$q_n = \phi_n(\theta) + h(y_1, \ldots, y_{n-1}, \theta), \qquad (20)$$

where $h(\cdot)$ is a scalar smooth function. Note that as soon as a control action makes the desired cycle invariant and the

[2]Assumptions guaranteeing the feasibility of computations done below are discussed in details in [20] for the case when $q_\star(t)$ is periodic in time with no jumps, and they are quite mild. Here, we shell have a little more restrictive assumptions (such as invertability of the matrix functions $L(\cdot)$ and $N(\cdot)$ in (22) and (23) in a vicinity of $q_\star(\cdot)$) to hold on the time interval $[0, T_h]$.

initial conditions are on the cycle, we must have

$$y_1 = 0, \quad \ldots, \quad y_{n-1} = 0, \qquad \text{and} \qquad \theta = \theta_\star(t), \quad (21)$$

where $\theta_\star(t)$ is defined by (5) and it is a solution of (7).

2) Rewrite the dynamics of (1) in new generalized coordinates (19): Compute the first and the second time derivatives of the original generalized coordinates q as functions the new coordinates and their time derivatives, i.e.

$$\dot q = L(\theta,y)\begin{bmatrix}\dot y\\ \dot\theta\end{bmatrix},\ \ddot q = L(\theta,y)\begin{bmatrix}\ddot y\\ \ddot\theta\end{bmatrix} + \dot L(\theta,y)\begin{bmatrix}\dot y\\ \dot\theta\end{bmatrix}, \quad (22)$$

$$L(\theta,y) = \begin{bmatrix}\mathbf{1}_{n-1},\ \mathbf{0}_{(n-1)\times 1}\\ \operatorname{grad} h\end{bmatrix} + \left[\mathbf{0}_{n\times(n-1)},\ \Phi'(\theta)\right]$$

$$\operatorname{grad} h = \left[\tfrac{\partial h}{\partial y_1}, \ldots, \tfrac{\partial h}{\partial y_{n-1}}, \tfrac{\partial h}{\partial \theta}\right],\ \Phi' = [\phi_1'(\theta), \ldots, \phi_n'(\theta)]^T$$

The dynamics of y are then computed from (3) as

$$\ddot y = R(y,\theta,\dot y,\dot\theta) + N(y,\theta)\,u, \quad (23)$$

where $N(\cdot) = \left[\mathbf{1}_{n-1}, \mathbf{0}_{(n-1)\times 1}\right] L^{-1}(\theta,y)\left[M^{-1}(q)\,B(q)\right]$,

$$R(\cdot) = \left[\mathbf{1}_{n-1},\ \mathbf{0}_{(n-1)\times 1}\right] L^{-1}(\theta,y)\,M^{-1}(q) \times \left(-C(q,\dot q)\dot q - G(q) - \dot L(\theta,y)\begin{bmatrix}\dot y^T & \dot\theta\end{bmatrix}^T\right)$$

The θ-dynamics can be computed from (3): If $B^\perp(\cdot)$ is an annihilator for the matrix function $B(\cdot)$, then pre-multiplying the equations (3) results in the scalar equation

$$B^\perp(q)\left[M(q)\ddot q + C(q,\dot q)\dot q + G(q)\right] = B^\perp(q)\left[B(q)\,u\right] = 0.$$

Collecting terms, this can be rewritten as

$$\begin{aligned}\alpha(\theta)\ddot\theta + \beta(\theta)\dot\theta^2 + \gamma(\theta) &= g_y(\theta,\dot\theta,\ddot\theta,y,\dot y)y + \\ &+ g_{\dot y}(\theta,\dot\theta,\ddot\theta,y,\dot y)\dot y + g_{\ddot y}(\theta,\dot\theta,y,\dot y)\ddot y\end{aligned} \quad (24)$$

with the functions $\alpha(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$, $g_y(\cdot)$, $g_{\dot y}(\cdot)$, $g_{\ddot y}(\cdot)$ defined by the Lagrangian and functions $\phi_1(\cdot), \ldots, \phi_{n-1}(\cdot)$.

3) Make a feedback transform to linearize the y-dynamics (23):

$$u = N(y,\theta)^{-1}[v - R(y,\theta,\dot y,\dot\theta)] \quad (25)$$

that brings the equation (23) into the form: $\ddot y = v$.

4) Obtain explicit formulae for the coefficients of the linear control system (17): $a_{11}(t) = \frac{-2\dot\theta_\star(t)\,\beta(\theta_\star(t))}{\alpha(\theta_\star(t))}$, and

$$a_{12} = \frac{2\dot\theta_\star(t)g_y\left(\theta_\star(t),\dot\theta_\star(t),\ddot\theta_\star(t),0,0\right)}{\alpha(\theta_\star(t))}$$
$$a_{13} = \frac{2\dot\theta_\star(t)g_{\dot y}\left(\theta_\star(t),\dot\theta_\star(t),\ddot\theta_\star(t),0,0\right)}{\alpha(\theta_\star(t))}, b_1 = \frac{2\dot\theta_\star(t)g_{\ddot y}\left(\theta_\star(t),\dot\theta_\star(t),0,0\right)}{\alpha(\theta_\star(t))}$$

To define a moving Poincaré section $\{S(t)\}_{t\in[0,T_h]}$ associated with the controlled transverse linearization (17) computed along the continuous-in-time sub-arc of solution $q_\star(t)$ of the system (1), one can proceed as follows:

- Change in a vicinity of the target motion the variables $[q^T, \dot q^T]$ of the system (1) into $[\theta, \dot\theta, y^T, \dot y^T]$.
- Change of variables from $[\theta, \dot\theta, y^T, \dot y^T]$ into $[\psi, I, y^T, \dot y^T]$, where I is defined by (8). This step introduces the scalar variable $\psi = \psi(\theta, \dot\theta)$ such that the target trajectory is $\{\psi = \psi_\star(t), I = 0, y = 0, \dot y = 0\}$, and $\psi_\star(t) := \psi(\theta_\star(t), \dot\theta_\star(t))$ monotonically[3] changes with time.
- After that, the moving Poincaré section is defined by

$$S(t) := \{[q,\dot q] : \psi(q,\dot q) - \psi_\star(t) = 0\}, \quad t \in [0, T_h] \quad (26)$$

Searching for the surfaces (26) and rewriting dynamics of (1) in terms of the variables $[\psi(\cdot), I(\cdot), y^T(\cdot), \dot y^T(\cdot)]$ is non-trivial. Fortunately, the surfaces $TS(t)$ can be readily found without introducing the variable ψ. Indeed, the linearization of the identities $I(\theta(t), \dot\theta(t), \theta_\star(T_b), \dot\theta_\star(T_b)) = 0$, $y(t) = 0$, $\dot y(t) = 0$ around the point $[q_\star(t), \dot q_\star(t)]$ corresponds to the normal vector $n(t)$ to $TS(t)$ and at the same time the direction of the vector field on the target motion. So, by construction, surfaces $S(t)$ are orthogonal to the target motion, and hence are transversal.

C. Step 3: Merging the Two Parts of Linearized Dynamics

Combining the linear mapping (16), i.e. the linearization of the update law, with the transverse linearization (17) requires certain care. Indeed, the linear mapping (16) acts between the hyperplanes $T\Gamma_-$ and $T\Gamma_+$, defined by the switching surfaces, while the linear differential equation (17) maps a vector on $TS(0)$ into a vector on $TS(T_h)$. Difficulties associated with computing alternative transverse linearizations to (17) analytically[4] motivate us to modify the linear mapping $dF(\cdot)$ so that it acts between switching hyper-planes $TS(T_h)$ and $TS(0)$.

Definition 2: Suppose the following are given:
1) The hyperplanes $TS(0)$, $TS(T_h)$ defined by a moving Poincaré section $\{S(t)\}_{t\in[0,T_h]}$ are transversal to the hybrid periodic motion $q_\star(t)$.
2) The hyperplanes $T\Gamma_+$ and $T\Gamma_-$, tangent to the switching surfaces Γ_+ and Γ_-, are transversal to the hybrid periodic motion defined at the end-point of the continuous-in-time sub-arc and are not orthogonal to $TS(0)$, $TS(T_h)$.
3) The linear mapping $dF : T\Gamma_- \to T\Gamma_+$ is given by (16). Denote by $P^+_{n(0)}$ the projection along $n(0)$ from $T\Gamma_+$ onto $TS(0)$. This operator can be introduced by the following rule: For a given $z_0 \in T\Gamma_+$ consider the line l_0 parallel to $n(0)$ that passes through z_0. Denote by y_0 the point of intersection of $TS(0)$ and this line l_0. Then, y_0 is the image of z_0 under the map $P^+_{n(0)}$. Similarly, let $P^-_{n(T_h)}$ be the projection along $n(T_h)$ from $TS(T_h)$ onto $T\Gamma_-$. From the condition both projection operators are well-defined and linear by construction. The operator

$$d^{TS}F :\ TS(T_h) \ni \xi \mapsto \left[P^+_{n(0)}\right] dF \left[P^-_{n(T_h)}\right]\xi = \eta \in TS(0)$$

is called *a linearization of $F(\cdot)$ associated with the moving Poincaré section $\{S(t)\}_{t\in[0,T_h]}$*. ■

Definition 3: Let $\mathcal{O}$ be an open subset of the manifold Z containing the orbit $\{(\theta_\star(t), \dot\theta_\star(t)) : t \in [0, T_h]\}$. We define $\mathcal{T}$ to be a function $\mathcal{T}([\theta, \dot\theta]) : \mathcal{O} \to \mathbb{R}/[0, T_h]$, which is smooth and satisfies the following relation

$$\mathcal{T}([\theta_\star(t), \dot\theta_\star(t)]) = t \qquad \forall\, t \in \mathbb{R}/[0, T_h]. \quad (27)$$

[3] This is possible due to the assumption $\dot q_\star^2(t) + \ddot q_\star^2(t) > 0 \ \forall\, t \in [T_b, T_e]$

[4] A numerical procedure, based on a concept of orthogonalizing transform, has been proposed in [21].

Theorem 2: *Consider the hybrid control system with solutions $\zeta = \zeta(t) \in \mathbb{R}^{2n-1}$ defined by the inductive rule:*

- *On the time intervals $(0, T_h)$, $(T_h, 2T_h)$, ..., $(kT_h, (k+1)T_h)$, ... the solution is defined by the linear control system*

$$\dot{\zeta}(t) = A(\tau)\,\zeta(t) + B(\tau)\,w(t), \quad \tau = (t \mod T_h) \quad (28)$$

where $A(\cdot)$, $B(\cdot)$ are from (17), and $w(t) \in \mathbb{R}^{n-1}$ is a vector of control inputs.

- *At each of the time moments $t_s = T_h$, $2T_h$, ..., kT_h, ..., the state $\zeta(t_s)$ of linear system (28) is instantaneously changed by the linear transformation*

$$\zeta(t_s-) \longmapsto \zeta(t_s+) := d^{TS} F\left(\zeta(t_s-)\right), \quad (29)$$

where the linear mapping $d^{TS}F(\cdot)$ is from Definition 2. After the update, the solution is defined by (28) until $t = t_s + T_h$, where the next instant update (29) occurs.

Suppose there is $K(\cdot) \in C^1[0, T_h]$ and the state feedback

$$w(t) = K(\tau)\zeta(t), \; \tau = (t \mod T_h), \quad (30)$$

that makes the origin $\zeta = 0$ of the closed-loop system (28), (29), (30) exponentially stable. Then, the feedback controller (25) with $s = \mathcal{T}([\theta(t), \dot{\theta}(t)])$ and

$$v(t) = K(s)\left[I(\theta(t), \dot{\theta}(t), \theta_\star(0), \dot{\theta}_\star(0)),\, y(t)^T,\, \dot{y}(t)^T\right]^T \quad (31)$$

makes the hybrid period motion $q_\star(t)$ of the hybrid mechanical system (1), (9) orbitally exponentially stable. Here the quantities θ, y are defined in (18); the function $I(\cdot)$ is from (8); the function $\mathcal{T}(\cdot)$ satisfies (27). ■

IV. Conclusion

The hybrid mechanical systems with n-degrees of freedom and $(n-1)$ independent control inputs are considered. We have shown how to plan a hybrid periodic motion with one jump. For a planned hybrid periodic motion, we have proposed a constructive analytical procedure for computing a hybrid transverse linearization for hybrid dynamics. The obtained comparison system is linear, of order $(2n-1)$, and with instantaneous linear updates of the states after fixed time intervals. We have proved that stabilization of this linear control system allows one to synthesize a feedback controller for the Euler-Lagrange system to achieve orbital exponential stability of the cycle and have shown how it can be done. The analytically constructed controlled transverse linearization is not based on introducing a moving Poincaré section but defines one and is derived from intrinsic structural properties of the Euler-Lagrange system. The method allows one not only to synthesize stabilizing feedback controllers but also to analyze various properties of the hybrid closed-loop systems.

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