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Continuous-Time Behavioral Portfolio Selection

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Abstract— **This paper formulates and studies a general continuous-time behavioral portfolio selection model under Kahneman and Tversky's (cumulative) prospect theory, featuring S-shaped utility (value) functions and probability distortions. The optimal terminal wealth positions, derived in fairly explicit forms, possess surprisingly simple structure: they resemble the payoff of a portfolio of two binary (or digital) options written on the state density price. An example with a two-piece CRRA utility is presented to illustrate the general results obtained, and is solved completely for all installations of the parameters. The effect of the behavioral criterion on the risky allocations is finally discussed.**

I. Introduction

Mean–variance and expected utility maximization have long been criticized to be inconsistent with the way people do decision making in the real world. Substantial experimental evidences have suggested a systematic violation of the EUT principles. Specifically, the following anomalies (as opposed to the assumed rationality in EUT) in human behaviors are evident from daily life:

- People evaluate assets on gains and losses (which are defined with respect to a reference point), not on final wealth positions;
- People are *not* uniformly risk averse: they are risk-averse on gains and risk-taking on losses;
- People overweight small probabilities and underweight large probabilities.

In addition, there are widely known paradoxes and puzzles that EUT fails to explain, including the Allais paradox [Allais (1953)], Ellesberg paradox [Ellesberg (1961)], Friedman and Savage puzzle [Friedman and Savage (1948)], and the equity premium puzzle [Mehra and Prescott (1985)].

Considerable attempts and efforts have been made to address the drawback of EUT, among them notably the so-called non-additive utility theory. Unfortunately, most of these theories are far too complicated to be analyzable and applicable, and some of them even lead to new paradoxes. In 1970s, Kahneman and Tversky (1979) proposed the *prospect theory* (PT) for decision making under uncertainty, incorporating human emotions and psychology into their theory. Later, Tversky and Kahneman (1992) fine tuned the PT to the *cumulated prospect theory* (CPT) in order to be consistent with the first-order stochastic dominance. Among many other ingredients, the key elements of Kahneman and Tversky's Nobel-prize-winning theory are

- A *reference point* (or *benchmark/breakeven point/status quo*) in wealth that defines *gains* and *losses*;
- A value function (which replaces the notion of utility function), *concave for gains* and *convex for losses*, and steeper for losses than for gains;
- A *probability distortion* that is a *nonlinear* transformation of the probability scale, which enlarges a small probability and diminishes a large probability.

There have been burgeoning research interests in incorporating the PT into portfolio choice; nonetheless they have been hitherto overwhelmingly limited to the single-period setting; see for example Benartzi and Thaler (1995), Shefrin and Statman (2000), and Gomes (2005), with emphases on qualitative properties and empirical experiments. *Analytical* research on *dynamic*, especially continuous-time, asset allocation featuring behavioral criteria is literally nil according to our best knowledge. [In this connection the only paper we know of that has some bearing on the PT for the continuous time setting is Berkelaar, Kouwenberg and Post (2004) where a very specific two-piece power utility function is considered; however, the probability distortion, which is one of the major ingredients of the PT and which causes the main difficulty, is absent in that paper.] Such a lack of study on continuous-time behavioral portfolio selection is certainly not because the problem is uninteresting or unimportant; rather it is because, we believe, that the problem is *massively* difficult as compared with the conventional expected utility maximization model. Many conventional and convenient approaches, such as convex optimization, dynamic programming, and stochastic control, fall completely apart in handling such a behavioral model: First, the utility function (or *value function* as called in the PT) is partly concave and partly convex (also referred to as an *S-shaped* function), whereas the global convexity/concavity is a necessity in traditional optimization. Second, the nonlinear distortion in probabilities abolishes virtually all the nice properties associated with the normal additive probability and linear expectation. In particular, the dynamic consistency of the conditional expectation with respect to a filtration, which is the foundation of the dynamic programming principle, is absent due to the distorted probability. Worse still, the coupling of these two ill-behaved features greatly amplifies the difficulty of the problem. Even the well-posedness of the problem[1] is no longer something that can be taken for granted.

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[1] A maximization problem is called *well-posed* if its supremum is finite; otherwise it is *ill-posed*. An ill-posed problem is a mis-formulated one: the trade-off is not set right so that one can always push the objective value to be arbitrarily high.

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This paper first establishes a general continuous-time portfolio selection model under the PT, involving behavioral criteria defined on possibly continuous random variables. The probability distortions lead to the involvement of the Choquet integrals [Choquet (1953/54)], instead of the conventional expectation. We then carry out, *analytically*, extensive investigations on the model while developing new approaches in deriving the optimal solutions. First of all, by assuming that the market is complete, the asset prices follow general Itô processes, and the individual behavior of the investor in question will not affect the market, we need only to consider an optimization problem in terms of the terminal wealth. This is the usual trick employed in the conventional utility maximization, which also enables us to get around the inapplicability of the dynamic programming in the current setting. Having said this, our main endeavor is to find the optimal terminal wealth, which is a fundamentally different and difficult problem due to the behavioral criterion. As mentioned earlier such a behavioral model could be easily ill-posed and, therefore, we first identify several general cases where the model is indeed ill-posed. Then we move on to finding optimal solutions for a well-posed model. In doing so we decompose the original problem into two sub-problems: one takes care of the gain part and the other the loss part, both parameterized by an initial budget that is the price of the gain part (i.e., the positive change) of the terminal payoff over the reference wealth position and an event when the terminal payoff represents a gain. At the outset the gain part problem is a constrained *non-concave* maximization problem due to the probability distortion; yet by changing the decision variable and taking a series of transformations, we turn it into a concave maximization problem where the Lagrange method is applicable. The loss part problem, nevertheless, is more subtle because it is to *minimize* a *concave* functional even after the similar transformations. We are able to characterize explicitly its solutions to be certain "corner points" via delicate analysis. There is yet one more twist in deriving the optimal solution to the original model given the solutions to the above two problems: one needs to find the "best" parameters – the initial budget and the event of a terminal gain – by solving another constrained optimization problem.

As mathematically complicated and sophisticated the solution procedure turns out to be, the final solutions are surprisingly and beautifully simple: the optimal terminal wealth resembles the payoff of a portfolio of two *binary* (or *digital*) *options* written on a mutual fund (induced by the state pricing density), characterized by a single number. This number, in turn, can be identified by solving a very simple two-dimensional mathematical programming problem. The optimal strategy is therefore a gambling policy, betting on good states of the market, by buying a contingent claim and selling another.

To summarize, the main contributions of this paper are: 1) we establish, for the first time, a *bona fide* continuous-time behavioral portfolio selection model *à la* CPT, featuring very general S-shaped utility functions and probability distortions; 2) we demonstrate that the well-posedness becomes an eminent issue for the behavioral model, and identify several ill-posed problems; 3) we develop an approach, fundamentally different from the existing ones for the expected utility model, to overcome the immense difficulties arising from the analytically ill-behaved utility functions and probability distortions. Some of the sub-problems solvable by this approach, such as constrained maximization and minimization of Choquet integrals, are interesting, in both theory and applications, in their own rights; and 4) we obtain fairly explicit solutions to a general model, and closed-form solutions for an important special case, based on which we are able to examine how the allocations to equity are influenced by behavioral criteria.

This paper is an abbreviated version of the full paper Jin and Zhou (2008). The rest of the paper is organized as follow. In Section 2 the behavioral model is formulated, and its possible ill-posedness is addressed in Section 3. The procedure of analytically solving the general model is developed in Sections 4 – 7.

II. The Model

In this paper T is a fixed terminal time and $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ is a fixed filtered complete probability space on which is defined a standard $\mathcal{F}_t$-adapted m-dimensional Brownian motion $W(t) \equiv (W^1(t), \cdots, W^m(t))'$ with $W(0) = 0$. It is assumed that $\mathcal{F}_t = \sigma\{W(s) : 0 \leq s \leq t\}$, augmented by all the null sets. Here and throughout the paper A' denotes the transpose of a matrix A.

We define a continuous-time financial market with $m+1$ assets being traded continuously. One of the assets is a bank account whose price process $S_0(t)$ is subject to the following equation:

$$dS_0(t) = r(t)S_0(t)dt; \quad S_0(0) = s_0 > 0, \tag{1}$$

where the interest rate $r(\cdot)$ is an $\mathcal{F}_t$-progressively measurable, scalar-valued stochastic process with $\int_0^T |r(s)|ds < +\infty$, a.s.. The other m assets are stocks whose price processes $S_i(t)$, $i = 1, \cdots, m$, satisfy the following stochastic differential equation (SDE):

$$\frac{dS_i(t)}{S_i(t)} = b_i(t)dt + \sum_{j=1}^{m} \sigma_{ij}(t)dW^j(t); \quad S_i(0) = s_i > 0, \tag{2}$$

where $b_i(\cdot)$ and $\sigma_{ij}(\cdot)$, the appreciation and dispersion (or volatility) rates, respectively, are scalar-valued, $\mathcal{F}_t$-progressively measurable stochastic processes.

Set the excess rate of return vector process $B(t) := (b_1(t) - r(t), \cdots, b_m(t) - r(t))'$, and define the volatility matrix process $\sigma(t) := (\sigma_{ij}(t))_{m\times m}$. Basic assumptions imposed on the market parameters throughout this paper are summarized as follows:

Assumption 2.1:
(i) There exists $c \in \mathbf{R}$ such that $\int_0^T r(s)ds \geq c$, a.s..
(ii) Rank $(\sigma(t)) = m$, a.e.$t \in [0, T]$, a.s..
(ii) There exists an $\mathbf{R}^m$-valued, uniformly bounded, $\mathcal{F}_t$-progressively measurable process $\theta(\cdot)$ such that $\sigma(t)\theta(t) = B(t)$, a.e.$t \in [0, T]$, a.s..

It is well known that under these assumptions the process

$$\rho(t) := e^{-\int_0^t \left[r(s)+\frac{1}{2}|\theta(s)|^2\right]ds - \int_0^t \theta(s)'dW(s)} \tag{3}$$

is the pricing kernel or state density price. Denote $\rho := \rho(T)$. It is clear that $0 < \rho < +\infty$ a.s., and $0 < E\rho < +\infty$.

ASSUMPTION 2.2: ρ admits no atom.

We are also going to use the following notation:

$$\begin{aligned} \bar{\rho} &\equiv \text{esssup}\ \rho := \sup\left\{a \in \mathbf{R} : P\{\rho > a\} > 0\right\}, \\ \underline{\rho} &\equiv \text{essinf}\ \rho := \inf\left\{a \in \mathbf{R} : P\{\rho < a\} > 0\right\}. \end{aligned} \tag{4}$$

Consider an agent, with an initial endowment $x_0 \in \mathbf{R}$, whose total wealth at time $t \geq 0$ is denoted by $x(t)$. Assume that the trading of shares takes place continuously in a self-financing fashion Then $x(\cdot)$ satisfies

$$\begin{aligned} &dx(t) = [r(t)x(t) + B'(t)\pi(t)]dt + \pi(t)'\sigma(t)dW(t), \\ &x(0) = x_0, \end{aligned} \tag{5}$$

where $\pi(\cdot) \equiv (\pi_1(\cdot), \cdots, \pi_m(\cdot))'$ is the *portfolio* of the agent with $\pi_i(t),\ \ i = 1, 2\cdots, m$, denoting the total market value of the agent's wealth in the i-th asset at time t. A portfolio $\pi(\cdot)$ is said to be *admissible* if it is an $\mathbf{R}^m$-valued, $\mathcal{F}_t$-progressively measurable process with

$$\int_0^T |\sigma(t)'\pi(t)|^2 dt < +\infty \ \text{ and } \ \int_0^T |B(t)'\pi(t)|dt < +\infty, \ \text{ a.s..}$$

An admissible portfolio $\pi(\cdot)$ is said to be *tame* if the corresponding discounted wealth process, $S_0(t)^{-1}x(t)$, is almost surely bounded from below (the bound may depend on $\pi(\cdot)$).

In this paper, we study a portfolio model featuring human behaviors by working within the CPT framework of Tversky and Kahneman (1992). First of all, in CPT there is a natural outcome or benchmark, assumed to be 0 in this paper without loss of generality, which serves as a base point to distinguish gains from losses. Next, we are given two utility functions $u_+(\cdot)$ and $u_-(\cdot)$, both mapping from $\mathbf{R}^+$ to $\mathbf{R}^+$, that measure the gains and losses respectively. There are two additional functions $T_+(\cdot)$ and $T_-(\cdot)$ from $[0, 1]$ to $[0, 1]$, representing the distortions in probability for the gains and losses respectively. The technical assumptions on these functions, which will be imposed throughout this paper, are summarized as follows.

ASSUMPTION 2.3: $u_+(\cdot)$ and $u_-(\cdot)$: $\mathbf{R}^+ \mapsto \mathbf{R}^+$, are strictly increasing, concave, with $u_+(0) = u_-(0) = 0$. Moreover, $u_+(\cdot)$ is strictly concave and twice differentiable, with the Inada conditions $u_+'(0+) = +\infty$ and $u_+'(+\infty) = 0$.

ASSUMPTION 2.4: $T_+(\cdot)$ and $T_-(\cdot)$: $[0, 1] \mapsto [0, 1]$, are differentiable and strictly increasing, with $T_+(0) = T_-(0) = 0$ and $T_+(1) = T_-(1) = 1$.

Now, given a contingent claim X, we define $V(X)$ by

$$V(X) = V_+(X^+) - V_-(X^-)$$

where

$$\begin{aligned} V_+(Y) &:= \textstyle\int_0^{+\infty} T_+(P\{u_+(Y) > y\})dy, \\ V_-(Y) &= \textstyle\int_0^{+\infty} T_-(P\{u_-(Y) > y\})dy \end{aligned}$$

for any random variable $Y \geq 0$, a.s.. It is evident that V is also non-decreasing.

Under this CPT framework, our portfolio selection problem is to find the most preferable portfolios, in terms of maximizing the value $V(x(T))$, by continuously managing the portfolio. The mathematical formulation is as follows:

$$\begin{array}{ll} \text{Maximize} & V(x(T)) \\ \text{subject to} & \left\{ \begin{array}{l} (x(\cdot), \pi(\cdot)) \text{ satisfies (5),} \\ \pi(\cdot) \ \text{ is admissible and tame.} \end{array} \right. \end{array} \tag{6}$$

In order to solve (6) one needs only first to solve the following optimization problem in the terminal wealth, X:

$$\begin{array}{ll} \text{Maximize} & V(X) \\ \text{subject to} & E[\rho X] = x_0, \\ & X \text{ is lower bounded \& } \mathcal{F}_T\text{-measurable.} \end{array} \tag{7}$$

Once (7) is solved with a solution X^*, the optimal portfolio is then the one replicating X^*. Therefore, in the rest of the paper we will focus on Problem (7).

Before we conclude this section, we recall the following definition. For any non-decreasing function f: $\mathbf{R}^+ \mapsto \mathbf{R}^+$, we define its inverse function

$$f^{-1}(x) := \inf\{y \in \mathbf{R}^+ : f(y) \geq x\}, \ \ x \in \mathbf{R}^+. \tag{8}$$

It is immediate that f^{-1} is non-decreasing and continuous on the left, and it holds always that

$$f^{-1}(f(y)) \leq y.$$

III. ILL-POSEDNESS

Well-posedness is an important issue from the modeling point of view. In classical portfolio selection literature the utility function is typically assumed to be globally concave along with other nice properties; thus the problem is guaranteed to be well-posed[2]. We now demonstrate that for the behavioral model (6) or (7) the well-posedness becomes a more significant issue, and that probability distortions in gains and losses play prominent, yet somewhat opposite, roles.

THEOREM 3.1: Problem (7) is ill-posed if there exists a nonnegative $\mathcal{F}_T$-measurable random variable X such that $E[\rho X] < +\infty$ and $V_+(X) = +\infty$.

This theorem says that the model is ill-posed if the value of a nonnegative claim with a finite price is infinite. Intuitively, in this case the agent can purchase such a claim initially (by taking out a loan if necessary) and reach the infinite value at the end. The following example shows that this could occur even with very "nice" parameters involved.

To exclude the ill-posed case identified by Theorem 3.1, we need the following assumption throughout this paper:

ASSUMPTION 3.1: $V_+(X) < +\infty$ for any nonnegative, $\mathcal{F}_T$-measurable random variable X satisfying $E[\rho X] < +\infty$.

[2] Even with a global concave utility function the underlying problem could still be ill-posed; see counter-examples and discussions in Korn and Kraft (2004) and Jin, Xu and Zhou (2006).

Assumption 3.1 is not sufficient to completely rule out the ill-posedness. The following theorem specifies another class of ill-posed problems.

THEOREM 3.2: If $u_+(+\infty) = +\infty$, $\bar{\rho} = +\infty$, and $T_-(x) = x$, then Problem (7) is ill-posed.

IV. SPLITTING

The key idea developed in this paper to solve (7) is based on the following observation: If X is a feasible solution of (7), then one can split X^+ and X^-. The former defines naturally an event $A := \{X \geq 0\}$ and an initial price $x_+ := E[\rho X^+]$, and the latter corresponds to A^C and $x_+ - x_0$, where A^C denotes the complement of the set A. An optimal solution to (7) should, therefore, induce the "best" such A and x_+ in certain sense. We now carry out this idea in the following two steps.

Step 1. In this step we consider two problems respectively:

- *Positive Part Problem*: A problem with parameters (A, x_+):

$$\begin{array}{ll} \text{Maximize} & V_+(X) \\ \text{subject to} & E[\rho X] = x_+, X \geq 0 \\ & X = 0 \text{ a.s. on } A^C, \end{array} \tag{9}$$

  where $x_+ \geq x_0^+$ (≥ 0) and $A \in \mathcal{F}_T$ with $P(A) \leq 1$ are given. Thanks to Assumption 3.1, $V_+(X)$ is a finite number for any feasible X. We define the optimal value of Problem (9), denoted $v_+(A, x_+)$, in the following way. If $P(A) > 0$, in which case the feasible region of (9) is non-empty [$X = x_+\mathbf{1}_A/\rho P(A)$ is a feasible solution satisfying all the constraints], then $v_+(A, x_+)$ is defined to be the supremum of (9). If $P(A) = 0$ and $x_+ = 0$, then (9) has only one feasible solution $X = 0$ a.s. and $v_+(A, x_+) := 0$. If $P(A) = 0$ and $x_+ > 0$, then (9) has no feasible solution, where we define $v_+(A, x) := -\infty$.

- *Negative Part Problem*: A problem with parameters (A, x_+):

$$\begin{array}{ll} \text{Minimize} & V_-(X) \\ \text{subject to} & \left\{ \begin{array}{l} E[\rho X] = x_+ - x_0, X \geq 0, \\ X = 0 \text{ a.s. on } A, \\ X \text{ is upper bounded a.s.}, \end{array} \right. \end{array} \tag{10}$$

  where $x_+ \geq x_0^+$ and $A \in \mathcal{F}_T$ with $P(A) \leq 1$ are given. Similarly to the positive part problem we define the optimal value $v_-(A, x_+)$ of Problem (10) as follows. When $P(A) < 1$ in which case the feasible region of (10) is non-empty, $v_-(A, x_+)$ is the infimum of (10). If $P(A) = 1$ and $x_+ = x_0$ where the only feasible solution is $X = 0$ a.s., then $v_-(A, x_+) := 0$. If $P(A) = 1$ and $x_+ \neq x_0$, then there is no feasible solution, in which case we define $v_-(A, x_+) := +\infty$.

Step 2. In this step we solve

$$\begin{array}{ll} \text{Maximize} & v_+(A, x_+) - v_-(A, x_+) \\ \text{subject to} & \left\{ \begin{array}{l} A \in \mathcal{F}_T,\ x_+ \geq x_0^+, \\ x_+ = 0 \text{ when } P(A) = 0, \\ x_+ = x_0 \text{ when } P(A) = 1. \end{array} \right. \end{array} \tag{11}$$

PROPOSITION 4.1: Given X^*, define $A^* := \{\omega : X^* \geq 0\}$ and $x_+^* := E[\rho(X^*)^+]$. Then X^* is optimal for Problem (7) if and only if (A^*, x_+^*) are optimal for Problem (11) and $(X^*)^+$ and $(X^*)^-$ are respectively optimal for Problems (9) and (10) with parameters (A^*, x_+^*).

Proposition 4.1 essentially shows that our problem (7) is completely equivalent to the set of problems (9) – (11).

Problem (11) is an optimization problem with the decision variables being a real number, x_+, and a random event, A, the latter being very hard to handle. We now show that one needs only to consider $A = \{\rho \leq c\}$, where c is a real number in certain range, when optimizing (11).

THEOREM 4.1: For any feasible pair (A, x_+) of Problem (11), $\exists\, c \in [\underline{\rho}, \bar{\rho}]$ such that $\bar{A} := \{\omega : \rho \leq c\}$ satisfies

$$v_+(\bar{A}, x_+) - v_-(\bar{A}, x_+) \geq v_+(A, x_+) - v_-(A, x_+). \tag{12}$$

Moreover, if Problem (9) admits an optimal solution with parameters (A, x_+), then the inequality in (12) is strict unless $P(A \cap \bar{A}^C) + P(A^C \cap \bar{A}) = 0$.

To simplify the notation, we now use $v_+(c, x_+)$ and $v_-(c, x_+)$ to denote $v_+(\{\omega : \rho \leq c\}, x_+)$ and $v_-(\{\omega : \rho \leq c\}, x_+)$ respectively.

In view of Theorem 4.1, one may replace Problem (11) by the following problem:

$$\begin{array}{ll} \text{Maximize} & v_+(c, x_+) - v_-(c, x_+) \\ \text{subject to} & \left\{ \begin{array}{l} \underline{\rho} \leq c \leq \bar{\rho},\ \ x_+ \geq x_0^+, \\ x_+ = 0 \text{ when } c = \underline{\rho}, \\ x_+ = x_0 \text{ when } c = \bar{\rho}. \end{array} \right. \end{array} \tag{13}$$

This is clearly a *much* simpler problem, being a constrained optimization problem (a mathematical programming problem) in $\mathbf{R}^2$.

Theorem 4.1 is one of the most important results in this paper. It discloses the form of a general solution to the behavioral model: the optimal wealth is the payoff of a combination of two binary options characterized by a single number c^*, as stipulated in the next theorem.

THEOREM 4.2: Given X^*, and define $c^* := F^{-1}(P\{X^* \geq 0\})$, $x_+^* := E[\rho(X^*)^+]$, where $F(\cdot)$ is the distribution function of ρ. Then X^* is optimal for Problem (7) if and only if (c^*, x_+^*) is optimal for Problem (13) and $(X^*)^+\mathbf{1}_{\rho \leq c^*}$ and $(X^*)^-\mathbf{1}_{\rho > c^*}$ are respectively optimal for Problems (9) and (10) with parameters $(\{\omega : \rho \leq c^*\}, x_+^*)$. Moreover, in this case $\{\omega : X^* \geq 0\}$ and $\{\omega : \rho \leq c^*\}$ are identical up to a zero probability set.

In the following two sections, we will solve the positive and negative part problems respectively to obtain $v_+(c, x_+)$ and $v_-(c, x_+)$. It turns out that the two problems require very different techniques to tackle.

V. POSITIVE PART PROBLEM

In this section we solve the positive part problem (9) for any $A = \{\omega : \rho \leq c\}$, $\underline{\rho} \leq c \leq \bar{\rho}$, and $x_+ \geq x_0^+$.

To solve Problem (9) for all $A = \{\omega : \rho \leq c\}$, we need the following assumption.

ASSUMPTION 5.1: $F^{-1}(z)/T'_+(z)$ is non-decreasing in $z \in (0,1]$, $\liminf_{x\to+\infty}\left(\frac{-xu''_+(x)}{u'_+(x)}\right) > 0$, and $E\left[u_+\left((u'_+)^{-1}(\frac{\rho}{T'_+(F(\rho))})\right)T'_+(F(\rho))\right] < +\infty$.

THEOREM 5.1: Let Assumption 5.1 hold. Given $A := \{\omega : \rho \le c\}$ with $\underline{\rho} \le c \le \bar{\rho}$, and $x_+ \ge x_0^+$.

(i) If $x_+ = 0$, then the optimal solution of (9) is $X^* = 0$ and $v_+(c, x_+) = 0$.

(ii) If $x_+ > 0$ and $c = \underline{\rho}$, then there is no feasible solution to (9) and $v_+(c, x_+) = -\infty$.

(iii) If $x_+ > 0$ and $\underline{\rho} < c \le \bar{\rho}$, then the optimal solution to (9) is $X^*(\lambda) = (u'_+)^{-1}\left(\frac{\lambda\rho}{T'_+(F(\rho))}\right)\mathbf{1}_{\rho\le c}$ with the optimal value $v_+(c, x_+) = E\left[u_+\left((u'_+)^{-1}(\frac{\lambda\rho}{T'_+(F(\rho))})\right)T'_+(F(\rho))\mathbf{1}_{\rho\le c}\right]$, where $\lambda > 0$ is the unique real number satisfying $E[\rho X^*(\lambda)] = x_+$.

Before we end this subsection, we state the following result which is useful in the sequel.

PROPOSITION 5.1: If $x_+ > 0$, then Problem (9) admits an optimal solution with parameters $(\{\rho \le c\}, x_+)$ only if $v_+(\bar{c}, x_+) > v_+(c, x_+)$ for any $\bar{c} > c$ satisfying $P\{c < \rho \le \bar{c}\} > 0$.

In other words, v_+ is *strictly* increasing in c.

VI. NEGATIVE PART PROBLEM

Now we turn to the negative part problem (10).

THEOREM 6.1: Assume that $u_-(\cdot)$ is strictly concave at 0. Given $A := \{\omega : \rho \le c\}$ with $\underline{\rho} \le c \le \bar{\rho}$, and $x_+ \ge x_0^+$.

(i) If $c = \bar{\rho}$ and $x_+ = x_0$, then the optimal solution of (10) is $X^* = 0$ and $v_-(c, x_+) = 0$.

(ii) If $c = \bar{\rho}$ and $x_+ \ne x_0$, then there is no feasible solution to (10) and $v_-(c, x_+) = +\infty$.

(iii) If $\underline{\rho} \le c < \bar{\rho}$, then $v_-(c, x_+) = \inf_{\bar{c}\in[c,\bar{\rho})} u_-\left(\frac{x_+-x_0}{E[\rho\mathbf{1}_{\rho>\bar{c}}]}\right)T_-(1-F(\bar{c}))$. Moreover, Problem (10) with parameters (A, x_+) admits an optimal solution X^* if and only if the following minimization problem

$$\min_{\bar{c}\in[c,\bar{\rho})} u_-\left(\frac{x_+-x_0}{E[\rho\mathbf{1}_{\rho>\bar{c}}]}\right)T_-(1-F(\bar{c})) \qquad (14)$$

admits an optimal solution $\bar{c}^*$, in which case $X^* = \frac{x_+-x_0}{E[\rho\mathbf{1}_{\rho>\bar{c}^*}]}\mathbf{1}_{\rho>\bar{c}^*}$, a.s..

VII. FINAL SOLUTION

We impose the following assumption in this section.

ASSUMPTION 7.1: $u_-(\cdot)$ is strictly concave at 0.

Consider the following auxiliary problem

$$\begin{array}{ll} \text{Maximize} & v_+(c,x_+) - u_-(\frac{x_+-x_0}{E[\rho\mathbf{1}_{\rho>c}]})T_-(1-F(c)) \\ \text{subject to} & \begin{cases} \underline{\rho} \le c \le \bar{\rho}, \ \ x_+ \ge x_0^+, \\ x_+ = 0 \text{ when } c = \underline{\rho}, \\ x_+ = x_0 \text{ when } c = \bar{\rho}, \end{cases} \end{array} \qquad (15)$$

here when $c = \bar{\rho}$ and $x_+ = x_0$, we define

$$u_-\left(\frac{x_+-x_0}{E[\rho\mathbf{1}_{\rho>c}]}\right)T_-(1-F(c)) := 0 \qquad (16)$$

THEOREM 7.1: We have the following conclusions:

(i) If X^* is optimal for Problem (7), then $c^* := F^{-1}(P\{X^* \ge 0\})$, $x_+^* := E[\rho(X^*)^+]$, where F is the distribution function of ρ, are optimal for Problem (15). Moreover, $\{\omega : X^* \ge 0\}$ and $\{\omega : \rho \le c^*\}$ are identical up to a zero probability set, and $(X^*)^- = \frac{x_+^*-x_0}{E[\rho\mathbf{1}_{\rho>c^*}]}\mathbf{1}_{\rho>c^*}$ a.s..

(ii) If (c^*, x_+^*) is optimal for Problem (15) and X_+^* is optimal for Problem (9) with parameters $(\{\rho \le c^*\}, x_+^*)$, then $X^* := (X^*)^+\mathbf{1}_{\rho\le c^*} - \frac{x_+^*-x_0}{E[\rho\mathbf{1}_{\rho>c^*}]}\mathbf{1}_{\rho>c^*}$ is optimal for Problem (7).

If in addition Assumption 5.1 holds, then $v_+(c, x_+)$ and the corresponding optimal solution X_+^* can be expressed more explicitly:

$$v_+(c,x_+) = E\left[u_+\left((u'_+)^{-1}\left(\tfrac{\lambda(c,x_+)\rho}{T'_+(F(\rho))}\right)\right)T'_+(F(\rho))\mathbf{1}_{\rho\le c}\right],$$
$$X_+^* = (u'_+)^{-1}\left(\tfrac{\lambda(c,x_+)\rho}{T'_+(F(\rho))}\right)\mathbf{1}_{\rho\le c},$$

where $\lambda(c, x_+)$ satisfies $E[(u'_+)^{-1}(\frac{\lambda(c,x_+)\rho}{T'_+(F(\rho))})\rho\mathbf{1}_{\rho\le c}] = x_+$. In this case Theorem 7.1 can be re-stated with the preceding explicit expressions properly substituted.

Under Assumption 5.1, the optimal terminal wealth to our behavioral model (6) is given explicitly as the following

$$X^* = (u'_+)^{-1}\left(\frac{\lambda(c^*,x_+^*)\rho}{T'_+(F(\rho))}\right)\mathbf{1}_{\rho\le c^*} - \frac{x_+^*-x_0}{E[\rho\mathbf{1}_{\rho>c^*}]}\mathbf{1}_{\rho>c^*}. \qquad (17)$$

This solution possesses some appealing features. On one hand, the terminal wealth having a gain or a loss is completely determined by the terminal state density price being lower or higher than a single threshold, c^*, which in turn can be obtained by solving (15). On the other hand, (17) is the payoff of a combination of two binary options, which can be easily priced;

In the light of Theorem 7.1, we have the following algorithm to solve Problem (7).

(1) Solve Problem (9) with given (c, x_+), where $\underline{\rho} \le c \le \bar{\rho}$ and $x_+ \ge x_0^+$, to obtain $v_+(c, x_+)$ and the optimal solution $X_+^*(c, x_+)$ (c.f. Theorem 5.1).

(2) Solve Problem (15) to get (c^*, x_+^*).

(3) (i) If $(c^*, x_+^*) = (\bar{\rho}, x_0)$, then $X_+^*(\bar{\rho}, x_0)$ solves Problem (7).

(ii) Else $X_+^*(c^*, x_+^*)\mathbf{1}_{\rho\le c^*} - \frac{x_+^*-x_0}{E[\rho\mathbf{1}_{\rho>c^*}]}\mathbf{1}_{\rho>c^*}$ solves Problem (7).

VIII. CONCLUDING REMARKS

In this paper, we introduce, for the first time in literature to our best knowledge, a general continuous-time portfolio selection model within the framework of the cumulative prospect theory, so as to account for human psychology and emotions in investment activities. The model features inherent difficulties, including non-convex/concave and non-smooth (overall) utility functions and probability distortions. Even the well-posedness of such a model becomes more an exception than a rule: We demonstrate that a well-posed model calls for a careful coordination among the underlying market, the utility function, and the probability distortions. We then develop an approach to solving the model thoroughly. Notwithstanding the complexity of the approach, the final solution turns out to be simply structured: the optimal terminal payoff is related to certain binary options characterized by a single number, and the optimal strategy is a gambling policy betting on good states of the market.

It should be emphasized that the agent under study in this paper is a "small investor" in that his behavior will not affect the market. Hence we can still comfortably assume some market properties, such as the absence of arbitrage and the market completeness, as usually imposed for the conventional utility model. (It remains an interesting problem to study a behavioral model in an incomplete market.) It is certainly a fascinating and challenging problem to study how the overall market might be changed by the joint behaviors of investors; e.g., a "behavioral" capital asset pricing model.

To conclude, this work is meant to be initiating and inspiring, rather than exhaustive and conclusive, for the research on intertemporal behavioral portfolio allocation.

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