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Constructive Lyapunov design of dynamic state feedback controllers

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Abstract— **In this paper we use filtered Lyapunov functions, introduced in previous works, to construct a general framework for the global stabilization of nonlinear systems. Filtered Lyapunov functions are Lyapunov functions which may depend on parameters satisfying differential equations. The main feature of filtered Lyapunov functions is that it is easy to construct and combine them even for nontriangular systems to obtain composite filtered Lyapunov functions which may be used for Lyapunov-based design of stabilizing controllers. Tools for the design of composite filtered Lyapunov functions are given and used to prove new global stabilization results via dynamic feedback.**

Index Terms— **Filtered Lyapunov functions, interconnected systems, dynamic feedback stabilization**

I. Introduction

The problem of constructing composite Lyapunov functions for interconnected systems has been the subject of many papers ([8], [12], [11],[14], [9]) and books ([15]). Dissipativity ([8]) and Lyapunov-based small-gain theorems ([12], [13] , [18], [3]) and finite gap conditions ([1]) establish elegant methods for the stability analysis of interconnected systems but do not point out any constructive procedure for a Lyapunov function of the interconnection, since in most cases the stability of the interconnection is proved by using a Lyapunov function defined on the system trajectories or resulting from smoothing out procedures ([17]). A constructive design of composite Lyapunov functions has been proposed in [14] and [11] in the case of triangular systems $\Sigma_1 : \dot{x} = f(x) + h(x,z)$, $\Sigma_2 : \dot{z} = a(z)$. In [14] the composite Lyapunov function is the sum of suitable nonlinear rescalings of the Lyapunov functions of each system Σ_1 and Σ_2 but it is assumed that $h(x,z)$ contains no linear terms in z (see also section 5.1.3 of [15]). In [11] no assumption is required on the term $h(x,z)$ and the composite Lyapunov function is the sum of the Lyapunov functions of each system Σ_1 and Σ_2 plus a suitable cross term. However, the definition of this cross term is given through a line integral along the trajectories of the interconnected system. Recently in [9] and [10] the constructive aspect of a composite Lyapunov function have been studied for nontriangular systems $\Sigma_1 : \dot{x} = f(x) + h(x,z)$, $\Sigma_2 : \dot{z} = a(z) + b(x,z)$, where each single system is ISS/iISS ([16], [2]) and a small-gain condition plus some additional conditions are satisfied. The proposed composite Lyapunov function is the sum of a nonlinear rescaling of the Lyapunov functions of each system Σ_1 and Σ_2. The state rescaling can be computed using certain maps which charaterize the ISS/iISS property of each system Σ_1 and Σ_2. However, while in the case of both ISS systems Σ_1 and Σ_2 the small-gain condition is the same one used in [12], [13] , [18] and [3] and no additional conditions are required, in the case of either Σ_1 or Σ_2 being iISS the required conditions are stronger than the tight ones given [1] (examples 4.2 and 4.1). In [1] it is shown that if either Σ_1 or Σ_2 is iISS then at least a "finite gap" condition must hold if we want to conclude stability of the interconnection on the basis of geometric conditions involving the nullclines of the dissipation inequalities characterizing the iISS properties of Σ_1 and Σ_2.

Recently a new type of Lyapunov function has been introduced in [4]. Filtered Lyapunov functions are Lyapunov functions which depend on time-varying parameters satisfying suitable differential equations. These differential equations can be implemented as dynamical filters. The flexibility of this new type of Lyapunov functions can be seen in the design of composite Lyapunov functions for interconnected systems $\Sigma_1 : \dot{x} = f(x) + h(x,z)$, $\Sigma_2 : \dot{z} = a(z) + b(x,z)$ consisting of a "filtered" combination of the Lyapunov functions $W_1(x)$ and $W_2(z)$ for Σ_1 and Σ_2, i.e. $\theta[W_1(x) + \mathfrak{d} W_2(z)]$ where $\mathfrak{d} > 0$ and θ is the output of a filter implemented by using the mixed terms in x and z appearing in the time derivatives $\dot{W}_1(x)$ and $\dot{W}_2(z)$.

Following the preliminary work [4], we want to give the following contributions in this paper.

- We establish more profound results on the construction of filtered Lyapunov functions than in [4]. We consider the case of triangular systems $\Sigma_1 : \dot{x} = f(x) + h(x,z)$, $\Sigma_2 : \dot{z} = a(z)$, where $h(x,z)$ may contain linear terms in z, and the case of nontriangular systems satisfying different types of small-gain and finite gap conditions ([12], [18], [3], [1]). In particular, in the case of both ISS systems Σ_1 and Σ_2 we show how a filtered Lyapunov function can be easily designed for the interconnection of Σ_1 and Σ_2: in this case, the constructive result of [9] and [10] gives already a Lyapunov function as the sum of suitable rescalings of the Lyapunov functions for Σ_1 and Σ_2. In the case of either Σ_1 and Σ_2 being iISS the conditions required in [9] and [10] are stronger than the finite gap conditions given in [1] (examples 4.2 and 4.1). In a more detailed paper we will show that we are able to construct a filtered Lyapunov function 1) if the finite gap condition of [1] holds and 2) the finite gap condition of [1] does not hold but global asymptotic stability of the interconnection is known and a local small-gain condition is satisfied. In this paper we limit ourselves to give some examples which let understand this.

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- Secondly, using the design of filtered Lyapunov functions, we obtain constructive results concerning global stabilization of different classes of nonlinear systems. In doing this, we follow closely [11]. As a first class of systems we consider

$$\begin{aligned}\dot{x} &= f(x) + h(x,z) + g(x,z)v \\ \dot{z} &= a(z) + c(x,z)v \end{aligned} \quad (1)$$

with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $v \in \mathbb{R}^r$ the control vector, $f : \mathbb{R}^n \to \mathbb{R}^n$, $a : \mathbb{R}^m \to \mathbb{R}^m$, $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n \times r}$ and $c : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m \times r}$ locally Lipschitz continuous functions. Note that the "core" system $\dot{x} = Fx + h(x,z)$, $\dot{z} = a(z)$ is the uncontrolled dynamics of (1). Our stabilization results are based on the knowledge of a filtered Lyapunov function for the core system, designed according to the lines stipulated in this paper. In particular, if a Lyapunov function $W(x,z)$ is known for the core system with nonpositive time derivative $\dot{W}(x,z)$ then a feedback control which preserves global stability in (1) is $v = -g^T(x,z)(\frac{\partial W}{\partial x}(x,z))^T - (c^T(x,z)\frac{\partial W}{\partial z}(x,z))^T$. By adding suitable detectability assumptions, we prove the global asymptotic stability of the closed-loop system.

As a second class of systems, we consider

$$\begin{aligned}\dot{x} &= f(x) + h(x,z) + g(x,z,y)y \\ \dot{z} &= a(z) + b(x,z) + c(x,z,y)y \\ \dot{y} &= v \end{aligned} \quad (2)$$

with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $y \in \mathbb{R}^r$, $v \in \mathbb{R}^r$ the control vector, $f : \mathbb{R}^n \to \mathbb{R}^n$, $a : \mathbb{R}^m \to \mathbb{R}^m$, $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $b : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^{n \times r}$ and $c : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^{m \times r}$ locally Lipschitz continuous functions. In this case the core system $\dot{x} = f(x) + h(x,z)$, $\dot{z} = a(z) + b(x,z)$ is the zero dynamics of (2) with output y. Although not shown in this paper, our stabilization results are based on the knowledge of a filtered Lyapunov function for the core system $\dot{x} = f(x) + h(x,z)$, $\dot{z} = a(z) + b(x,z)$, designed according to the lines stipulated in this paper. Since the relative degree of (2)with output y is one and the zero dynamics of (2) with output y is globally stable together with a Lyapunov function obtained by our constructive procedure, we use the feedback passivation approach ([6]) for stabilizing (2).

II. Notations

Before going further, we give some notations extensively used throughout the paper.

- $\|v\| = \sqrt{v^T v}$ denotes the euclidean norm of any given vector v and $\|v\|_A := \sqrt{v^T A v}$ for any positive semidefinite matrix A. Let $\mathbb{R}^s$ be the vector space of s-dimensional real column vectors; $\mathbb{R}_+$ (resp. $\mathbb{R}_\geq$) denotes the set of positive (resp. nonnegative) real numbers and $\mathbb{C}_-$ (resp. $\mathbb{C}_\leq$) the set of complex numbers with negative (resp. nonpositive) real part; $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ matrices. For any matrix $A \in \mathbb{R}^{n \times n}$ we denote by $\sigma(A)$ its spectrum.
- By $f \circ g$ we denote composition of functions f and g and by fg we denote product of functions f and g.
- Let $\mathscr{X} \subset \mathbb{R}^q$ and $\mathscr{Y} \subset \mathbb{R}^r$. We denote by $\mathbf{C}^0(\mathscr{X},\mathscr{Y})$ the set of continuous functions $f : \mathscr{X} \to \mathscr{Y}$, $\mathbf{C}^j(\mathscr{X},\mathscr{Y})$, $j = 1,\ldots,\infty$, the set of j times continuously differentiable functions $f : \mathscr{X} \to \mathscr{Y}$, by $\mathbf{L}_j(\mathscr{X},\mathscr{Y})$, $j = 1,\ldots,p < +\infty$, the set of continuous functions $f : \mathscr{X} \to \mathscr{Y}$ such that $(\int_0^\infty \|f(s)\|^j ds)^{\frac{1}{j}} < +\infty$ and by $\mathbf{L}_\infty(\mathscr{X},\mathscr{Y})$ the set of continuous functions $f : \mathscr{X} \to \mathscr{Y}$ such that $\sup_{s \in \mathscr{X}} \|f(s)\| < +\infty$.
- A function $\alpha \in \mathbf{C}^0([0,r],\mathbb{R}_\geq)$, $r \in (0,\infty]$, is said to be of class $\mathscr{K}_+$ (or $\alpha \in \mathscr{K}_+$) if $\alpha(0) \in \mathbb{R}_\geq$ and it is nondecreasing. A function $\alpha \in \mathbf{C}^0([0,r],\mathbb{R}_\geq)$, $r \in (0,\infty]$, is said to be of class $\mathscr{K}$ (or $\alpha \in \mathscr{K}$) if $\alpha(0) = 0$ and it is increasing; a function $\alpha \in \mathbf{C}^0([0,r],\mathbb{R}_\geq)$, $r \in (0,\infty]$, is said to be of class $\mathscr{K}_\infty$ (or $\alpha \in \mathscr{K}_\infty$) if $\alpha \in \mathscr{K}$ and $\lim_{s \to +\infty} \alpha(s) = +\infty$. A function $\alpha \in \mathbf{C}^0([0,r],\mathbb{R}_\geq)$, $r \in (0,\infty]$, is said to be of class $\mathscr{L}$ (or $\alpha \in \mathscr{L}$) if $\alpha(0) = 0$ and it is strictly decreasing and $\lim_{s \to +\infty} \alpha(s) = 0$.

III. Filtered Lyapunov functions

In a context in which a system has exogenous inputs and some of these may be the states of some other interconnected system, it is natural to introduce Lyapunov functions which may depend on *parameters satisfying differential equations*. When interconnecting two systems Σ_1 and Σ_2 with Lyapunov functions W_1 and W_2, a simple combination $W_1\theta_1 + W_2\theta_2$, $\theta_1, \theta_2 > 0$, may be not a candidate Lyapunov function for the interconnection Σ of Σ_1 and Σ_2 (see a detailed discussion in [15]). In this section we investigate the problem of constructing simple Lyapunov functions for *triangular* Σ.

A. Introducing filtered Lyapunov functions

Consider the system

$$\Sigma : \begin{cases} \Sigma_1 : \dot{x} = f(x) + h(x,z) \\ \Sigma_2 : \dot{z} = a(z) \end{cases} \quad (3)$$

with $f(0) = h(x,0) = 0$, $a(0) = 0$, $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$, $f \in \mathbf{C}^0(\mathbb{R}^n,\mathbb{R}^n)$, $h \in \mathbf{C}^0(\mathbb{R}^n \times \mathbb{R}^m,\mathbb{R}^n)$ and $a \in \mathbf{C}^0(\mathbb{R}^m,\mathbb{R}^m)$ locally Lipschitz functions. Assume

(A) *the existence of proper and positive definite* $W \in \mathbf{C}^2(\mathbb{R}^n,\mathbb{R}^\geq)$ *and* $V \in \mathbf{C}^2(\mathbb{R}^m,\mathbb{R}_\geq)$, *with* $\frac{\partial^2 V}{\partial z^2}(0) = P > 0$ *(i.e.* $V(z)$ *is locally quadratic around zero),* $\tau \in \mathscr{K}_+$, $\varphi \in \mathbf{C}^0(\mathbb{R}^n,\mathbb{R}_\geq)$ *and* $\kappa \in \mathscr{K}$, *with* $\kappa(s) = \kappa_0 s$ *for all* s *near the origin (i.e.* $\kappa(s)$ *is linear near zero), for which along the trajectories of* Σ

$$\begin{aligned}&\dot{W}(x) \leq -\varphi(x) \\ &+(\tau \circ V)(z)\Big[(\kappa \circ V)(z) + \sqrt{(\kappa \circ V)(z)}W(x)\Big] \\ &\dot{V}(z) \leq -(\kappa \circ V)(z). \end{aligned} \quad (4)$$

Remark *3.1:* By following the proof of lemma A.1 of [11] it is possible to show that the the trajectories $(x(t), z(t))$ of Σ are bounded for all $t \geq 0$.□

A composite Lyapunov function $W(x) + \Psi(x, z) + V(x)$ for Σ can be designed as pointed out in [15] by defining the cross term $\Psi(x, z)$ as a suitable line integral along the trajectories of Σ. However, the calculation of the cross term $\Psi(x, z)$ is not feasible in most cases, since it is defined on the state trajectories $(x(t), z(t))$. Also the simpler Lyapunov function $\alpha(W(x)) + \beta(V(x))$ proposed in [14] cannot be used for Σ, since $h(x, z)$ may contain linear terms in z (see section 5.1.3 of [15]). Finally, since (A) does not imply that Σ_1 be either iISS or ISS ([16], [2]) even the constructive procedure of [9] and [10] cannot be invoked.

In order to investigate different and simpler ways of implementing a Lyapunov function for Σ, consider the *filtered* combination of the Lyapunov functions $W(x)$ and $V(z)$

$$\tilde{W}(x, z, \theta) = \theta[W(x) + \mathfrak{d}V(z)], \tag{5}$$

with $\mathfrak{c} \in \mathbb{R}_+$, $\mathfrak{d} > \tau(\mathfrak{c})$ and θ being the output of the following dynamical filter

$$\dot{\theta}(t) = -\Big[\frac{\max\{(\tau \circ V)(z(t)) - \tau(\mathfrak{c}), 0\}(\kappa \circ V)(z(t))}{\mathfrak{c}\mathfrak{d}} + (\tau \circ V)(z(t))\sqrt{(\kappa \circ V)(z(t))}\Big]\theta(t), \theta(0) = 1 \tag{6}$$

Therefore, on account of (A) along the trajectories of (3)-(6)

$$\dot{\tilde{W}}(x, z, \theta) \leq -\theta\{\varphi(x) + [\mathfrak{d} - \tau(\mathfrak{c})](\kappa \circ V)(z)\} \tag{7}$$

We want to prove the following claim.

Claim *3.1: $\theta(t)$ is positive, bounded by 1 and nonincreasing for all $t \geq 0$ and for each trajectory $(x(t), z(t))$ of Σ.*

Proof: Fix a trajectory $(x(t), z(t))$ of (3). By the second of (4) it is easy to conclude that for each $\mathfrak{c} > 0$ the trajectory $z(t)$ is captured in finite time $T \geq 0$ by the set $\{z : 0 \leq V(z) \leq \mathfrak{c}\}$. Therefore, since $\tau \in \mathscr{K}_+$, $(\tau \circ V)(z(t)) \leq \tau(\mathfrak{c})$ for all $t \geq T$ and

$$\exp\{\int_0^t \frac{\max\{(\tau \circ V)(z(s)) - \tau(\mathfrak{c}), 0\}(\kappa \circ V)(z)}{\mathfrak{c}\mathfrak{d}} ds\} = \exp\{\int_0^T \frac{\max\{(\tau \circ V)(z(s)) - \tau(\mathfrak{c}), 0\}(\kappa \circ V)(z)}{\mathfrak{c}\mathfrak{d}} ds\} \tag{8}$$

for all $t \geq T$. Moreover, by the second of (4) and since $\kappa(s) = \kappa_0 s$ for all s near the origin, $\mathfrak{c} \in \mathbb{R}_+$ can be selected so that

$$V(z(t)) \leq \mathfrak{c}\exp\{-\kappa_0(t - T)\} \tag{9}$$

for all $t \geq T$. On account of (8)-(9) and linearity of $\kappa(s)$ near the origin, integration of (6) yields that $\theta(t)$ is defined and positive for all $t \geq 0$. By (6) $\theta(t)$ is also decreasing and bounded by 1 for all $t \geq 0$. ■

The fact that $\theta(t)$ is bounded, positive and nonincreasing for all $t \geq 0$ however does not prevent $\theta(t)$ from going to zero as $t \to \infty$. This fact is crucial when using the filtered Lyapunov function $\tilde{W}(x, z, \theta)$ for concluding asymptotic convergence to zero of $(x(t), z(t))$ (section III-C). The answer to this issue follows from the local exponential stability of Σ_2 and it is contained in the next claim (not proved here).

Claim *3.2: Along each trajectory $(x(t), z(t))$ we have $\inf_{t \geq 0}\theta(t) > 0$.*

Example *3.1:* Consider the system ([11])

$$\begin{aligned}\dot{x} &= \frac{x^2 z}{1 + x^2}\\ \dot{z} &= -z\end{aligned} \tag{10}$$

It is easy to see that along the trajectories of (10) $\dot{W}(x) \leq 2\sqrt{V(z)}W(x)$ and $\dot{V}(z) \leq -V(z)$, with $V(z) = z^2$ and $W(x) = x^2$. Therefore, we define $\tilde{W}(x, z, \theta) = \theta[x^2 + 4z^2]$, with $\dot{\theta} = -2|z|\theta$ and $\theta(0) = 1$. Note that $z(t) = z(0)e^{-t}$ and therefore $\theta(t) = \exp\{2|z(0)|(e^{-t} - 1)\}$ with $\inf_{t \geq 0}\theta(t) = \exp\{-2|z(0)|\} > 0$. The simpler construction of a composite Lyapunov function $\alpha(W(x)) + \beta(V(x))$ proposed in [14] cannot be applied, since the term $h(x, z) = \frac{x^2 z}{1+x^2}$ is linear in z.□

However if $h(x, z)$ contains a term like $H_0 z$, then the first of (4) cannot be satisfied even in simple cases, while this is not a limitation in [11]. By restricting to $f(x) = Fx$ ([11]) and noting that in virtue of (4) $A = \frac{\partial a}{\partial z}(0)$ and F satisfy a nonresonance condition, in the next section we show how to add a term $z^T\Psi_0 z + z^T\Psi_1 x$ in our filtered Lyapunov function to fix this limitation. The idea of introducing such a term to cope with linear terms $H_0 z$ in $h(x, z)$ has been suggested in [15].

B. Linear cross term

Consider the system

$$\Sigma : \begin{cases} \Sigma_1 : \dot{x} = Fx + h(x, z) \\ \Sigma_2 : \dot{z} = a(z) \end{cases} \tag{11}$$

with $h(x, 0) = 0$, $a(0) = 0$, $\frac{\partial h}{\partial z}(0, 0) = H_0$, $\frac{\partial a}{\partial z}(0) = A$, $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$, $F \in \mathbb{R}^{n\times n}$, $h \in \mathbf{C}^0(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ and $a \in \mathbf{C}^0(\mathbb{R}^m, \mathbb{R}^m)$ locally Lipschitz functions such that

$$\|h(x, z) - H_0 z\| \leq \gamma_1(\|z\|) + \gamma_2(\|z\|)\|x\| \tag{12}$$

for all x, z and for some locally at 0 Lipschitz $\gamma_1, \gamma_2 \in \mathscr{K}$ ((A1) in [11])). Assume

(B1) *the existence of proper and positive definite $W \in \mathbf{C}^2(\mathbb{R}^n, \mathbb{R}^{\geq})$ and $V \in \mathbf{C}^2(\mathbb{R}^m, \mathbb{R}^{\geq})$, with $\frac{\partial^2 W}{\partial x^2}(0) = Q > 0$ and $\frac{\partial^2 V}{\partial z^2}(0) = P > 0$, $\tau \in \mathscr{K}^+$, $\varphi \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}_{\geq})$, $M \in \mathbb{R}^{n\times m}$, $\tau \in \mathscr{K}_+$ and $\kappa \in \mathscr{K}$, with $\kappa(s) = \kappa_0 s$ for all s near the origin, such that along the trajectories of Σ*

$$\begin{aligned}&\dot{W}(x) \leq -\varphi(x) + x^T M z\\ &+(\tau \circ V)(z)\Big[(\kappa \circ V)(z) + \sqrt{(\kappa \circ V)(z)}W(x)\Big]\\ &\dot{V}(z) \leq -(\kappa \circ V)(z).\end{aligned} \tag{13}$$

Remark *3.2:* By following the proof of lemma A.1 of [15]) it is possible to show that the trajectories $(x(t), z(t))$ of (11) are bounded for all $t \geq 0$. □

By the first of (13) with $z = 0$ we conclude that $\sigma(F) \subset \mathbb{C}_{\leq}$ and by the second of (13) we conclude that $\sigma(A) \subset \mathbb{C}_-$.

Let $\Psi_0 \in \mathbb{R}^{m\times m}$ symmetric and $\Psi_1 \in \mathbb{R}^{m\times n}$ be such that

$$\Psi_0 A + A^T\Psi_0 = -\frac{1}{2}[\Psi_1 H_0 + H_0^T \Psi_1^T]$$
$$\Psi_1 F + A^T\Psi_1 = -M^T \qquad (14)$$

The matrix equations (14) admit a unique solution on account of the fact that $\sigma(A) \subset \mathbb{C}_-$ and $\sigma(A) \cap \sigma(-F) = \{\emptyset\}$. Moreover, on account of (12) and linearity of $\kappa \in \mathscr{K}$ near the origin, we can find $\lambda \in \mathscr{K}_+$ such that for all x, z

$$\|z^T\Psi_1[h(x,z) - H_0 z] + (2z^T\Psi_0 + x^T\Psi_1^T)(a(z) - Az)\| \leq (\lambda \circ V)(z)\Big[(\kappa\circ V)(z) + \sqrt{(\kappa\circ V)(z)}W(x)\Big]. \qquad (15)$$

Define

$$\tilde{W}(x,z,\theta) = \theta[W(x) + z^T\Psi_0 z + z^T\Psi_1 x + \mathfrak{d}V(z)] \qquad (16)$$
$$\dot{\theta}(t) = -\Big[\frac{\max\{(\tilde{\tau}\circ V)(z(t)) - \tilde{\tau}(\mathfrak{c}), 0\}(\kappa\circ V)(z(t))}{\mathfrak{c}\mathfrak{d}} + (\tilde{\tau}\circ V)(z(t))\sqrt{(\kappa\circ V)(z(t))}\Big]\theta(t), \theta(0) = 1 \qquad (17)$$

with $\tilde{\tau}(s) := \tau(s) + \lambda(s)$ and $\mathfrak{c} \in \mathbb{R}_+$ and $\mathfrak{d} > \tilde{\tau}(c)$ such that $W(x) + z^T\Psi_0 z + z^T\Psi_1 x + \mathfrak{d}V(z)$ is positive definite. The choice of such a $\mathfrak{d}$ is possible since $W(x) + z^T\Psi_0 z + z^T\Psi_1 x + \mathfrak{d}V(z)$ is locally quadratic around zero on account of $W(x)$ and $V(z)$ being locally quadratic around zero by (B1).

By (B1) we have along the trajectories of (11)-(17)

$$\dot{\tilde{W}}(x,z,\theta) \leq -\theta\{\varphi(x) + [\mathfrak{d} - \tilde{\tau}(\mathfrak{c})](\kappa\circ V)(z)\}. \qquad (18)$$

From this point we can argue as in section III-A to draw similar conclusions. We sum up the discussion of sections III-A and III-B into the following proposition.

Proposition *3.1: Consider the system (11) with* $a(0) = 0$, $h(x,0) = 0$, $\frac{\partial h}{\partial \xi}(0,0) = H_0$, $\frac{\partial a}{\partial \xi}(0) = A$, $x \in \mathbb{R}^n$ *and* $\xi \in \mathbb{R}^m$, $F \in \mathbb{R}^{n\times n}$, $h \in \mathbf{C}^0(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ *and* $a \in \mathbf{C}^0(\mathbb{R}^m, \mathbb{R}^m)$ *locally Lipschitz function such that (12) holds for all* x, ξ *and for some locally at* 0 *Lipschitz* $\gamma_1, \gamma_2 \in \mathscr{K}$*. Under assumptions (B1) the function* $\widetilde{W}(x,\xi,\theta)$*, defined in (16)-(17), satisfies (18) for some* $\tilde{\tau} \in \mathscr{K}_+$ *and* $\mathfrak{d} \in \mathbb{R}_+$ *and, in addition,* $\inf_{t\geq 0}\theta(t) > 0$ *along each trajectory of (11).*

If Fx *is replaced by any locally Lipschitz function* $f : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $f(0) = 0$ *and* $M = 0$ *and* $Q \geq 0$ *in (B1), the function* $\widetilde{W}(x,\xi,\theta)$*, defined in (5)-(6), satisfies (7) for some* $\mathfrak{d} \in \mathbb{R}_+$ *and, in addition,* $\inf_{t\geq 0}\theta(t) > 0$ *along each trajectory of (11).*

A filtered Lyapunov function can be used for designing *dynamic state feedback* stabilizing control laws. The main idea is illustrated through the following paradigm.

C. Using filtered Lyapunov functions for asymptotic stabilization with stable uncontrolled system

Consider the system (1) with $a(0) = 0$, $h(x,0) = 0$, $g(x,0) = g_0$, $c(x,0) = c_0$, $\frac{\partial h}{\partial z}(x,0) = H_0$, $\frac{\partial a}{\partial z}(0) = A$, $f(x) = Fx$, $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $v \in \mathbb{R}^r$ is the control vector, $F \in \mathbb{R}^{n\times n}$, $a \in \mathbf{C}^0(\mathbb{R}^m, \mathbb{R}^m)$, $h \in \mathbf{C}^0(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$, $g \in \mathbf{C}^0(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^{n\times r})$ and $c \in \mathbf{C}^0(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^{m\times r})$ locally Lipschitz functions with $h(x,z)$ and H_0 satisfying (12) for all x, z and for some locally at 0 Lipschitz $\gamma_1, \gamma_2 \in \mathscr{K}$. Assume (B1) and, in addition,

(B2) $W(x)$ *is quadratic*

(B3) *the pair* $\left(\begin{pmatrix} F & H_0 \\ 0 & A\end{pmatrix}, \begin{pmatrix} g_0 \\ c_0\end{pmatrix}\right)$ *is stabilizable.*

Remark *3.3:* Note that the uncontrolled system (1) is exactly equal to the core system (11). Condition (B3) guarantees that a certain zero-state detectability property with respect to the output v can be achieved for (11) after feedback.□

As remarked, the uncontrolled system (1) is globally stable. We want to find a state feedback controller which globally asymptotically stabilizes (1). Let Ψ_0 and Ψ_1 be as in (14), $\tilde{W}(x,z,\theta)$ as in (16)-(17), with $\mathfrak{d} > \tilde{\tau}(c)$ such that $W(x) + z^T\Psi_0 z + z^T\Psi_1 x + \mathfrak{d}V(z)$ is positive definite and $\tilde{\tau}(s) := \tau(s) + \lambda(s)$, where $\lambda \in \mathscr{K}_+$ is a continuous increasing function satisfying (15) for all x, z. Define the following *dynamic* state feedback control law

$$v = -\Big\{g^T(x,z)[(\frac{\partial W}{\partial x}(x))^T + \Psi_1^T z] + c^T(x,z)[\mathfrak{d}(\frac{\partial V}{\partial z}(z))^T + 2\Psi_0 z + \Psi_1 x]\Big\}\theta$$
$$\dot{\theta} = -\Big[\frac{\max\{(\tilde{\tau}\circ V)(z) - \tilde{\tau}(\mathfrak{c}), 0\}\sqrt{(\kappa\circ V)(z)}}{\mathfrak{c}\mathfrak{d}} + (\tilde{\tau}\circ V)(z)\Big]\sqrt{(\kappa\circ V)(z)}\theta, \theta(0) = 1. \qquad (19)$$

(any other control v as in (19) multiplied by some gain $\mathfrak{p} \in \mathbb{R}_+$ or some positive definite function of θ can be assumed as well). We want to prove that the closed-loop system (1)-(19) is globally asymptotically stable [1]. By direct calculations, we have along the trajectories of (1)-(19)

$$\dot{\tilde{W}}(x,z,\theta) \leq -\theta\{\varphi(x) + [\mathfrak{d} - \tilde{\tau}(\mathfrak{c})](\kappa\circ V)(z)\} - \|v\|^2 \qquad (20)$$

First, we prove boundedness of the trajectories $(x(t), z(t))$ of (1)-(19). By LaSalle invariance principle and using assumptions (B1)-(B3) we easily prove the following claims.

Claim *3.3: The trajectories* $(x(t), z(t))$ *of (1)-(19) are bounded for all* $t \geq 0$ *and along each such trajectory* $\inf_{t\geq 0}\theta(t) > 0$.

Claim *3.4: The trajectories* $(x(t), z(t))$ *of (1)-(19) globally asymptotically and locally exponentially tend to the origin.*

Claims 3.3 and 3.4 prove that the closed-loop system trejectories $(x(t), z(t))$ of (1)-(19) are bounded for all times and asymptotically driven to the origin. Therefore, global asymptotic stability of (1)-(19) follows if it is Lyapunov stable (partially with respect to x, z). From (20) and since $\theta(0) = 1$ it follows that

$$\theta(t)[W(x(t)) + \mathfrak{d}V(z(t))] \leq W(x(0)) + \mathfrak{d}V(z(0)) \qquad (21)$$

On the other hand by local exponential convergence to the origin of $z(t)$ and the existence for each trajectory $z(t)$ of $T \in \mathbb{R}_+$ such that $(\tilde{\tau}\circ V)(z(t)) \leq \tilde{\tau}(\mathfrak{c})$ for all $t \geq T$, we obtain $\psi \in \mathscr{L}$ such that $\theta(t) \geq \psi(\|z(0)\|)$ for all $t \geq 0$.

[1]By global (asymptotic) stability of (1)-(19) we mean partial global (asymptotic) stability with respect to x and z: def. 55.2 of [7]

Therefore, since V and W are proper and positive definite, there exist $\zeta_1, \zeta_2 \in \mathscr{K}_\infty$ such that

$$\begin{aligned}&\zeta_1(\|(x(t), z(t)\|) \leq W(x(t)) + \mathfrak{d}V(z(t))\\ &\leq \frac{W(x(0)) + \mathfrak{d}V(z(0))}{\psi(z(0))} \leq \zeta_2(\|(x(0), z(0)\|) \qquad (22)\end{aligned}$$

which implies Lyapunov stability of (1)-(19). We can collect the above conclusion into the following result.

Theorem *3.1: Consider the system (1) with $h(x,0) = 0$, $a(0) = 0$, $g(x,0) = g_0$, $c(x,0) = c_0$, $\frac{\partial h}{\partial z}(x,0) = H_0$, $\frac{\partial a}{\partial z}(0) = A$, $f(x) = Fx$ for all $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $F \in \mathbb{R}^{n\times n}$, and $a : \mathbb{R}^m \to \mathbb{R}$, $h \in \mathbf{C}^0(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$, $g \in \mathbf{C}^0(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^{n\times r})$ and $c \in \mathbf{C}^0(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^{m\times r})$ locally Lipschitz functions with $h(x,z)$ and H_0 satisfying (12) for all x, z. Under assumptions (B1)-(B3) the closed-loop system (1)-(19) is globally asymptotically stable and locally exponentially stable.*

D. Filtered Lyapunov function: definitions

The discussions of sections III-A, III-B and III-C motivate the following definition. This definition has been given under different but equivalent form in [4]. Let $\Sigma(z,\chi)$ be a given system with state $z \in \mathbb{R}^n$ and inputs $\chi \in \mathbb{R}^m$.

Definition *3.1: We say that $\tilde{W} \in \mathbf{C}^\infty(\mathbb{R}^n \times \mathbb{R}_\geq, \mathbb{R}_\geq)$ is a smooth filtered Lyapunov function for $\Sigma(z,\chi)$ if*
(i) *$\tilde{W}(z,\theta) = \theta W(z)$ with proper and positive definite $W \in \mathbf{C}^\infty(\mathbb{R}^n, \mathbb{R}^\geq)$,*
(ii) *there exists locally Lipschitz $\Gamma \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}_\geq)$ such that $\dot\theta(t) = -\Gamma(z(t))\theta(t)$, $\theta(0) \in \mathbb{R}_+$, for each trajectory $z(t)$ of $\Sigma(z,\chi)$,*
(iii) *there exist $\varphi_j \in \mathbf{C}^\infty(\mathbb{R}^n, \mathbb{R}_\geq)$ and $\alpha_j : \mathbf{C}^\infty \in (\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}_\geq)$ such that $\varphi_j(0) = \alpha_j(z,0) = 0$ for all z and along the trajectories of $\Sigma(z,\chi)$*

$$\dot{\tilde{W}}(z,\theta) \leq \theta\{-\varphi_j(z) + \alpha_j(z,\chi)\}. \qquad (23)$$

In the next section we see how to construct filtered Lyapunov functions for nontriangular interconnections, using the filtered Lyapunov functions of the simpler dynamics (core systems) in which these systems can be decomposed.

IV. Filtered Lyapunov functions for interconnected systems: general conditions

Let $\Sigma_j(z_j,\chi_j)$, $j = 1,2$, be given systems with smooth filtered Lyapunov functions $\tilde{W}_j(z_j,\theta_j)$. We can assume that $\chi_j = (z_i,\chi)$, $j \neq i$, for some exogenous inputs $\chi \in \mathbb{R}^m$, in other words we consider (z_i,χ) as inputs for Σ_j, and let $z = (z_1, z_2)$. Therefore, by definition 3.1 there exist $\Gamma_j \in \mathbf{C}^0(\mathbb{R}^{\mathfrak{n}_j}, \mathbb{R}_\geq)$, $j = 1,2$, such that $\dot\theta_j = -\Gamma_j(z_j)\theta_j$ and $\varphi_j \in \mathbf{C}^\infty(\mathbb{R}^{\mathfrak{n}_j}, \mathbb{R}_\geq)$ and $\alpha_j \in \mathbf{C}^\infty(\mathbb{R}^{\mathfrak{n}_j} \times \mathbb{R}^{\mathfrak{n}_i} \times \mathbb{R}^m, \mathbb{R}_\geq)$, $j = 1,2$, such that $\varphi_j(0) = \alpha_j(z_j,z_i,0) = 0$, $i \neq j$, for all z and along the trajectories of $\Sigma_j(z_j,\chi_j)$

$$\dot{\tilde{W}}_j(z_j,\theta_j) \leq \theta_j[-\varphi_j(z_j) + \alpha_j(z_j,\chi_j)] \qquad (24)$$

By smoothness of φ_j and α_j, $j = 1,2$, and since $\alpha_j(z_j,z_i,0) = 0$ for all z, we can assume the existence of $\gamma_j \in \mathbf{C}^\infty(\mathbb{R}^{\mathfrak{n}_j} \times \mathbb{R}^{\mathfrak{n}_i}, \mathbb{R}_\geq)$, $\tilde\varphi_{ji} \in \mathbf{C}^\infty(\mathbb{R}^{\mathfrak{n}_j}, \mathbb{R}_\geq)$, $i = 1 \ldots, \mathfrak{n}_j$, and $\tilde\alpha_j \in \mathbf{C}^\infty(\mathbb{R}^{\mathfrak{n}_j} \times \mathbb{R}^{\mathfrak{n}_i} \times \mathbb{R}^m, \mathbb{R}_\geq)$, $j = 1,2$, such that

$$\begin{aligned}\varphi_j(z_j) &= \textstyle\sum_{i=1}^{\mathfrak{n}_j} \tilde\varphi_{ji}(z_j) z_{ji}^2,\\ \alpha_j(z_j,\chi_j) &= \gamma_j(z) + \tilde\alpha_j(z,\chi)\end{aligned}$$

The main result of this section is inspired by the ideas presented in section III and points out the construction of a filtered Lyapunov function for the interconnection $\Sigma(z,\chi)$ of $\Sigma_1(z_1,\chi_1)$ and $\Sigma_2(z_2,\chi_2)$ as a " filtered" linear combination of $\tilde{W}_1(z_1,\theta_1)$ and $\tilde{W}_2(z_2,\theta_2)$. The conditions under which this construction is possible are the following: 1) the trajectories $z(t)$ are bounded for all times and are captured in finite time by some neighbourhood of the origin and 2) a local small-gain condition is satisfied. Property 1) follows straightforwardly from global asymptotic stability of the interconnection, which should be ascertained by using stability analysis tools such as small-gain or iISS/ISS theorems. Property 2) can be easily checked from the maps γ_j and φ_j, $j = 1,2$. Under conditions 1)-2) the parameter $\theta(t)$ of the filtered Lyapunov function of the interconnection Σ is not guaranteed to have a positive limit set. Therefore, we need the additional: 3) integrability of the maps $\Gamma_j(z_j)$, $j = 1,2$, along the trajectories $z(t)$. To a deeper analysis, this is not at all a strong requirement especially if we put our result in perspective of an iterative design: at a first level we construct a filtered Lyapunov function for two systems Σ_1 and Σ_2 with Lyapunov functions W_1 and W_2 and, therefore, we simply take $\theta_1 = \theta_2 = 1$ and $\Gamma_1(z_1) = \Gamma_2(z_2) \equiv 0$, which trivially satisfy condition 3) (sections III-A-III-B). At a second level we try to construct a filtered Lyapunov function for the interconnection of Σ_1 and Σ_2, on one hand, and Σ_3, on the other. Also in this case we simply take $\theta_3 = 1$ and $\Gamma_3(z_3) \equiv 0$ and we are left with checking condition 3) on the map $\Gamma(z_1,z_2)$ associated to the filtered Lyapunov function constructed for the interconnection of Σ_1 and Σ_2. On the other hand, this follows directly from boundedness and local exponential stability of the trajectories $(z_1(t), z_2(t))$, which should be ascertained through the Jacobian linearization of Σ around the origin.

The following result has been already proved under different forms in [4] and includes as a particular case the second part of proposition 3.1.

Theorem *4.1: Assume that Σ_j, $j = 1,2$, has smooth filtered Lyapunov function $\tilde{W}_j(z_j,\theta_j)$. Assume also that the existence of $\mathfrak{c}_i > 0$ and $\tau_i \in \mathbf{C}^0(\mathbb{R}_\geq \times \mathbb{R}_\geq, \mathbb{R}_\geq)$, $i = 1,2$, with $\tau_i(r,\cdot), \tau_i(\cdot,s) \in \mathscr{K}_+$ for each s and r, such that*
(i) *$\gamma_j(z) \leq \tau_j(W_1(z_1), W_2(z_2))\varphi_i(z_i)$, $j,i = 1,2$, $j \neq i$, for all z*
(ii) *$\tau_1(\mathfrak{c}_1,\mathfrak{c}_2)\tau_2(\mathfrak{c}_1,\mathfrak{c}_2) < 1$,*
(iii) *each trajectory $z(t)$ of $\Sigma(z,\chi)$ is bounded for all $t \geq 0$ and it is captured in finite time by the set $\mathscr{R} = \{(z_1,z_2) : \tau_j(W_1(z_1), W_2(z_2)) \leq \tau_j(\mathfrak{c}_1,\mathfrak{c}_2), j = 1,2\}$.*
There exist $\mathfrak{d}_1 \in \mathbb{R}_+$, $\mathfrak{d}_2 \in (\mathfrak{d}_1\tau_1(\mathfrak{c}_1,\mathfrak{c}_2), \frac{\mathfrak{d}_1}{\tau_2(\mathfrak{c}_1,\mathfrak{c}_2)})$ ($\mathfrak{d}_2 \in (\mathfrak{d}_1\tau_1(\mathfrak{c}_1,\mathfrak{c}_2), +\infty)$ if $\tau_2(s,r) \equiv 0$)

$$\tilde{W}(z,\theta) = \theta[\mathfrak{d}_1 W_1(z_1) + \mathfrak{d}_2 W_2(z_2)] \qquad (25)$$

is a smooth filtered Lyapunov function for $\Sigma(z,\chi)$. *If in addition*

(iv) $\Gamma_j(z_j) \in \mathbf{L}_1(\mathbb{R}_\geq,\mathbb{R}_\geq)$, $j = 1,2$, *for each trajectory of* $\Sigma(z,\chi)$,

then along each trajectory of $\Sigma(z,\chi)$

$$\inf_{t\geq 0}\theta(t) > 0. \tag{26}$$

Remark *4.1:* Assumption i) requires that γ_1 can be factored as $\varphi_2\tau_1$ and γ_2 as $\varphi_1\tau_2$. A sufficient condition for this being true is that $\frac{\varphi_2(0)}{\gamma_1(0)}, \frac{\varphi_1(0)}{\gamma_2(0)} < \infty$. If this is not the case, one can replace in (24) the functions $\varphi_1, \varphi_2 \in \mathscr{K} \cap \mathbf{C}^\infty(\mathbb{R}_\geq,\mathbb{R}_\geq)$ with some other $\tilde{\varphi}_1, \tilde{\varphi}_2 \in \mathscr{K} \cap \mathbf{C}^\infty(\mathbb{R}_\geq,\mathbb{R}_\geq)$ such that $\tilde{\varphi}_j(s) \leq \varphi_j(s)$, $j = 1,2$, for all $s \geq 0$ and $\frac{\tilde{\varphi}_2(0)}{\gamma_1(0)}, \frac{\tilde{\varphi}_1(0)}{\gamma_2(0)} < \infty$. □

Theorem 4.1 allows to prove global stabilization results for minimum phase systems with relative degree one such as (2) (with output y), where it is possible to identify a core system (the zero dynamics) for which we can construct a filtered Lyapunov function. This is possible under different small-gain and finite gap conditions, which guarantee the conditions of theorems 4.1. The filtered Lyapunov function of the core system can be used to establish constructive global stabilization results for (2) by using the feedback passivation approach ([6]). For lack of space we limit ourselves to give some examples of core systems for which a filtered Lyapunov functions can be designed in virtue of theorem 4.1 and no other constructive procedure applies. In particular, neither small-gain conditions of theorems 2 and 3 of [10] nor conditions of theorem 3 of [9] are met. Yet, a simple filtered Lyapunov function can be designed in virtue of theorem 4.1.

Example *4.1:* (*Local small-gain conditions with no escape to infinity*). Consider

$$\dot{W} = -\mathfrak{a}\frac{(1+W)^2-1}{(1+W)^2} + \mathfrak{b}[1+V^8][(1+V)^2-1],$$
$$\dot{V} = -\mathfrak{h}\frac{(1+V)^2-1}{(1+V)^2} + \mathfrak{g}\frac{(1+W)^2-1}{(1+W)^2}, \tag{27}$$

where (W,V) is assumed in the positive orthant and with $\mathfrak{a},\mathfrak{b},\mathfrak{h},\mathfrak{g} \in \mathbb{R}_+$ such that

$$\frac{(1+\mathfrak{c}_2)^2-1}{(1+\mathfrak{c}_2)^2} > \frac{\mathfrak{g}}{\mathfrak{a}\mathfrak{h}}, [1+\mathfrak{c}_2^8](1+\mathfrak{c}_2)^2 < \frac{\mathfrak{h}\mathfrak{a}}{\mathfrak{b}\mathfrak{g}},$$
$$\frac{(1+\mathfrak{c}_1)^2-1}{(1+\mathfrak{c}_1)^2} > \frac{\mathfrak{b}}{\mathfrak{a}}[1+\mathfrak{c}_2^8][(1+\mathfrak{c}_2)^2-1] \tag{28}$$

for some $\mathfrak{c}_1,\mathfrak{c}_2 \in \mathbb{R}_+$. These interconnections arise in the stabilization problem of wide classes of nonlinear systems including feedforward systems ([5]). In this case conditions of [1] are not satisfied.□

Example *4.2:* (*Global small-gain with infinite/finite gap conditions*). Consider the system (section 4.1 of [1]))

$$\dot{W} \leq -\frac{2W}{W+1} + \frac{V}{V+1}, \dot{V} \leq -\frac{2V}{V+1} + W. \tag{29}$$

where (W,V) is assumed in the positive orthant. In this case also conditions of [1] are satisfied.□

Example *4.3:* (*No gap conditions*). Consider the system (section 4.1 of [1])

$$\dot{W} = -\frac{W}{1+W} + V, \dot{V} = -V + \Big(\frac{W^2}{1+W+W^2}\Big)^n, \tag{30}$$

for $n \geq 1$ (if $n \in (0,1)$ unbounded trajectories do exist). This is an example in which the nullclines of the left hand sides of (30) have no gap at infinity. It should be clear from this example that our constructive procedure can be applied independently of how we guarantee the conditions of theorem 4.1 (small-gain, finite gap and so on).□

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