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H_∞ Boundary Control of Semilinear Heat Processes and Distributed Mechanical Oscillators: an LMI Approach

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Abstract— **Exponential stability analysis and L_2-gain analysis are developed for uncertain distributed parameter systems. Scalar heat processes and distributed mechanical oscillators, governed by semilinear partial differential equations of parabolic and, respectively, hyperbolic types, are chosen for treatment. Sufficient exponential stability conditions with a given decay rate are derived in the form of Linear Matrix Inequalities (LMIs) for an uncertain heat conduction equation and for an uncertain wave equation. These conditions are then utilized to synthesize H_∞ static output-feedback boundary controllers of the systems in question.**

Keywords: distributed parameter systems, stability, H_∞ control, Lyapunov functional, LMI.

I. Introduction

Many important plants, such as flexible manipulators and heat transfer processes are governed by partial differential equations and are often described by models with a significant degree of uncertainties. The existing results [2], [4], [5], [7] on robust control of distributed parameter systems, operating under uncertainty conditions, extend the state space or the frequency domain H_∞ approach and are confined to the linear case. It is thus of interest to develop consistent methods that are capable of utilizing nonlinear distributed parameter models and of providing the desired system performance in spite of significant model uncertainties. The LMI approach [3] is definitely among such methods and its extension to uncertain distributed parameter systems is the primary concern of the present paper.

In our recent papers [6], [11] we have introduced LMI approach to the stability analysis of linear heat and wave equations with the Dirichlet boundary conditions. In the present paper we extend the LMI approach to the Neumann boundary control stabilization and H_∞ control of uncertain systems. The paper is organized as follows. Exponential stability analysis and L_2-gain analysis are developed side by side in Sections 2 and 3 for scalar heat processes and, respectively, for distributed mechanical oscillators, governed by semilinear partial differential equations of parabolic and of hyperbolic types. Sufficient exponential stability conditions with a given decay rate are derived in the form of LMIs for these systems. Capabilities of the LMI approach are then tested for designing H_∞ static output-feedback boundary controllers of the systems in question.

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A. *Notation and Preliminaries*

The notation used throughout is fairly standard. The superscript 'T' stands for matrix transposition, $\mathbf{R}^n$ denotes the n dimensional Euclidean space with the norm $|\cdot|$, $\mathbf{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathbf{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$.

Functions, continuous in all arguments and, respectively, continuously differentiable in all arguments, are referred to as of class C and of class C^1.

$L_2(a,b)$ is the Hilbert space of square integrable functions $z(\xi)$, $\xi\in[a,b]$ with the corresponding norm. $L_2(0,\infty;L_2(0,1))$ is the Hilbert space of square integrable functions $w(\cdot,t)\in L_2(0,\infty)$ with values $w(\xi,\cdot)\in L_2(a,b)$ and with the corresponding norm.

For later use, we recall the following.

Lemma 1: [12]. Let $z\in W^{1,2}([a,b],R)$ be a scalar function with $z(a)=0$. Then

$$\int_a^b z^2(\xi)d\xi \le \frac{(b-a)^2}{2}\int_a^b (z'(\xi))^2 d\xi. \tag{1}$$

II. Boundary Stabilization of a Semilinear Heat Equation

A. *Exponential Stability*

Consider the heat equation

$$\begin{aligned} z_t(\xi,t) &= \tfrac{\partial}{\partial\xi}[a(\xi,t,z(\xi,t))z_\xi(\xi,t)] + r_0(\xi,t,z(\xi,t))z(\xi,t) \\ &+ r_1(\xi,t,z(\xi,t))z(1,t),\ t\ge t_0,\ 0\le\xi\le 1 \end{aligned} \tag{2}$$

where $t_0\in R$ is an initial time instant, $a(\xi,t,z)$ and $r_i(\xi,t,z),\ i=0,1$ are functions of class C^1, which may be unknown and which satisfy the inequalities

$$|r_i|\le\beta_i,\ a\ge a_1>0 \tag{3}$$

for all $(\xi,t,z)\in[0,1]\times R^2$ and for some constants $\beta_0\ge 0,\beta_1\ge 0,\ a_1>0$, known *a priori*. Hereinafter, the dependence on time t and spatial variable ξ is suppressed whenever possible and the functions a and r_i will be written without arguments.

Equation (2) describes the nonlinear propagation of heat in a one-dimensional rod. Due to the presence of the boundary-value term $z(1,t)$ in the state equation, the above model particularly captures significant features of thermal instability in solid propellant rockets [1].

Let the above equation be coupled to the mixed boundary condition

$$z(0,t)=0,\quad z_\xi(1,t)=-kz(1,t),\ t\ge t_0 \tag{4}$$

with a parameter $k \geq 0$ and to the initial condition

$$z(\xi, t_0) = \phi(\xi) \in L_2(0,1). \quad (5)$$

Since the nonlinear terms in (2) locally meet the Lipshitz condition, a unique strong solution of the boundary-value problem (2)–(5) turns out to locally exist [9]. Throughout, only strong solutions are under study.

It is well-known [1] that the linear system (2), (4) with $k = 0$ and with constant coefficients $r_0 = 0, a = 1$ and $r_1 > 2$ is unstable. We are looking for exponential stability conditions for uncertain nonlinear system (2), (4) with $k \geq 0$.

Consider the following Lyapunov-Krasovskii functional

$$V(z(\cdot,t)) = \int_0^1 z^2(\xi,t)d\xi. \quad (6)$$

We aim to find conditions guaranteeing that along the solutions $z(\xi,t)$ of (2), (4) the inequality

$$\frac{d}{dt}V(z(\cdot,t)) + 2\delta V(z(\cdot,t)) \leq 0 \quad (7)$$

holds. Then by the comparison principle argument (Khalil, 1992), it would follow

$$\begin{aligned}&\int_0^1 z^2(\xi,t)d\xi = V(z(\cdot,t)) \leq V(z(\cdot,t_0))e^{-2\delta(t-t_0)}\\ &= e^{-2\delta(t-t_0)}\int_0^1 \phi^2(\xi)d\xi.\end{aligned}$$

Therefore, solutions of the boundary value problem (2), (4) would be globally continuable to the right and would satisfy the inequality

$$\int_0^1 z^2(\xi,t)d\xi \leq e^{-2\delta(t-t_0)}\int_0^1 \phi^2(\xi)d\xi, \quad \forall t \geq t_0. \quad (8)$$

The heat process (2), (4) would thus be exponentially stable in $L_2(0,1)$ with the decay rate δ.

Differentiating V along (2), integrating by parts and taking into account (4), we find that

$$\begin{aligned}&\tfrac{d}{dt}V + 2\delta V = 2\int_0^1 z(\xi,t)z_t(\xi,t)d\xi + 2\delta\int_0^1 z^2(\xi,t)d\xi\\ &= 2\int_0^1 z(\xi,t)[\tfrac{\partial}{\partial\xi}[az_\xi(\xi,t)] + r_0 z(\xi,t) + r_1 z(1,t)]d\xi\\ &+2\delta\int_0^1 z^2(\xi,t)d\xi = -2kaz^2(1,t) - 2\int_0^1 az_\xi^2(\xi,t)d\xi\\ &+2\int_0^1(\delta + r_0)z^2(\xi,t)d\xi + 2r_1\int_0^1 z(\xi,t)z(1,t)d\xi\\ &\leq -2ka_1 z^2(1,t) - 2\int_0^1 a_1 z_\xi^2(\xi,t)d\xi\\ &+2(\delta+\beta_0)\int_0^1 z^2(\xi,t)d\xi + 2r_1\int_0^1 z(\xi,t)z(1,t)d\xi.\end{aligned} \quad (9)$$

Applying inequality (1), we have

$$-2a_1\int_0^1 z_\xi^2 d\xi \leq -4a_1\int_0^1 z^2(\xi,t)d\xi.$$

We thus derive that

$$\frac{d}{dt}V + 2\delta V \leq \int_0^1 [z(\xi,t)\ z(1,t)]\Psi[z(\xi,t)\ z(1,t)]^T d\xi \leq 0$$

provided that the following LMI

$$\Psi \triangleq \begin{bmatrix} -4a_1 + 2(\delta+\beta_0) & r_1 \\ r_1 & -2ka_1 \end{bmatrix} \leq 0 \quad (10)$$

is feasible. Since LMI (10) is affine in r_1 and $r_1 \in [-\beta_1, \beta_1]$, the latter LMI is feasible if the following LMI

$$\begin{bmatrix} -4a_1 + 2(\delta+\beta_0) & \beta_1 \\ \beta_1 & -2ka_1 \end{bmatrix} \leq 0 \quad (11)$$

is feasible.

We note that the condition $\beta_0 < 2a_1$ is necessary for the feasibility of (11). For $\beta_1 = 0$ the system (2)–(5) is exponentially stable for all $k \geq 0$ and the resulting δ is given by

$$\delta = 2a_1 - \beta_0. \quad (12)$$

For $\beta_1 > 0$ (2)–(5) is exponentially stable with the decay rate $0 < \delta < 2a_1 - \beta_0$ for large enough $k > 0$ that can be found from the inequality

$$-4a_1 + 2(\delta+\beta_0) + \frac{\beta_1^2}{2ka_1} \leq 0. \quad (13)$$

Summarizing, the following result is concluded.

Theorem 1: Consider the boundary-value problem (2)–(5) with the assumptions above and with $\beta_0 < 2a_1$. Given $\delta \in (0, 2a_1 - \beta_0]$, let there exist k such that LMI (11) is feasible. Then a unique solution of (2)–(5) is globally continuable to the right and it satisfies (8).

B. H_∞ Boundary Control

Let us, along with the homogeneous heat process (2), consider its perturbed version

$$\begin{aligned}&z_t(\xi,t) = \tfrac{\partial}{\partial\xi}[az_\xi(\xi,t)] + r_0 z(\xi,t) + r_1 z(1,t) + bw(\xi,t),\\ &t \geq t_0,\ 0 \leq \xi \leq 1\end{aligned} \quad (14)$$

where $w(\xi,t) \in L_2(0,\infty; L_2(0,1))$ is an external disturbance; $b = b(\xi,t,z)$ is a function of class C^1, which is assumed to be uniformly bounded, i.e., $|b(\xi,t,z)| \leq b_1$ for all $(\xi,t,z) \in [0,1] \times R^2$ and some $b_1 > 0$.

While internally stabilizing the heat process, the influence of the admissible external disturbance $w(\xi,t) \in L_2(0,\infty; L_2(0,1))$ on the controlled output

$$\bar{z}(\xi,t) = [\alpha(\xi,t,z(\xi,t))z(\xi,t) \qquad d(t,z(1,t))u(t)]^T, \quad (15)$$

is to be attenuated through the actuation at the right-hand end

$$z(0,t) = 0, \quad z_\xi(1,t) = u(t),\ t \geq t_0. \quad (16)$$

Hereinafter, $u(t)$ is the control input, d and α are continuous functions, which are uniformly bounded

$$|\alpha(\xi,t,z)| \leq \alpha_1,\ |d(t,z)| \leq d_1, \quad (17)$$

for all $(\xi,t,z) \in [0,1] \times R^2$, where $\alpha_1 \geq 0$ and $d_1 \geq 0$ are some constants. Collocated sensing $y(t) = z(1,t)$ at the boundary $\xi = 1$ is the only available information on the process.

The following H_∞ – control problem is thus under study. Given $\gamma > 0$, it is required to find a linear static output feedback

$$u(t) = -kz(1,t), \quad (18)$$

that exponentially stabilizes the unperturbed process (4), (14) and leads to a negative performance index

$$J = \int_{t_0}^{\infty}\int_0^1 [\bar{z}^T(\xi,t)\bar{z}(\xi,t) - \gamma^2 w^2(\xi,t)]d\xi dt < 0, \quad (19)$$

for all $0 \neq w(\xi,t) \in L_2(0,\infty; L_2(0,1))$ and for all solutions of (14), (16), being initialized with zero data $z(\xi,t_0) = 0$ and being globally continuable to the right.

In order to solve the problem we carry out conditions that guarantee the following:

$$W(t) \triangleq p\frac{d}{dt}V + \int_0^1 [\bar{z}^T(\xi,t)\bar{z}(\xi,t) - \gamma^2 w^2(\xi,t)]d\xi < 0, \tag{20}$$

where $p > 0$, V is given by (6) and the temporal derivative is computed along the trajectories of closed-loop system (4), (14). Then integrating (9) in t from t_0 to ∞ and taking into account that $V \geq 0$ and $V(0) = 0$ would yield (19).

It is worth noticing that

$$\int_0^1 \bar{z}^T(\xi,t)\bar{z}(\xi,t)d\xi = \int_0^1 \alpha^2 z^2(\xi,t)d\xi + d^2k^2z^2(1,t).$$

Then similar to the previous section, we obtain

$$\begin{array}{l} W = \int_0^1 [2pz(\xi,t)z_t(\xi,t) + \bar{z}^T(\xi,t)\bar{z}(\xi,t) - \gamma^2 w^2(\xi,t)]d\xi \\ = \int_0^1 \Big[2pz(\xi,t)[\frac{\partial}{\partial\xi}[az_\xi(\xi,t)] + r_0 z(\xi,t) + r_1 z(1,t) \\ +bw(\xi,t)] + \bar{z}^T(\xi,t)\bar{z}(\xi,t) - \gamma^2 w^2(\xi,t)\Big]d\xi \\ \leq (-2ka_1p + d_1^2k^2)z^2(1,t) - 2p\int_0^1 a_1 z_\xi^2(\xi,t)d\xi \\ +2\beta_0 p \int_0^1 z^2(\xi,t)d\xi + \int_0^1 \Big[2pz(\xi,t)[r_1 z(1,t) \\ +bw(\xi,t)] + \alpha_1^2 z^2(\xi,t) - \gamma^2 w^2(\xi,t)\Big]d\xi. \end{array} \tag{21}$$

Furthermore, applying inequality (1) and setting $\zeta^T = [z(\xi,t)\ z(1,t)\ w(\xi,t)]$, we find that

$$W \leq \int_0^1 \zeta^T \Psi_\gamma \zeta d\xi < 0$$

if

$$\Psi_\gamma \triangleq \begin{bmatrix} -4a_1p + 2\beta_0 p + \alpha_1^2 & r_1 p & bp \\ * & -2kap + d_1^2k^2 & 0 \\ * & * & -\gamma^2 \end{bmatrix} < 0 \tag{22}$$

is feasible. By Schur complements, the latter inequality holds if

$$\begin{bmatrix} -4a_1p + 2\beta_0p + \alpha_1^2 & r_1p & bp & 0 \\ * & -2ka_1p & 0 & d_1k \\ * & * & -\gamma^2 & 0 \\ * & * & * & -1 \end{bmatrix} < 0. \tag{23}$$

Multiplying (23) by $diag\{p^{-1}, p^{-1}, 1, 1\}$ from the right and from the left, we denote $q = p^{-1}$ and $g = p^{-1}k$. By Schur complements formula we arrive at

$$\begin{bmatrix} -4a_1q + 2\beta_0q & r_1 & b & 0 & q\alpha_1^2 \\ * & -2ga_1 & 0 & d_1g & 0 \\ * & * & -\gamma^2 & 0 & 0 \\ * & * & * & -1 & 0 \\ * & * & * & * & -\alpha_1^2 \end{bmatrix} < 0. \tag{24}$$

LMI (24) is affine in r_1 and b and it is therefore feasible for all $r_1 \in [-\beta_1, \beta_1,\ b \in [-b_1, b_1]$ if it is feasible for $r_1 = \pm\beta_1$ and $b = \pm b_1$, thereby yielding 4 LMIs. It is easy to see that these 4 LMIs are equivalent to the following LMI

$$\begin{bmatrix} -4a_1q + 2\beta_0q & \beta_1 & b_1 & 0 & q\alpha_1^2 \\ * & -2ga_1 & 0 & d_1g & 0 \\ * & * & -\gamma^2 & 0 & 0 \\ * & * & * & -1 & 0 \\ * & * & * & * & -\alpha_1^2 \end{bmatrix} < 0. \tag{25}$$

Thus, we proved the following.

Theorem 2: Consider the perturbed input-output system (14)–(16) with the assumptions above and with $\beta_0 < 2a_1$. Given $\gamma > 0$, let there exist $q > 0$ and g such that the LMI (25) is satisfied. Then the static output feedback (18) with $k = q^{-1}g$ internally exponentially stabilizes the boundary-value problem (14), (16) and attenuates the admissible perturbations $w(\xi,t) \in L_2(0,\infty; L_2(0,1))$ in the sense of (19).

C. Example

Consider (14)–(17) with

$$a_1 = 1,\ b_1 = 1,\ \beta_0 = 1,\ \beta_1 = 3,\ d_1 = 0.1,\ \alpha_1 = 1.$$

In this example $\beta_0 < 2a_1$ and $\beta_1 > 0$. Therefore, by Theorem 1 the static output feedback (18) with large enough $k > 0$ internally exponentially stabilizes the system which appears to be unstable for $k = 0$ (since $\beta_1 > 2a_1$ cf. [1]). By using LMI toolbox of Matlab to verify the feasibility of LMI (25), we find that the static output feedback (18) with $k = 10.1744$ internally exponentially stabilizes the system and leads to the disturbance attenuation level $\gamma = 3$. Substituting the resulting k into (13), we find that this gain exponentially stabilizes the system with $\delta = 0.7789$.

A lower L_2-gain $\gamma = 1.1$ is achieved by a higher gain $k = 106.01$. The decay rate by the latter gain is found to be $\delta = 0.9788$.

III. Boundary Stabilization of a Semilinear Wave Equation

A. Exponential Stability

Consider the wave equation

$$\begin{array}{l} z_{tt}(\xi,t) = \frac{\partial}{\partial\xi}[az_\xi(\xi,t)] + r_0 z_t(\xi,t) + r_1 z_t(1,t) \\ +r_2 z_\xi(\xi,t), \\ t \geq t_0,\ 0 \leq \xi \leq 1 \end{array} \tag{26}$$

where $a = a(\xi)$ and $r_i = r_i(\xi,t,z,z_t),\ i = 0,1,2$ are functions of class C^1. Equation (26) describes nonlinear oscillations of a string. As in the heat equation (2), the functions a and $r_i,\ i = 0,1,2$ are admitted to be unknown subject to inequalities (3) that hold for all $(\xi,t,z,z_t) \in [0,1] \times R^3$ with *a priori* known constants $\beta_i \geq 0, i = 0,1,2$ and $a_1 > 0$ whereas due to technical reasons, $a(\xi)$ does not depend of t, z, z_t anymore.

To facilitate exposition, we have ignored restoring stiffness of the string, implicitly assuming that the corresponding terms (such as $r(\xi,t,z,z_t)z(\xi,t)$ and $r(\xi,t,z,z_t)z(1,t)$) are negligible. Since the above simplified model captures all the essential features of the general treatment, the extension to a wave model with a nontrivial stiffness is indeed possible. For the sake of generality, we included the boundary-value term $r_1 z_t(1,t)$, similar to that of the parabolic equation (2). However, in contrast to the parabolic case, the stabilization problem still persists even without this term, because the wave equation (26) subject to the boundary condition (27), given below, may present instabilities under $r_1 \equiv 0$.

Let equation (26) be coupled to the mixed boundary condition

$$\begin{array}{l} z(0,t) = 0, \\ z_\xi(1,t) = -k z_t(1,t),\ t \geq 0, \end{array} \tag{27}$$

with a parameter $k > 0$ and to the initial condition

$$z(\xi,t_0) = \phi(\xi) \in L_2(0,1),\ z_t(\xi,t_0) = \phi_1(\xi) \in L_2(0,1). \tag{28}$$

As the nonlinear terms in (26) locally meet the Lipshitz condition, a unique strong solution of the boundary-value problem (26)–(28) turns out to locally exist [9]. As in the heat equation case, only strong solutions of the boundary-value problem (26)–(28) are under study.

It should be pointed out that the linear system (26), (27), specified with $k = 0$, $a = 1$, and $r_i = 0,\ i = 0,1,2$, generates oscillating solutions and it is therefore asymptotically unstable. As in the case of heat equation, we are looking for exponential stability conditions for uncertain nonlinear system (26), (27) with $k > 0$.

On solutions of (26)–(28), consider the Lyapunov-Krasovskii functional

$$\begin{array}{l} V(z_\xi(\cdot,t),z_t(\cdot,t)) = p\int_0^1 a z_\xi^2(\xi,t)d\xi + p\int_0^1 z_t^2(\xi,t)d\xi \\ +2\chi\int_0^1 \xi z_\xi(\xi,t) z_t(\xi,t)d\xi \end{array} \tag{29}$$

proposed in [10] with some constants $p > 0$ and χ such that

$$\begin{array}{l} V(z_\xi(\cdot,t),z_t(\cdot,t)) \geq \int_0^1 [z_\xi\ z_t] \left[\begin{array}{cc} a_1 p & \chi\xi \\ \chi\xi & p \end{array}\right] [z_\xi\ z_t]^T d\xi \\ > \varepsilon\int_0^1 [z_\xi^2(\xi,t) + z_t^2(\xi,t)]d\xi \end{array} \tag{30}$$

for some $\varepsilon > 0$. The latter inequality holds if

$$\left[\begin{array}{cc} a_1 p & \chi\xi \\ \chi\xi & p \end{array}\right] > 0,\ \forall\xi \in [0,1],$$

i.e., if

$$\left[\begin{array}{cc} a_1 p & \chi \\ \chi & p \end{array}\right] > 0. \tag{31}$$

Our aim is to find conditions that would guarantee that along (26) the inequality $\frac{d}{dt}V + 2\delta V \leq 0$ holds. Then by the comparison principle argument (Khalil, 1992), it would follow that

$$\begin{array}{l} \varepsilon\int_0^1 [z_\xi^2(\xi,t) + z_t^2(\xi,t)]d\xi = V(z_\xi(\cdot,t),z_t(\cdot,t)) \\ \leq V(z_\xi(\cdot,t_0),z_t(\cdot,t_0))e^{-2\delta(t-t_0)} \\ \leq Me^{-2\delta(t-t_0)}\int_0^1 [\phi_\xi^2(\xi) + \phi_1^2(\xi)]d\xi \end{array} \tag{32}$$

for some $M > 0$. Therefore, the solutions of the boundary value problem (26), (27) would be globally continuable to the right and would satisfy the inequality

$$\int_0^1 [z_\xi^2(\xi,t) + z_t^2(\xi,t)]d\xi \leq \frac{M}{\varepsilon}e^{-2\delta(t-t_0)}\int_0^1 [\phi_\xi^2(\xi) + \phi_1^2(\xi)]d\xi, \tag{33}$$

for all $t \geq t_0$. The boundary value problem (26), (27) would thus be exponentially stable with the decay rate δ.

For later use, we derive that

$$\begin{array}{l} \frac{d}{dt}\left(2\int_0^1 \xi z_t z_\xi d\xi\right) = 2\int_0^1 \xi z_{tt} z_\xi d\xi + 2\int_0^1 \xi z_t z_{\xi t} d\xi \\ = 2\int_0^1 \xi\frac{\partial}{\partial\xi}[a z_\xi(\xi,t)]z_\xi d\xi + 2\int_0^1 \xi z_t z_{\xi t} d\xi \\ +2\int_0^1 \xi[r_0 z_t(\xi,t) + r_1 z_t(1,t) + r_2 z_\xi(\xi,t)]z_\xi d\xi \\ = \int_0^1 \frac{1}{a}\xi\frac{\partial}{\partial_\xi}(a z_\xi)^2 d\xi + 2\int_0^1 \xi z_t z_{\xi t} d\xi \\ +2\int_0^1 \xi[r_0 z_t(\xi,t) + r_1 z_t(1,t) + r_2 z_\xi(\xi,t)]z_\xi d\xi \\ = -\int_0^1 a z_\xi^2 d\xi + a_{|\xi=1} z_\xi^2(1,\ t) + 2\int_0^1 \xi z_t(\xi,t) z_{\xi t} d\xi \\ +2\int_0^1 \xi[r_0 z_t(\xi,t) + r_1 z_t(1,t) + r_2 z_\xi(\xi,t)]z_\xi d\xi. \end{array}$$

After integrating by parts, we obtain

$$2\int_0^1 \xi z_t z_{\xi t} d\xi = -2\int_0^1 \xi z_{\xi t} z_t d\xi - 2\int_0^1 z_t^2 d\xi + 2z_t^2(1,\ t).$$

Therefore,

$$2\int_0^1 \xi z_t z_{\xi t} d\xi = -\int_0^1 z_t^2 d\xi + z_t^2(1,\ t),$$

which yields

$$\begin{array}{l} \frac{d}{dt}\left(2\int_0^1 \xi z_t z_\xi d\xi\right) = -\int_0^1 (z_t^2 + a z_\xi^2)d\xi + z_t^2(1,\ t) \\ +a_{|\xi=1} z_\xi^2(1,\ t) + 2\int_0^1 \xi[r_0 z_t(\xi,t) + r_1 z_t(1,t) \\ +r_2 z_\xi(\xi,t)]z_\xi d\xi. \end{array}$$

Thus, differentiating V along (26), we obtain

$$\begin{array}{l} \frac{d}{dt}V + 2\delta V \leq 2p\int_0^1 a z_\xi(\xi,t) z_{t\xi}(\xi,t)d\xi \\ +2p\int_0^1 z_t(\xi,t) z_{tt}(\xi,t)d\xi + \frac{d}{dt}\left(2\chi\int_0^1 \xi z_t z_\xi d\xi\right) \\ +\int_0^1 2\delta[ap z_\xi^2(\xi,t) + 2\chi\xi z_\xi(\xi,t) z_t^2(\xi,t) + p z_t^2(\xi,t)]d\xi \\ = 2p\int_0^1 [a z_\xi(\xi,t) z_{t\xi}(\xi,t) + z_t(\xi,t)\frac{\partial}{\partial\xi}[a z_\xi(\xi,t)]]d\xi \\ +2p\int_0^1 z_t(\xi,t)[r_0 z_t(\xi,t) + r_1 z_t(1,t) + r_2 z_\xi(\xi,t)]d\xi \\ +\chi\Big[-\int_0^1 (z_t^2 + a z_\xi^2)d\xi + z_t^2(1,\ t) + a_{|\xi=1} k^2 z_t^2(1,t) \\ +2\int_0^1 \xi[r_0 z_t(\xi,t) + r_1 z_t(1,t) + r_2 z_\xi(\xi,t)]z_\xi d\xi\Big] \\ +\int_0^1 2\delta[a z_\xi^2(\xi,t) + 2\chi\xi z_\xi(\xi,t) z_t(\xi,t) + z_t^2(\xi,t)]d\xi. \end{array}$$

Now integrating by parts and taking into account (26) and (27) yield

$$\begin{array}{l} \frac{d}{dt}V + 2\delta V \leq -2a_{|\xi=1} kp z_t^2(1,t) \\ +2p\int_0^1 z_t(\xi,t)[r_0 z_t(\xi,t) + r_1 z_t(1,t) + r_2 z_\xi(\xi,t)]d\xi \\ +\chi\Big[-\int_0^1 (z_t^2 + a z_\xi^2)d\xi + (1 + a_{|\xi=1} k^2) z_t^2(1,t) \\ +2\int_0^1 \xi[r_0 z_t(\xi,t) + r_1 z_t(1,t) + r_2 z_\xi(\xi,t)]z_\xi d\xi\Big] \\ +\int_0^1 2\delta[ap z_\xi^2(\xi,t) + 2\chi\xi z_\xi(\xi,t) z_t(\xi,t) + p z_t^2(\xi,t)]d\xi. \end{array} \tag{34}$$

Since

$$\begin{array}{l} 2\int_0^1 \chi\xi[r_0 z_t(\xi,t) + r_1 z_t(1,t) + r_2 z_\xi(\xi,t)]z_\xi(\xi,t)d\xi \\ \leq \int_0^1 \Big[\frac{\beta_0^2}{s_0} z_t^2(\xi,t) + \frac{\beta_1^2}{s_1} z_t^2(1,t) + (s_0 + s_1 \\ +2\chi\xi\beta_2) z_\xi^2(\xi,t)\Big]d\xi \end{array}$$

for some $s_0 > 0$ and $s_1 > 0$, by setting $\zeta^T(\xi,t) = [z_t(1,t)\ \ z_\xi(\xi,t)\ \ z_t(\xi,t)]$ and by taking into account that $a \geq a_1$, we conclude that

$$\frac{d}{dt}V + 2\delta V \leq \int_0^1 \zeta^T(\xi,t)\Psi\zeta(\xi,t)d\xi < 0,$$

if

$$\Psi = \begin{bmatrix} \psi_1 + \frac{\beta_1^2}{s_1} & 0 & pr_1 \\ * & \psi_2 & 2\chi\delta\xi + pr_2 \\ * & * & \psi_3 + \frac{\beta_0^2}{s_0} \end{bmatrix} < 0 \tag{35}$$

where

$$\begin{array}{l} \psi_1 = -2a_1kp + (1 + a_1k^2)\chi, \\ \psi_2 = -a_1\chi + 2\delta a_1 p + s_0 + s_1 + 2\chi\xi\beta_2, \\ \psi_3 = -\chi + 2p\beta_0 + 2\delta p. \end{array} \tag{36}$$

By Schur complements (35) holds if

$$\begin{bmatrix} \psi_1 & 0 & pr_1 & \beta_1 & 0 \\ * & \psi_2 & 2\chi\delta\xi + pr_2 & 0 & 0 \\ * & * & \psi_3 & 0 & \beta_0 \\ * & * & * & -s_1 & 0 \\ * & * & * & * & -s_0 \end{bmatrix} < 0. \tag{37}$$

It is worth noticing that given k, (37) is LMI which is affine in $\xi \in [0, 1]$, $r_i \in [-\beta_i, \beta_i], i = 1, 2$. Therefore, LMI (37) is feasible if the following LMIs in the eight vertices are feasible:

$$\begin{array}{l} \begin{bmatrix} \psi_1 & 0 & pr_1^{(i)} & \beta_1 & 0 \\ * & \psi_2^{(j)} & 2\chi\delta\xi^{(j)} + pr_2^{(l)} & 0 & 0 \\ * & * & \psi_3 & 0 & \beta_0 \\ * & * & * & -s_1 & 0 \\ * & * & * & * & -s_0 \end{bmatrix} < 0, \\ \psi_2^{(j)} = -a_1\chi + 2\delta a_1 p + s_0 + s_1 + 2\chi\xi^{(j)}\beta_2, \\ i = 1, 2;\ j = 1, 2;\ l = 1, 2; \\ r_m^{(1)} = \beta_m,\ r_m^{(2)} = -\beta_m,\ m = 1, 2;\ \xi^{(1)} = 0,\ \xi^{(2)} = 1. \end{array} \tag{38}$$

Since LMIs (38) for $r_1^{(i)} = \pm\beta_1$ are equivalent, it is sufficient to check the feasibility of the four LMIs (38), where $r_1^{(i)} = \beta_1$ and $j = 1, 2,\ l = 1, 2$. We note that for the stability analysis p can be chosen to be 1. Summarizing, the following result is obtained.

Theorem 3: Given $k > 0$ and $\delta > 0$, let the LMIs (31) and (38) with notations (36) and $p = 1$ hold for some χ, s_0 and s_1. Then the boundary-value problem (26), (27) is exponentially stable with the decay rate δ.

B. H_∞ Boundary Control

In addition to the wave equation (26), let us now consider its perturbed version

$$\begin{array}{l} z_{tt}(\xi, t) = \frac{\partial}{\partial\xi}[az_\xi] + r_0 z_t(\xi, t) + r_1 z_t(1, t) \\ +r_2 z_\xi(\xi, t) + bw(\xi, t), \\ t \geq 0,\ 0 \leq \xi \leq 1 \end{array} \tag{39}$$

where $w(\xi, t) \in L_2(0, \infty; L_2(0, 1))$ is an external disturbance; $b = b(\xi, t, z)$ is a function of class C^1, which is assumed to be uniformly bounded, i.e., $|b(\xi, t, z)| \leq b_1$ for all $(\xi, t, z) \in [0, 1] \times R^2$ and some $b_1 > 0$. While internally stabilizing the wave process, the influence of the admissible external disturbance on the controlled output

$$\bar{z}(\xi, t) = [\alpha z(\xi, t) \quad \bar{\alpha} z_t(\xi, t) \qquad d(t, z_t(1, t))u(t)]^T, \tag{40}$$

is to be attenuated through the actuation at the right-hand end

$$z(0, t) = 0, \quad z_\xi(1, t) = u(t),\ t \geq t_0. \tag{41}$$

Hereinafter, $u(t)$ is the control input, d and $\alpha = \alpha(\xi, t, z)$, $\bar{\alpha} = \bar{\alpha}(\xi, t, z)$ are continuous functions, which are uniformly bounded

$$|\alpha(\xi, t, z, z_t)| \leq \alpha_0,\ |\bar{\alpha}(\xi, t, z, z_t)| \leq \alpha_1,\ |d(t, z)| \leq d_1,$$

for all $(\xi, t, z, z_t) \in [0, 1] \times R^3$, where $\alpha_i \geq 0$, $i = 0, 1$ and $d_1 \geq 0$ are some constants. Collocated sensing $y(t) = z_t(1, t)$ at the boundary $\xi = 1$ is the only available information on the process.

The H_∞ –control problem of interest is stated as follows. Given $\gamma > 0$, it is required to find a linear static output feedback

$$u(t) = -kz_t(1, t), \tag{42}$$

that exponentially stabilizes the unperturbed system (26), (27) and leads to a negative performance index

$$J = \int_{t_0}^{\infty} \int_0^1 [\bar{z}^T(\xi, t)\bar{z}(\xi, t) - \gamma^2 w^2(\xi, t)]d\xi dt < 0 \tag{43}$$

for all $0 \neq w(\xi, t) \in L_2(0, \infty; L_2(0, 1))$ and for all solutions of (39), (41), being initialized with zero data $z(\xi, t_0) = z_t(\xi, t_0) = 0$ and being globally continuable to the right.

For solving the stated problem, let us find conditions that guarantee the following:

$$W(t) \triangleq \frac{d}{dt}V + \int_0^1 [\bar{z}^T(\xi, t)\bar{z}(\xi, t) - \gamma^2 w^2(\xi, t)]d\xi < 0, \tag{44}$$

where V is given by (29), and the temporal derivative is computed along the closed-loop system (39), (41). We have

$$\begin{array}{l} \int_0^1 \bar{z}^T(\xi, t)\bar{z}(\xi, t)d\xi \leq \int_0^1 \Big[\alpha_0^2 z^2(\xi, t) + \alpha_1^2 z_t^2(\xi, t) \\ +d_1^2 k^2 z_t^2(1, t)\Big]d\xi \leq \int_0^1 \Big[\frac{1}{2}\alpha_0^2 z_\xi^2(\xi, t) + \alpha_1^2 z_t^2(\xi, t) \\ +d_1^2 k^2 z_t^2(1, t)\Big]d\xi, \end{array}$$

where inequality (1) has been used. In analogy to (34), we have

$$\begin{array}{l} \frac{d}{dt}V \leq -2a_{|\xi=1}kpz_t^2(1, t) \\ +2p\int_0^1 z_t(\xi, t)[r_0 z_t(\xi, t) + r_1 z_t(1, t) + r_2 z_\xi(\xi, t) + bw]d\xi \\ +\chi\Big[-\int_0^1 (z_t^2 + az_\xi^2)d\xi + (1 + a_{|\xi=1}k^2)z_t^2(1, t) \\ +2\int_0^1 \xi[r_0 z_t(\xi, t) + r_1 z_t(1, t) + r_2 z_\xi(\xi, t) + bw]z_\xi d\xi\Big]. \end{array}$$

Since

$$\begin{array}{l} 2\int_0^1 \chi\xi[r_0 z_t(\xi, t) + r_1 z_t(1, t) + r_2 z_\xi(\xi, t) + bw]z_\xi(\xi, t)d\xi \\ \leq \int_0^1 \Big[\frac{\beta_0^2}{s_0}z_t^2(\xi, t) + \frac{\beta_1^2}{s_1}z_t^2(1, t) + \frac{b_1^2}{s_2}w^2 \\ +(s_0 + s_1 + s_2 + 2\chi\beta_2)z_\xi^2(\xi, t)\Big]d\xi \end{array}$$

for some $s_0 > 0$ and $s_1 > 0$. By taking into account that $a \geq a_1$, we conclude that

$$W = \frac{d}{dt}V + \int_0^1 [\bar{z}^T\bar{z} - \gamma^2 w^2]d\xi \leq \bar{\zeta}^T\Psi_\gamma\bar{\zeta},$$

where

$$\bar{\zeta}^T = [z_t(1,t)\ z_\xi(\xi,t)\ z_t(\xi,t)\ w(\xi,t)],$$

$$\Psi_\gamma = \begin{bmatrix} \psi_1 + d_1^2k^2 + \frac{\beta_1^2}{s_1} & 0 & pr_1 & 0 \\ * & \psi_{2\gamma} & pr_2 & 0 \\ * & * & \psi_{3\gamma} + \frac{\beta_0^2}{s_0} & pb \\ * & * & * & -\gamma^2 + \frac{b_1^2}{s_2} \end{bmatrix},$$

and

$$\begin{array}{l} \psi_1 = -2a_1kp + (1 + a_1k^2)\chi, \\ \psi_{2\gamma} = -a_1\chi + \sum_{i=0}^{2} s_i + \frac{1}{2}\alpha_0^2 + 2\chi\beta_2, \\ \psi_{3\gamma} = -\chi + 2p\beta_0 + \alpha_1^2. \end{array} \tag{45}$$

Therefore $W < 0$ if $\Psi_\gamma < 0$, i.e. by Schur complements, if

$$\begin{bmatrix} \psi_1 + d_1^2k^2 & 0 & pr_1 & 0 & \beta_1 & 0 & 0 \\ * & \psi_{2\gamma} & pr_2 & 0 & 0 & 0 & 0 \\ * & * & \psi_{3\gamma} & pb & 0 & \beta_0 & 0 \\ * & * & * & -\gamma^2 & 0 & 0 & b_1 \\ * & * & * & * & -s_1 & 0 & 0 \\ * & * & * & * & * & -s_0 & 0 \\ * & * & * & * & * & * & -s_2 \end{bmatrix} < 0. \tag{46}$$

LMI (46) is affine in $r_i \in [-\beta_i, \beta_i], i = 1, 2$ and $b \in [-b_1, b_1]$. Therefore, it is feasible if it holds in the vertices $r_i = \pm\beta_i$ and $b = \pm b_1$. It is easy to see that the eight LMIs in the vertices are equivalent to the following one

$$\begin{bmatrix} \psi_1 + d_1^2k^2 & 0 & p\beta_1 & 0 & \beta_1 & 0 & 0 \\ * & \psi_{2\gamma} & p\beta_2 & 0 & 0 & 0 & 0 \\ * & * & \psi_{3\gamma} & pb_1 & 0 & \beta_0 & 0 \\ * & * & * & -\gamma^2 & 0 & 0 & b_1 \\ * & * & * & * & -s_1 & 0 & 0 \\ * & * & * & * & * & -s_0 & 0 \\ * & * & * & * & * & * & -s_2 \end{bmatrix} < 0. \tag{47}$$

We note that if (47) is feasible, then the LMIs (38) for exponential stability hold with small enough $\delta > 0$. We thus proved the following.

Theorem 4: Consider the perturbed input-output system (39)–(41) with the assumptions above. Given $\gamma > 0$ and $k > 0$, let there exist $p > 0, \chi, s_0, s_1$ and s_2 such that the LMIs (31) and (47) are satisfied with the notations given by (45). Then the static output feedback (42) internally exponentially stabilizes the boundary value problem (39), (41) and attenuates the external disturbances $w(\xi,t) \in L_2(0,\infty; L_2(0,1))$ in the sense of (43).

C. Example

Consider (39)–(41) with

$$\begin{array}{l} a_1 = 2, \quad \beta_0 = \beta_1 = 0.3, \quad \beta_2 = 0.4, \\ \alpha_0 = \alpha_1 = 1, \quad d_1 = 0.1. \end{array}$$

As mentioned above, the open-loop system is unstable. By using LMI toolbox of Matlab to verify LMIs (31) and (47), we find that the static output feedback (42) with $k = 1$ internally exponentially stabilizes the system and attenuates the external disturbances with $\gamma = 8.9$. By verifying (38) in the four vertices, we find that the resulting closed-loop system is internally exponentially stable with the decay rate $\delta = 0.06$.

IV. Conclusions

In the present paper an LMI approach is extended to H_∞ boundary control of uncertain semilinear heat equations and wave equations. The uncertainties are admitted to be time-, space- and state-dependent with *a priori* known upper/lower bounds. Sufficient conditions for static output feedback stabilization are given in terms of LMIs. Numerical examples illustrate the efficiency of the method.

The proposed method seems to be extendible to dynamic output feedback H_∞ control and to other classes of distributed parameter systems. LMIs are thus expected to provide effective tools for robust control of distributed parameter systems.

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