Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Stabilization of Closed Sets for Passive Systems, Part I: Reduction Principles

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***Abstract*— Given an unforced nonlinear system and two nested closed and invariant sets $\Gamma \subset \mathcal{O}$, we present reduction principles allowing one to extrapolate the properties of stability, attractivity, and asymptotic stability of Γ from analogous properties of the system restricted to $\mathcal{O}$. As a corollary to our reduction principles, we present a stability criterion for cascade-connected systems which generalizes well-known results in the literature. Using the reduction principles, in Part II of this paper we present a comprehensive theory for passivity-based stabilization of closed sets.**

I. Introduction

In this paper we investigate the reduction problem for finite dimensional dynamical systems described by Lipschitz continuous ODEs, $\dot{x} = f(x)$. Suppose we are given two closed sets, Γ and $\mathcal{O}$, with $\Gamma \subset \mathcal{O}$, that are invariant for the system. Consider the restriction of the system dynamics to the invariant set $\mathcal{O}$, and suppose that Γ is stable, attractive, or asymptotically stable for such restriction to $\mathcal{O}$. *When is it that Γ is stable, attractive, or asymptotically stable with respect to the whole state space?*

The above reduction problem, originally formulated by Seibert and Florio in 1970 [1], [2] and investigated in more depth in [3] (see also the related work in [4]), is fundamentally important in control theory, as it often arises whenever one wants to infer stability properties of a control system based on its properties on a subset of the state space. A particularly important application of this kind is the problem of stabilizing closed sets for passive systems. Given a passive system with input u, output function $h(x)$, and storage function $V(x)$, suppose that Γ is a closed and open-loop invariant set such that $\Gamma \subset V^{-1}(0)$. Consider a passivity-based feedback, that is, a feedback $u = -\varphi(y)$, with $\varphi(0) = 0$ and $y^\top \varphi(y) > 0$ for all $y \neq 0$. *When is it that Γ is stable, attractive, or asymptotically stable for the closed-loop system?* The answer to this question is known in two cases: when Γ is an equilibrium point and V is positive definite (so that $\Gamma = V^{-1}(0)$), see [5]; and when Γ is a compact set and $\Gamma = V^{-1}(0)$, see [6]. In [7], we initiated the investigation of the more general case when $\Gamma \subset V^{-1}(0)$ and Γ is not necessarily compact. Later, in [8], we presented a theorem (without proof) extending some of the theory in [7] and presented applications of the theory to two maneuvering problems for the kinematic unicycle. It turns out that the foregoing question concerning the passivity-based stabilization of Γ is precisely a special instance of the first question, the reduction problem; this is shown in Part II of this work.

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This research was supported by the National Sciences and Engineering Research Council of Canada.

In this paper, we review Seibert and Florio's reduction principles in [1], [2], [3] dealing with stability and asymptotic stability in the case of compact Γ. We then present a novel reduction principle for attractivity, and an extension of Seibert and Florio's reduction principles for stability and asymptotic stability to the case of closed but not necessarily compact Γ. As a corollary to our result, we give a novel set stability criterion for cascade-connected systems. In Part II of this work we apply the reduction principles presented in this paper to the solution of the passivity-based set stabilization problem.

II. Preliminaries and Problem Statement

In this paper we consider the dynamical system

$$\Sigma : \dot{x} = f(x) \tag{1}$$

with state space $\mathcal{X} \subset \mathbb{R}^n$ and f a smooth vector field. We assume that $\mathcal{X}$ is either an open subset of $\mathbb{R}^n$ or a smooth submanifold thereof. In both cases, the restriction of a metric $\|\cdot\| : \mathbb{R}^n \to [0, +\infty)$ to $\mathcal{X}$ gives a metric on $\mathcal{X}$.

A. Notation

Let $\mathbb{R}^+$ denote the positive real line $[0, +\infty)$. We denote by $\phi(t, x_0)$ the unique solution of (1) with initial condition x_0. Given an interval I of the real line and a set $S \in \mathcal{X}$, we denote by $\phi(I, S)$ the set $\phi(I, S) := \{\phi(t, x_0) : t \in I, x_0 \in S\}$. Given a closed nonempty set $S \subset \mathcal{X}$ and a point $x \in \mathcal{X}$, the point-to-set distance $\|x\|_S$ is defined as $\|x\|_S := \inf\{\|x - y\| : y \in S\}$. Given two subsets S_1 and S_2 of $\mathcal{X}$, the maximum distance of S_1 to S_2, $d(S_1, S_2)$, is defined as $d(S_1, S_2) := \sup\{\|x\|_{S_2} : x \in S_1\}$. For a constant $\alpha > 0$, a point $x \in \mathcal{X}$, and a set $S \subset \mathcal{X}$, define the open sets $B_\alpha(x) = \{y \in \mathcal{X} : \|y - x\| < \alpha\}$ and $B_\alpha(S) = \{y \in \mathcal{X} : \|y\|_S < \alpha\}$. We denote by $\mathrm{cl}(S)$ the closure of the set S, and by $\mathcal{N}(S)$ a generic open neighbourhood of S, that is, an open subset of $\mathcal{X}$ containing S.

B. Set stability and attractivity

Here, we present the basic notions of set stability and attractivity used in this paper. Let $\Gamma \subset \mathcal{X}$ be a closed positively invariant set for Σ in (1).

Definition II.1 (Set stability and attractivity). (i) Γ is *stable* for Σ if for all $\varepsilon > 0$ there exists a

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neighbourhood $\mathcal{N}(\Gamma)$ such that $\phi(\mathbb{R}^+, \mathcal{N}(\Gamma)) \subset B_\varepsilon(\Gamma)$.

(ii) Γ is *uniformly stable* for Σ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\phi(\mathbb{R}^+, B_\delta(\Gamma)) \subset B_\varepsilon(\Gamma)$.

(iii) Γ is a *semi-attractor* for Σ if there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that, for all $x_0 \in \mathcal{N}(\Gamma)$, $\lim_{t\to\infty} \|\phi(t, x_0)\|_\Gamma = 0$.

(iv) Γ is an *attractor* for Σ if there exists $\delta > 0$ such that, for all $x_0 \in B_\delta(\Gamma)$, $\lim_{t\to\infty} \|\phi(t, x_0)\|_\Gamma = 0$.

(v) Γ is a *global attractor* for Σ if it is an attractor with $\delta = +\infty$ or a semi-attractor with $\mathcal{N}(\Gamma) = \mathcal{X}$.

(vi) Γ is a *uniform semi-attractor* for Σ if for all $x \in \Gamma$, there exists $\lambda > 0$ such that, for all $\varepsilon > 0$, there exists $T > 0$ yielding $\phi([T, +\infty), B_\lambda(x)) \subset B_\varepsilon(\Gamma)$.

(vii) Γ is a *uniform attractor* for Σ if there exists $\lambda > 0$ such that, for all $\varepsilon > 0$, there exists $T > 0$ yielding $\phi([T, +\infty), B_\lambda(\Gamma)) \subset B_\varepsilon(\Gamma)$.

(viii) Γ is *[globally] semi-asymptotically stable* for Σ if it is stable and semi-attractive [globally attractive] for Σ.

(ix) Γ is *[globally] asymptotically stable* for Σ if it is a uniformly stable and [globally] attractive for Σ.

All definitions above, except that of a uniform semi-attractor, are standard and can be found in [9].

Remark. If Γ is a compact positively invariant set, then the concepts of stability, semi-attractivity, and uniform semi-attractivity are respectively equivalent to those of uniform stability, attractivity, and uniform attractivity. Moreover, a compact positively invariant set Γ is a uniform attractor if, and only if, it is asymptotically stable (see Theorems V.1.15 and V.1.16 in [10]). In general, when Γ is closed but unbounded, uniform attractivity of Γ implies stability, and hence semi-asymptotic stability, of Γ (see Theorem 1.6.24 in [9]), but not vice versa.

Definition II.2 (Relative set stability and attractivity)**.** Let $\mathcal{O} \subset \mathcal{X}$ be such that $\mathcal{O} \cap \Gamma \neq \emptyset$. We say that Γ is *stable relative to* $\mathcal{O}$ for Σ if, for any $\varepsilon > 0$, there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $\phi(\mathbb{R}^+, \mathcal{N}(\Gamma) \cap \mathcal{O}) \subset B_\varepsilon(\Gamma)$. Similarly, one modifies all other notions in Definition II.1 by restricting initial conditions to lie in $\mathcal{O}$.

Definition II.3 (Local stability and attractivity near a set)**.** Let Γ and $\mathcal{O}$, $\Gamma \subset \mathcal{O} \subset \mathcal{X}$, be positively invariant sets. The set $\mathcal{O}$ is *locally stable near* Γ if for all $x \in \Gamma$, for all $c > 0$, and all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_0 \in B_\delta(\Gamma)$ and all $t > 0$, whenever $\phi([0, t], x_0) \subset B_c(x)$ one has that $\phi([0, t], x_0) \subset B_\varepsilon(\mathcal{O})$. The set $\mathcal{O}$ is *locally (semi-) attractive near* Γ if there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that, for all $x_0 \in \mathcal{N}(\Gamma)$, $\phi(t, x_0) \to \mathcal{O}$ at $t \to +\infty$.

The property of local stability can be rephrased as follows. Given an arbitrary ball $B_c(x)$ centered at a point x in Γ, trajectories originating in $B_c(x)$ sufficiently close to Γ cannot travel far away from $\mathcal{O}$ before first exiting $B_c(x)$. It is not difficult to see that if Γ is stable, then $\mathcal{O}$ is locally stable near Γ.

Finally, we present the following definition used in the sequel.

Definition II.4 (Local uniform boundedness)**.** The system Σ is *locally uniformly bounded near* Γ if for each $x \in \Gamma$ there exist positive scalars λ and m such that $\phi(\mathbb{R}^+, B_\lambda(x)) \subset B_m(x)$.

Remark. If Γ is a stable compact set, then Σ is locally uniformly bounded near Γ. For, the stability of Γ implies the existence of a compact neighbourhood S of Γ which is positively invariant for Σ. Let $\lambda > 0$ and $m > 0$ be such that, for all $x \in \Gamma$, $B_\lambda(x) \subset S \subset B_m(x)$ (λ and m exist by compactness). Then, for any $x \in \Gamma$, the ball $B_\lambda(x)$ is contained in S, and thus by positive invariance, $\phi(\mathbb{R}^+, B_\lambda(x)) \subset S \subset B_m(x)$.

The next lemma, proved in Appendix I, clarifies the relationship between uniform semi-attractivity and semi-asymptotic stability.

Lemma II.5. *Let* Γ *be a closed set which is positively invariant for* Σ *in* (1) *and let* $U \supset \Gamma$ *be a closed set. If* Γ *is a uniform semi-attractor [relative to* U*], then it is semi-asymptotically stable [relative to* U*]. Furthermore, if* Σ *is locally uniformly bounded near* Γ*, then* Γ *is semi-asymptotically stable [relative to* U*] if, and only if, it is a uniform semi-attractor [relative to* U*].*

C. Limit Sets

In order to characterize the asymptotic properties of bounded solutions, we will use the well-known notion of limit set, due to G. D. Birkhoff (see [11]), and that of prolongational limit set, due to T. Ura (see [12]). Given a point $x_0 \in \mathcal{X}$, the *positive limit set* (or ω-limit set) of the solution $\phi(t, x_0)$ is defined as

$$L^+(x_0) := \{p \in \mathcal{X} : (\exists \{t_n\} \subset \mathbb{R}^+)\, t_n \to +\infty, \\ \phi(x_0, t_n) \to p\}.$$

The *negative limit set* (or α-limit set) $L^-(x_0)$ of $\phi(t, x_0)$ is defined using time sequence diverging to $-\infty$. We let $L^+(S) := \bigcup_{x_0 \in S} L^+(x_0)$.

The *prolongational limit set* $J^+(x_0)$ of a solution $\phi(t, x_0)$ is defined as

$$J^+(x_0) := \{p \in \mathcal{X} : (\exists \{(x_n, t_n)\} \subset \mathcal{X} \times \mathbb{R}^+),\, x_n \to x_0, \\ t_n \to +\infty,\, \phi(x_n, t_n) \to p\}.$$

If $U \subset \mathcal{X}$, the *prolongational limit set of* $\phi(t, x_0)$ *relative to* U is defined as

$$J^+(x_0, U) := \{p \in \mathcal{X} : (\exists \{(x_n, t_n)\} \subset U \times \mathbb{R}^+),\, x_n \to x_0, \\ t_n \to +\infty,\, \phi(x_n, t_n) \to p\}.$$

We let

$$J^+(S) := \bigcup_{x_0 \in S} J^+(x_0),\; J^+(S, U) := \bigcup_{x_0 \in S} J^+(x_0, U).$$

Obviously, $L^+(x_0) \subset J^+(x_0)$. Moreover, if $x_0 \in U$, $L^+(x_0) \subset J^+(x_0, U) \subset J^+(x_0)$.

The following results, Propositions II.6 and II.7, present useful relations for prolongational limit sets. These relations will be used in the development of the main results in Section III.

Proposition II.6 (Theorem II.4.3 and Lemma V.1.10 in [10])**.** *Consider the dynamical system Σ in* (1)*. For any $x \in \mathcal{X}$, $J^+(x)$ is closed and invariant. Moreover, for any $\omega \in L^+(x)$, $J^+(x) \subset J^+(\omega)$.*

Remark. The results in Proposition II.6 still hold if one replaces $J^+(x)$ by $J^+(x, U)$, with $U \subset \mathcal{X}$.

While $L^+(x_0)$ is used to characterize the asymptotic convergence properties of $\phi(t, x_0)$, $J^+(x_0)$ is used to characterize *uniform* convergence, as shown next.

Proposition II.7. *Suppose that Σ in* (1) *is locally uniformly bounded near a closed and positively invariant set Γ. Let $U \subset \mathcal{X}$ be a closed set, $\Gamma \subset U$. Then, for each x in some neighbourhood of Γ, $J^+(x) \neq \emptyset$ [$J^+(x, U) \neq \emptyset$]. Moreover, Γ is a uniform semi-attractor [relative to U] for Σ if, and only if, there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $J^+(\mathcal{N}(\Gamma)) \subset \Gamma$ [$J^+(\mathcal{N}(\Gamma), U) \subset \Gamma$].*

The proof of sufficiency can be found in Appendix II, while that of necessity is omitted because it is not used in the sequel.

D. Problem Statement

This paper addresses the following problem.

Reduction Problem ([1], [2])**.** Consider the dynamical system Σ in (1). Let Γ and $\mathcal{O}$ be two closed positively invariant sets such that $\Gamma \subset \mathcal{O} \subset \mathcal{X}$. Assume that Γ is, respectively, stable, semi-attractive, and semi-asymptotically stable relative to $\mathcal{O}$. Find what additional conditions are needed to guarantee that Γ is, respectively, stable, semi-attractive, and semi-asymptotically stable for Σ. We also seek to solve the global version of each of the problems above.

This problem was originally formulated by P. Seibert in 1969-1970. Seibert and Florio developed reduction principles for stability and asymptotic stability (but not attractivity) for dynamical systems on metric spaces assuming that Γ is compact. Their conditions first appeared in [1] and [2], while the proofs are found in [3] (see also the work in [4] for related results). In the next section we first review the results by Seibert and Florio. We then present the main results of the paper. These are novel reduction principles for semi-attractivity and, for the case of unbounded Γ, stability and semi-asymptotic stability.

III. Reduction principles

In this section we address the *Reduction Problem* given in Section II-D. Consider the dynamical system Σ given in (1). Let Γ and $\mathcal{O}$, $\Gamma \subset \mathcal{O} \subset \mathcal{X}$, be closed sets which are positively invariant for system Σ.

A. Previous Results

First, we present Seibert and Florio's reduction principles for compact sets.

Theorem III.1 (Theorem 3.4 in [3])**.** *Suppose that Γ is a compact set. Then, Γ is uniformly stable if the following conditions hold:*

(i) Γ is asymptotically stable relative to $\mathcal{O}$,
(ii) $\mathcal{O}$ is locally stable near Γ.

Theorem III.2 (Theorem 4.13 and Corollary 4.11 in [3])**.** *Assume that Γ is a compact set. If, and only if, the following conditions hold:*

(i) Γ is asymptotically stable relative to $\mathcal{O}$,
(ii) $\mathcal{O}$ is locally stable near Γ,
(iii) $\mathcal{O}$ is locally attractive near Γ,

then Γ is asymptotically stable for Σ. Furthermore, if

(iv) all trajectories of Σ are bounded,

and conditions (i) and (iii) are replaced by

(i)' Γ is globally asymptotically stable relative to $\mathcal{O}$,
(iii)' $\mathcal{O}$ is a global attractor for Σ,

then Γ is globally asymptotically stable for Σ.

Remark. The notion of local stability used by Seibert and Florio in [3] for compact Γ is slightly different than that in our Definition II.3. Specifically, if the conditions in Definition II.3 hold, then $\mathcal{O}$ is locally stable near Γ in the sense of Seibert and Florio. Since local stability of $\mathcal{O}$ near Γ in the sense of Definition II.3 is a *necessary condition* for stability of Γ, the assumptions in Theorems III.1 and III.2 are equivalent to the conditions in [3].

B. Main Results

In this section we present the main results of the paper. In Theorems III.3, III.4, and III.7 we provide novel reduction theorems for semi-attractivity, stability, and semi-asymptotic stability of non-compact sets. We also present a number of implications of these results.

Theorem III.3 (Reduction principle for semi-attractivity)**.** *The closed set Γ is semi-attractive if the following conditions hold:*

(i) Γ is semi-asymptotically stable relative to $\mathcal{O}$
(ii) $\mathcal{O}$ is locally semi-attractive near Γ,
(iii) there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that, for all initial conditions in $\mathcal{N}(\Gamma)$, the associated solutions are bounded and such that the set $\mathrm{cl}(\phi(\mathbb{R}^+, \mathcal{N}(\Gamma))) \cap \mathcal{O}$ is contained in the domain of attraction of Γ relative to $\mathcal{O}$.

The set Γ is globally attractive if:

(i)' Γ is globally semi-asymptotically stable relative to $\mathcal{O}$,
(ii)' $\mathcal{O}$ is a global attractor,
(iii)' all trajectories in $\mathcal{X}$ are bounded.

Proof: By assumption (ii), there exists a neighbourhood $\mathcal{N}_1(\Gamma)$ of Γ such that all trajectories originating there asymptotically approach $\mathcal{O}$ in positive time. Let $\mathcal{N}_2(\Gamma)$ be the neighbourhood in assumption (iii), and define $\mathcal{N}_3(\Gamma) = \mathcal{N}_1(\Gamma) \cap \mathcal{N}_2(\Gamma)$. Clearly, $\mathcal{N}_3(\Gamma)$ is a neighbourhood of Γ. By construction, for all $x_0 \in \mathcal{N}_3(\Gamma)$, the solution is bounded and approaches $\mathcal{O}$. Therefore, the positive limit set $L^+(x_0)$ is non-empty, compact, invariant, and $L^+(x_0) \subset \mathcal{O}$. Moreover, by definition of positive limit set, and by assumption (iii) we have the following inclusion,

$$L^+(x_0) \subset \mathrm{cl}(\phi(\mathbb{R}^+, x_0)) \cap \mathcal{O} \subset \{\text{domain of attraction of } \Gamma \text{ rel. to } \mathcal{O}\}. \tag{2}$$

We need to show that $L^+(x_0) \subset \Gamma$. Assume, by way of contradiction, that there exists $\omega \in L^+(x_0)$ and $\omega \notin \Gamma$. By the invariance of $L^+(x_0)$, $\phi(\mathbb{R}, \omega) \subset L^+(x_0)$, and therefore $L^-(\omega) \subset L^+(x_0)$. By the inclusion in (2), all trajectories in $L^-(\omega)$ asymptotically approach Γ in positive time, and so since $L^-(\omega)$ is closed, $L^-(\omega) \cap \Gamma \neq \emptyset$. Let $p \in L^-(\omega) \cap \Gamma$. Pick $\varepsilon > 0$ such that $\|\omega\|_\Gamma > \varepsilon$. By the stability of Γ relative to $\mathcal{O}$, there exists a neighbourhood $\mathcal{N}_4(\Gamma)$ of Γ such that $\phi(\mathbb{R}^+, \mathcal{N}_4(\Gamma) \cap \mathcal{O}) \subset B_\varepsilon(\Gamma)$. Since $p \in L^-(\omega)$, there exists a sequence $\{t_k\} \subset \mathbb{R}^+$, with $t_k \to +\infty$, such that $\phi(-t_k, \omega) \to p$ at $k \to +\infty$. Since $p \in \Gamma$, we can pick $k^\star$ large enough that $\phi(-t_{k^\star}, \omega) \in \mathcal{N}_4(\Gamma)$. Let $T = t_{k^\star}$ and $z = \phi(-t_{k^\star}, \omega)$. We have thus obtained that $z \in \mathcal{N}_4(\Gamma)$, but $\phi(T, z) = \omega$ is not in $B_\varepsilon(\Gamma)$. This contradicts the stability of Γ, and therefore, for all $x_0 \in \mathcal{N}_3(\Gamma)$, $L^+(x_0) \subset \Gamma$, proving that Γ is a semi-attractor for Σ.

To prove global attractivity of Γ it is sufficient to notice that by assumptions (ii)' and (iii)', for all $x_0 \in \mathcal{X}$, $L^+(x_0)$ is non-empty and $L^+(x_0) \subset \mathcal{O}$. On $\mathcal{O}$, by assumption (i)' all trajectories approach Γ, so by the contradiction argument above we conclude that $L^+(x_0) \subset \Gamma$. ■

Part of the previous proof was inspired by the stability results using positive semidefinite Lyapunov functions presented in [13] and by the proof of Lemma 1 in [14]. Being of a rather technical nature, Assumption (iii) is difficult to check and of limited practical use. It has, however, theoretical significance because it is used to prove the reduction principle for semi-asymptotic stability stated in the sequel. If condition (i) is replaced by the stronger (i)', then one can replace (iii) by the simpler requirement that trajectories in some neighbourhood of Γ be bounded.

Theorem III.4 (Reduction principle for stability)**.** *The closed set Γ is stable if the following conditions hold:*

(i) Γ is semi-asymptotically stable relative to $\mathcal{O}$,
(ii) $\mathcal{O}$ is locally stable near Γ,
(iii) If Γ is unbounded, then Σ is locally uniformly bounded near Γ.

To prove the theorem we need the following lemma whose proof is omitted due to space limitations.

Lemma III.5. *Let $\Gamma \subset \mathcal{X}$ be a closed set which is positively invariant set for Σ in* (1)*. If Γ is unstable, then there exist $\varepsilon > 0$, a bounded sequence $\{x_i\} \subset \mathcal{X}$, and a sequence $\{t_i\} \subset \mathbb{R}^+$, such that $x_i \to \bar{x} \in \Gamma$, and $\|\phi(t_i, x_i)\|_\Gamma = \varepsilon$.*

Proof of Theorem III.4: By way of contradiction, suppose that Γ is unstable. Then, by Lemma III.5, there exist $\varepsilon > 0$, a bounded sequence $\{x_i\} \subset \mathcal{X}$, with $x_i \to \bar{x} \in \Gamma$, and a sequence $\{t_i\} \subset \mathbb{R}^+$, such that $\|\phi(t_i, x_i)\|_\Gamma = \varepsilon$, and $\phi([0, t_i), x_i) \in B_\varepsilon(\Gamma)$. By local uniform boundedness of Σ near Γ, there exist two positive numbers λ and m such that $\phi(\mathbb{R}^+, B_\lambda(\bar{x})) \subset B_m(\bar{x})$. We can assume $\{x_i\} \subset B_\lambda(\bar{x})$. Take a decreasing sequence $\{\varepsilon_i\} \subset \mathbb{R}^+$, $\varepsilon_i \to 0$. By assumption (ii), $\mathcal{O}$ is locally stable near Γ. Using the definition of local stability with $c = m$ and $\varepsilon = \varepsilon_i$, there exists $\delta_i > 0$ such that for all $x_0 \in B_{\delta_i}(\bar{x})$ and all $t > 0$, if $\phi([0,t], x_0) \subset B_m(\bar{x})$, then $\phi([0,t], x_0) \subset B_{\varepsilon_i}(\mathcal{O})$. By taking $\delta_i \leq \lambda$ we have $(\forall x_0 \in B_{\delta_i}(\bar{x}))\ \phi(\mathbb{R}^+, x_0) \subset B_{\varepsilon_i}(\mathcal{O})$. By passing, if needed, to a subsequence we can assume without loss of generality that, for all i, $x_i \in B_{\delta_i}(\bar{x})$ so that $\limsup_{i\to\infty} d(\phi([0,t_i], x_k), \mathcal{O}) = 0$.

Using assumptions (i) and (iii) (if Γ is unbounded), by Lemma II.5 it follows that Γ is a uniform semi-attractor relative to $\mathcal{O}$. Therefore,

$$(\forall x \in \Gamma)(\exists \mu > 0)(\forall \varepsilon' > 0)(\exists T > 0) \text{ s.t. } \phi([T, +\infty), B_\mu(x) \cap \mathcal{O}) \subset B_{\varepsilon'}(\Gamma). \tag{3}$$

Consider the set $\Gamma' = \Gamma \cap \mathrm{cl}(B_{2m}(\bar{x}))$. Since Γ' is compact, using (3) we infer the existence of $\mu > 0$ such that

$$(\forall x \in \Gamma')(\forall \varepsilon' > 0)(\exists T > 0)\ \phi([T, +\infty), B_\mu(x) \cap \mathcal{O}) \subset B_{\varepsilon'}(\Gamma). \tag{4}$$

By reducing, if necessary, ε in the instability definition, we may assume that $\varepsilon < \mu$. Now choose $\varepsilon' < \varepsilon/2$. Using again a compactness argument, by (4) one infers the following condition

$$(\exists T > 0)(\forall x \in \Gamma')\phi([T, +\infty), B_\mu(x) \cap \mathcal{O}) \subset B_{\varepsilon'}(\Gamma). \tag{5}$$

We claim that $B_\mu(\Gamma) \cap B_m(\bar{x}) \subset B_\mu(\Gamma')$. For, if $\mu \geq m$, then $B_\mu(\Gamma) \cap B_m(\bar{x}) = B_m(\bar{x}) \subset B_\mu(\bar{x}) \subset B_\mu\Big(\Gamma \cap \mathrm{cl}(B_{2m}(\bar{x}))\Big)$. If $\mu < m$, then $x \in B_\mu(\Gamma) \cap B_m(\bar{x})$ if and only if $\|x\|_\Gamma < \mu$ and $\|x - \bar{x}\| < m$; in particular, there exists $y \in \Gamma$ such that $\|x - y\| < \mu$. Since $\|y - \bar{x}\| \leq \|x - y\| + \|x - \bar{x}\| \leq \mu + m < 2m$, we have that $y \in \Gamma \cap \mathrm{cl}(B_{2m}(\bar{x}))$, and thus $x \in B_\mu(\Gamma \cap \mathrm{cl}(B_{2m}(\bar{x})))$.

Using (5) and the claim we've just proved we obtain

$$(\forall x \in B_\mu(\Gamma) \cap B_m(\bar{x}) \cap \mathcal{O})\ \phi([T, +\infty), x) \subset B_{\varepsilon'}(\Gamma). \tag{6}$$

Now, since $\{t_k\}$ is unbounded there exists $K_1 > 0$ such that $t_k > T$ for all $k \geq K_1$. Since $\phi([0, t_k), x_k) \subset B_\varepsilon(\Gamma)$ we have $\phi(t_k - T, x_k) \in B_\varepsilon(\Gamma)$ for all $k \geq K_1$. Let $y_k = \phi(t_k, x_k)$, and $z_k = \phi(t_k - T, x_k)$. Thus, $y_k = \phi(T, z_k)$, $\|y_k\|_\Gamma = \varepsilon$ and $z_k \in B_\varepsilon(\Gamma)$. By local uniform boundedness, it also holds that $z_k \in B_m(\bar{x})$. Pick $\delta \in (0, \mu - \varepsilon)$. Since $z_k \in \phi([0, t_k), x_k) \subset B_m(\bar{x})$, and since $\limsup_{k\to\infty} d(\phi([0, t_k], x_k), \mathcal{O}) = 0$, then there exists $K_2 \geq K_1$ such that, for all $k \geq K_2$, there exists $z'_k \in B_m(\bar{x}) \cap \mathcal{O}$ such that $\|z_k - z'_k\| < \delta$. Since $z_k \in B_\varepsilon(\Gamma)$, then

$$z'_k \in B_{\varepsilon+\delta}(\Gamma) \cap B_m(\bar{x}) \cap \mathcal{O} \subset B_\mu(\Gamma) \cap B_m(\bar{x}) \cap \mathcal{O}$$

and, by (6), $\phi([T, +\infty), z'_k) \subset B_{\varepsilon'}(\Gamma)$. By continuous dependence on initial conditions, δ can be chosen small enough that

$$(\forall x \in B_m(\bar{x}))(\forall x_0 \in B_\delta(x))\ \|\phi(T, x) - \phi(T, x_0)\| < \varepsilon/2.$$

We have $z_k \in B_m(\bar{x})$ and $\|z_k - z'_k\| < \delta$, hence $\|\phi(T, z_k) - \phi(T, z'_k)\| < \varepsilon/2$, which implies

$$y_k \in B_{\varepsilon/2}(\phi(T, z'_k)) \subset B_{\varepsilon/2+\varepsilon'}(\Gamma) \subset B_\varepsilon(\Gamma),$$

contradicting $\|y_k\|_\Gamma = \varepsilon$. ■

Part of this proof was inspired by that of of Lemma 2.1 in [3]. As mentioned in Section II-B, the local stability condition (ii) in Theorem III.4 is necessary. Condition (i) cannot be relaxed by just requiring that Γ be stable relative to $\mathcal{O}$. This fact was already pointed out by Seibert and Florio in [3] a simple counter-example.

By noting that if $\mathcal{O}$ is stable for Σ, then it is also locally stable near Γ, we get the following useful corollary.

Corollary III.6. *The closed set Γ is stable if conditions (i) and (iii) in Theorem III.4 hold and condition (ii) is replaced by the following one:*

(ii)' $\mathcal{O}$ is stable.

The stability of $\mathcal{O}$ in condition (ii)' is not necessary for the stability of Γ, as shown by Seibert and Florio in [3, Example 3].

By combining Theorems III.3 and III.4 we obtain a reduction principle for semi-asymptotic stability.

Theorem III.7 (Reduction principle for semi-asymptotic stability)**.** *The closed set Γ is [globally] semi-asymptotically stable if the following conditions hold:*

(i) Γ is [globally] semi-asymptotically stable relative to $\mathcal{O}$,
(ii) $\mathcal{O}$ is locally stable near Γ,
(iii) $\mathcal{O}$ is locally semi-attractive near Γ [$\mathcal{O}$ is globally attractive],
(iv) if Γ is unbounded, then Σ is locally uniformly bounded near Γ,
(v) [all trajectories of Σ are bounded.]

Conditions (i), (ii), and (iii) in the theorem above are necessary.

Proof: If Γ is compact, the theorem coincides with Theorem III.2. Suppose that Γ is unbounded. That the "global" assumptions imply global semi-asymptotic stability is a direct consequence of Theorems III.3 and III.4. To prove that the "local" assumptions imply semi-asymptotic stability of Γ, we need to show that assumption (iii) in Theorem III.3 is satisfied.

Assumptions (i), (ii), and (iv) in imply that Γ is stable. Moreover, by assumption (i), Γ is semi-attractive relative to $\mathcal{O}$. Let $N \subset \mathcal{O}$ denote the domain of attraction of Γ relative to $\mathcal{O}$. By assumption (iv), for each $x \in \Gamma$ there exist two positive numbers $\lambda(x)$ and $m(x)$ such that $\phi(\mathbb{R}^+, B_{\lambda(x)}(x)) \subset B_{m(x)}(x)$. Fix $x \in \Gamma$, and let $\varepsilon(x) > 0$ be small enough that

$$\mathrm{cl}\left(B_{\varepsilon(x)}(\Gamma) \cap B_{m(x)}(x)\right) \cap \mathcal{O} \subset N.$$

The constant ε is guaranteed to exist because the set on left-hand side of the inclusion is compact and can be made arbitrarily small. Since Γ is stable, there exists a neighbourhood $\mathcal{N}_x(\Gamma)$ such that $\phi(\mathbb{R}^+, \mathcal{N}_x(\Gamma)) \subset B_{\varepsilon(x)}(\Gamma)$. Now define

$$U = \bigcup_{x\in\Gamma} B_{\lambda(x)}(x) \cap \mathcal{N}_x(\Gamma).$$

Clearly, U is a neighbourhood of Γ. By definition, for each $y \in U$, there exists $x \in \Gamma$ such that $y \in B_{\lambda(x)}(x) \cap \mathcal{N}_x(\Gamma)$, so that the solution originating in y is bounded and

$$\phi(\mathbb{R}^+, y) \subset B_{\varepsilon(x)}(\Gamma) \cap B_{m(x)}(x).$$

Therefore, $\mathrm{cl}(\phi(\mathbb{R}^+, y)) \cap \mathcal{O} \subset \mathrm{cl}\left(B_{\varepsilon(x)}(\Gamma) \cap B_{m(x)}(x)\right) \cap \mathcal{O} \subset N$. ■

By combining Theorem III.7 and Corollary III.6 we obtain the following corollary.

Corollary III.8. *The closed set Γ is [globally] semi-asymptotically stable if conditions (i), (iii), (iv) [and (v)] in Theorem III.7 hold, and condition (ii) is replaced by the following one:*

(ii)' $\mathcal{O}$ is stable.

In Part II of this paper we use the reduction principles introduced above to provide results for stabilizing closed sets for passive systems. The usefulness of reduction principles is not limited to the stabilization of closed sets. As a matter of fact, stability theorems for cascade-connected systems of the form

$$\begin{aligned}\dot{x} &= f(x, y)\\ \dot{y} &= g(y),\end{aligned} \tag{7}$$

well-known in the control literature (see [15][Theorem 3.1], [16][Corollary 5.2], [17][Corollaries 10.3.2, 10.3.3]), are consequences of Seibert and Florio's reduction theory, specialized to the case when Γ is the origin and $\mathcal{O} = \{(x, y) : y = 0\}$. Motivated by this observation, we present a straightforward application of Corollary III.8 which has independent interest.

Corollary III.9. *Consider system* (7)*, with $x \in \mathbb{R}^{n_1}$, $y \in \mathbb{R}^{n_2}$, and let $\Gamma \subset \mathbb{R}^{n_1}$ be a positively invariant set for system $\dot{x} = f(x, 0)$. Suppose that $g(0) = 0$. Then, $\tilde{\Gamma} := \{(x, y) : x \in \Gamma, y = 0\}$ is [globally] semi-asymptotically stable for* (7) *if the following conditions hold:*

(i) Γ is [globally] semi-asymptotically stable for $\dot{x} = f(x, 0)$,
(ii) $y = 0$ is a [globally] asymptotically stable equilibrium of $\dot{y} = g(y)$,
(iii) if Γ is unbounded, then (7) *is locally uniformly bounded near $\tilde{\Gamma}$,*
(iv) [all trajectories of (7) *are bounded.]*

IV. Conclusions

Novel reduction principles have been presented for semi-attractivity, stability and semi-asymptotic asymptotic stability of closed invariant sets for nonlinear systems. A corollary to these principles is a stability criterion for cascade systems. The reduction principles are used in Part II of this paper to develop a theory for passivity-based stabilization of closed sets.

APPENDIX I
PROOF OF LEMMA II.5

We first show that if Γ is a uniform semi-attractor, then it is semi-asymptotically stable. Suppose, by way of contradiction, that Γ is unstable. This implies that there exists $\varepsilon > 0$ and sequences $\{x_i\} \subset \mathcal{X}$ and $\{t_i\} \subset \mathbb{R}^+$, with $\|x_i\|_\Gamma \to 0$ such that $\|\phi(t_i, x_i)\|_\Gamma = \varepsilon$. By Lemma III.5, we can assume, without loss of generality, that $\{x_i\}$ is bounded and has a limit $\bar{x} \in \Gamma$. Using $\bar{x}$ and ε in the definition of uniform semi-attractivity, we get $\lambda > 0$ and $T > 0$ such that $\phi([T, +\infty), B_\lambda(\bar{x})) \subset B_\varepsilon(\Gamma)$. For sufficiently large i, $x_i \in B_\lambda(\bar{x})$ and therefore, necessarily, $0 < t_i < T$. Having established that $\{t_i\}$ is a bounded sequence, we can assume that t_i has a limit $\tau < \infty$. Since Γ is positively invariant, $\phi(\tau, \bar{x}) \in \Gamma$. This gives a contradiction since $\phi(t_i, x_i) \to \phi(\tau, \bar{x})$ and, for all i, $\|\phi(t_i, x_i)\|_\Gamma = \varepsilon$.

Next we show that if Σ is locally uniformly bounded near Γ and Γ is semi-asymptotically stable, then Γ is a uniform semi-attractor for Σ. By Proposition II.7, we need to show that there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $J^+(\mathcal{N}(\Gamma)) \subset \Gamma$. By local uniform boundedness, for all x in a neighbourhood of Γ, $J^+(x) \neq \emptyset$. Moreover, since Γ is a semi-attractor, by Proposition II.6 we have $J^+(x) \subset J^+(L^+(x)) \subset J^+(\Gamma)$. Therefore, to prove uniform semi-attractivity it is enough to show that $J^+(\Gamma) \subset \Gamma$. Consider an arbitrary point $x \in \Gamma$, and let $p \in J^+(x)$. By local uniform boundedness, there exist positive constants λ and m such that $\phi(\mathbb{R}^+, B_\lambda(x)) \subset B_m(x)$. By the definition of prolongational limit set, there exist sequences $\{x_n\} \subset \mathcal{X}$ and $\{t_n\} \subset \mathbb{R}^+$, with $x_n \to x$ and $t_n \to +\infty$, such that $\phi(t_n, x_n) \to p$. Without loss of generality, we can assume that $\{x_n\} \subset B_\lambda(x)$. Take a decreasing sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$, with $\varepsilon_n \to 0$. By the stability of Γ, there exists a nested sequence of neighborhoods $\mathcal{N}_{n+1}(\Gamma) \subset \mathcal{N}_n(\Gamma)$ such that $\phi(\mathbb{R}^+, \mathcal{N}_n(\Gamma)) \subset B_{\varepsilon_n}(\Gamma)$. Since $\mathcal{N}_n(\Gamma) \cap B_\lambda(x)$ is a bounded set, for each n there exists $\delta_n > 0$ such that $B_{\delta_n}(\Gamma) \cap B_\lambda(x) \subset \mathcal{N}_n(\Gamma) \cap B_\lambda(x)$. We thus obtain a decreasing sequence $\{\delta_n\}$, $\delta_n \to 0$, such that $\phi(\mathbb{R}^+, B_{\delta_n}(x)) \subset B_m(x) \cap B_{\varepsilon_n}(\Gamma)$. Take subsequences $\{x_{n_k}\}$ and $\{B_{\delta_{n_k}}(x)\}$ such that, for each k, $x_{n_k} \in B_{\delta_{n_k}}(x)$. Since $x_n \to x \in \Gamma$, for each n there are infinitely many x_n's in $B_{\delta_n}(x)$, and therefore the subsequences just defined have infinite elements. We have that $\phi(t_{n_k}, x_{n_k}) \to p$ and, by construction, $\phi(t_{n_k}, x_{n_k}) \in B_{\varepsilon_{n_k}}(\Gamma)$. This implies that $p \in \Gamma$, and so $J^+(x) \subset \Gamma$.

The proofs of the statements involving relative stability concepts are identical. ■

APPENDIX II
PROOF OF PROPOSITION II.7

We only prove sufficiency. Assume that there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $J^+(\mathcal{N}(\Gamma)) \subset \Gamma$. By local uniform boundedness, we can assume that all trajectories on $\mathcal{N}(\Gamma)$ are bounded, and hence for each $x \in \mathcal{N}(\Gamma)$, $L^+(x) \neq \emptyset$. Since $L^+(\mathcal{N}(\Gamma)) \subset J^+(\mathcal{N}(\Gamma), U) \subset J^+(\mathcal{N}(\Gamma))$, we have that for each $x \in \mathcal{N}(\Gamma)$, $J^+(x)$ and $J^+(x, U)$ are not empty. To prove that Γ is a uniform semi-attractor, we need to show that, for all $x \in \Gamma$, $(\exists \delta > 0)(\forall \varepsilon > 0)(\exists T > 0)$ s.t. $\phi([T, +\infty), B_\delta(x)) \subset B_\varepsilon(\Gamma)$. Suppose, by way of contradiction, that there exists $x \in \Gamma$ such that

$$(\forall \delta > 0)(\exists \varepsilon > 0) \text{ s.t. } (\forall T > 0)(\exists \bar{x} \in B_\delta(x), \exists \bar{t} \geq T) \text{ s.t. } \|\phi(\bar{t}, \bar{x})\|_\Gamma \geq \varepsilon. \tag{8}$$

By the local uniform boundedness assumption, there exist positive λ and m such that $\phi(\mathbb{R}^+, B_\lambda(x)) \subset B_m(x)$. We can take small enough δ that $\delta \leq \lambda$ and $\mathrm{cl}(B_\delta(x)) \subset \mathcal{N}(\Gamma)$. Let $\varepsilon > 0$ be as in (8). Take a sequence $\{T_i\} \subset \mathbb{R}^+$, with $T_i \to \infty$. By (8), there exist sequences $\{\bar{x}_i\} \subset B_\delta(x)$ and $\{\bar{t}_i\} \subset \mathbb{R}^+$, with $\bar{t}_i \to \infty$, such that $\|\phi(\bar{t}_i, \bar{x}_i)\|_\Gamma \geq \varepsilon$. Since $\bar{x}_i \in B_\delta(x) \subset B_\lambda(x)$, then $\phi(\bar{x}_i, \bar{t}_i) \in B_m(x)$. By boundedness of $\{\bar{x}_i\}$ and $\{\phi(\bar{t}_i, \bar{x}_i)\}$, we can assume that $\bar{x}_i \to x^\star \in \mathrm{cl}(B_\delta(x))$, and $\phi(\bar{t}_i, \bar{x}_i) \to p$, with $\|p\|_\Gamma \geq \varepsilon$. We have thus obtained that there exists $x^\star \in \mathrm{cl}(B_\delta(x))$ such that $J^+(x^\star) \not\subset \Gamma$. However, $\mathrm{cl}(B_\delta(x)) \subset \mathcal{N}(\Gamma)$, and so $J^+(\mathrm{cl}(B_\delta(x))) \subset \Gamma$, a contradiction.

The proof that Γ is a uniform semi-attractor relative to U if and only if there exists $\mathcal{N}(\Gamma)$ such that $J^+(\mathcal{N}(\Gamma), U) \subset \Gamma$ is identical.

■

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