Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Decentralized robust state estimation via a data-rate constrained sensor network

Teddy M. Cheng, Veerachai Malyavej, Andrey V. Savkin

Abstract— **This paper considers a decentralized robust state estimation problem for uncertain systems via a data-rate constrained sensor network. The uncertainties of the systems satisfy an integral quadratic constraint. The sensor network consists of spatially distributed sensors that take the measurements of a system, and a fusion center where the state estimation is carried out. The communications from the sensors to the fusion center are through data-rate constrained communication channels. We propose an estimation scheme which involves coders and a decoder-estimator, and their construction is based on the robust Kalman filtering techniques.**

I. Introduction

Recently, the use of sensor networks for state estimation and control has been widely applied and its research has been actively pursued [5]. There are a number of advantages of using sensor networks or multisensor, e.g., more information can be gathered through the use of sensor fusion techniques, geographical constraints can be overcome by using a number of spatially distributed sensors, reliability is improved from some degree of redundancy of sensors, etc.. For instance, sensor fusion has been widely applied to vehicle and missile guidance, see e.g. [7], [12], [14].

The communications in a sensor network are often implicitly assumed to be infinite precision or to have infinite bit rate. Due to the enormous growth in communication technology, it is becoming more common to employ data-rate constrained communication networks for exchange of information between system components. However, classical estimation theory cannot be applied since the measurement information is sent via data-rate constrained communication channels, hence, the estimator only observes the transmitted sequence of finite-valued symbols. There has been a significant interest in the problems of state estimation and control via data-rate constrained communication channels in recent years (see, e.g., [1]–[4], [6], [8]–[11], [13], [16], [18]–[20]).

In terms of robust state estimation, the works of [8], [20] provide algorithms that allow one to reliably estimate states of an uncertain system through communication networks. The algorithms were developed using the robust Kalman filtering technique of [15]. Their proposed coding schemes are based on a centralized approach requiring that *all* the measurement information is available to a single centralized coder. The coder uses the information to obtain a state estimate that is then encoded and sent to a decoder. However, this scheme may not be practical in a sensor network or multisensor setting since the sensors may be spatially distributed or geographically separated. The transmission of all the measurement information to a centralized coder, and the transmission of the full state estimate to a decoder, will take up a significant amount of bandwidth, as bandwidth is always a constraint in a communication network. Therefore, it is more realistic to transmit each sensor's measurement to the decoder directly, rather than collecting all the measurements from the sensors and processing them at a centralized coder before transmission to a fusion center.

In this paper, we consider a decentralized robust state estimation via a data-rate constrained sensor network. The sensors in the network are spatially distributed. Instead of transmitting all the measurements to a centralized coder as in [8], [20], we employ a decentralized scheme and design a coder for each individual sensor. Each encoded measurement is sent to a remotely-located fusion center where a decoder and a robust state estimator are embedded. The fusion center combines all the received codewords from the remote sensors and produces a set-valued state estimate that over-bounds the true set of possible states of the uncertain system.

The improved feature of this decentralized scheme as compared to the other schemes previously proposed in the area of state estimation over communication networks (see, e.g., [2], [3], [8], [16], [19], [20]) is that here we only require a simple quantization at the coders within the sensors, and the use of central processing unit (CPU) is *not* necessary in the coders. We only need a CPU at the fusion center to perform state estimation. In contrast, the coding schemes in references [2], [3], [8], [16], [19], [20] and many other papers require that the centralized coder is equipped with a state estimator, and hence the use of a CPU in the coder is necessary. Therefore, our proposed decentralized scheme is more applicable than the previous scheme, since we only need simple coders in the sensors.

The paper is organized as follows. In Section II, we formulate the problem of decentralized robust state estimation via a data-rate constrained sensor network. Some useful preliminary results are presented in Section III. In Section IV, a design of coders and a decoder-estimator that solves the proposed problem is presented. Finally, an example is included to demonstrate the effectiveness of the proposed algorithms. Due to page constraint, we leave the proofs of the results in the full version of this paper.

This work was supported by the Australian Research Council.

Teddy M. Cheng and Andrey V. Savkin are with the School of Electrical Engineering and Telecommunications, the University of New South Wales, Sydney, NSW 2052, Australia.

Veerachai Malyavej is with the Faculty of Engineering, Mahanakorn University of Technology, Bangkok, Thailand

Corresponding author: Teddy M. Cheng (email: t.cheng@ieee.org)

II. Problem Statement

Consider the time-varying uncertain system defined over the finite time interval $[0, NT]$:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)w(t), \\ w(t) &= \phi(z(t)), \quad z(t) = K(t)x(t) \end{aligned} \tag{1}$$

where $N > 0$ is an integer, $T > 0$ is a given constant, $x \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^p$ is an *uncertainty input*, $z(t) \in \mathbb{R}^q$ is the *uncertainty output* and $A(\cdot)$, $B(\cdot)$, and $K(\cdot)$ are bounded piecewise continuous matrix functions defined on $[0, NT]$.

The decentralized estimation problem studied in this paper is to robustly estimate the state of system (1) by a sensor network consisting of l sensors, namely $\{\Omega_1, \Omega_2, \ldots, \Omega_l\}$, that are spatially distributed. For each sensor Ω_i, the observation or measurement $y_i(\cdot) \in \mathbb{R}^{m_i}$ is corrupted by a noise $v_i(\cdot) \in \mathbb{R}^{m_i}$ and it is given by

$$y_i(t) = C_i(t)x(t) + v_i(t), \quad i = 1, 2, \ldots, l, \tag{2}$$

where the measurement matrix $C_i(\cdot) \in \mathbb{R}^{m_i \times n}$ is bounded and piecewise continuous over the time interval $[0, NT]$.

The information of the measurement $y_i(\cdot)$ from each sensor Ω_i is passed on to a *fusion center* that is remotely located from the sensors. The only way of communicating information from the sensors to the fusion center is via digital communication channels. In other words, each sensor Ω_i not only observes the measurement $y_i(\cdot)$, but also converts it into a finite-length codeword for transmitting the information to the fusion center. In order to convert the measurement $y_i(\cdot)$ into a finite-length codeword, each sensor Ω_i is equipped with a *coder* $\mathcal{F}_i$ that takes the measurement $y_i(\cdot)$ and encodes this measurement into a codeword $h_i(\cdot)$.

The channel connecting the sensor Ω_i to the fusion center carries one discrete-valued symbol $h_i(kT)$ at time kT, selected from a coding alphabet $\mathcal{H}_i$ of size ν_i. Here $T > 0$ is a given period and $k = 0, 1, 2, \ldots, N$. This restricted number ν_i of codewords $h_i(kT)$ is determined by the transmission data rate of the channel. We assume that the channel is a perfect noiseless channel and there is no time delay. Using this communication channel, the codeword $h_i(kT)$ produced by the coder $\mathcal{F}_i$ is transmitted to the fusion center.

A decoder and a robust state estimator are embedded in the fusion center. The job of the fusion center is to combine all the received codewords $h_1(kT), h_2(kT), \ldots, h_l(kT)$ from the remote sensors and to produce a set-valued state estimate $\mathcal{X}_t$, for all $t \in [kT, (k+1)T)$, that overbounds the true set of possible state $x(t)$ of system (1) over the time interval $[kT, (k+1)T)$. The decoder and the state estimator within the fusion center are called *decoder-estimator* $\mathcal{G}$.

We define the total number of measurements from all the sensors as $\bar{m} := m_1 + m_2 + \ldots + m_l$. Let $h(\cdot) = [h_1(\cdot) \quad h_2(\cdot) \quad \ldots \quad h_l(\cdot)]' \in \mathbb{R}^{\bar{m}}$ be the vector of codewords produced by the sensors. Then the coders and the decoder-estimator are in the form: for $k = 0, 1, 2, \ldots, N$, **Coders** ($i = 1, 2, \ldots, l$): $h_i(kT) = \mathcal{F}_i\left(y_i(\cdot)|_0^{kT}\right)$; **Decoder-estimator:** $\mathcal{X}_t = \mathcal{G}\left(h(T), h(2T), ..., h(kT)\right), \forall t \in [kT, (k+1)T)$. A schematic of the proposed robust state estimation via a data-rate constrained sensor network is illustrated in Figure 1.

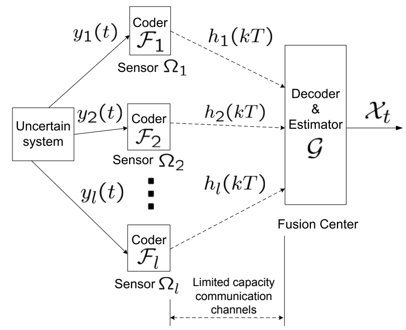

Fig. 1. Robust state estimation via a data-rate constrained sensor network

Notation 2.1: Let $x = [x_1 \quad x_2 \quad \cdots \quad x_n]'$ be a vector from $\mathbb{R}^n$. Then $\|x\|_\infty := \max_{j=1,\ldots,n} |x_j|$. Furthermore, $\|\cdot\|$ denotes the standard Euclidean vector norm: $\|x\| := \sqrt{\sum_{j=1}^n x_j^2}$.

Notation 2.2: The set $\mathcal{F} = \{\mathcal{F}_1, \ \mathcal{F}_2, \ \ldots \ , \mathcal{F}_l\}$ denotes the collection of coders of the sensors Ω_i, $i = 1, 2, \ldots, l$. The vectors $y = [y_1 \ \ y_2 \ldots y_l]' \in \mathbb{R}^{\bar{m}}$ and $v = [v_1 \ \ v_2 \ \ldots \ v_l]' \in \mathbb{R}^{\bar{m}}$ denote the augmented measurement vector from all the sensors and the augmented measurement noise vector, respectively. The measurement matrix $C(\cdot) \in \mathbb{R}^{\bar{m} \times n}$ is defined as $C(\cdot) := \left[C_1'(\cdot) \vdots C_2'(\cdot) \vdots \ldots \vdots C_l'(\cdot)\right]'$.

To solve our proposed estimation problem, we make the following assumption on the uncertain system (1) and the measurement noise in (2).

Assumption 2.1: The uncertainty $w(t)$ vector in system (1) and the augmented measurement noise vector $v(t)$ satisfy the following integral quadratic constraint (IQC). Let $Y_0 = Y_0' > 0$ be a given matrix, $x_0 \in \mathbb{R}^n$ be a given vector, $d > 0$ be a given constant, and $Q(\cdot) = Q(\cdot)'$ and $R(\cdot) = R(\cdot)'$ be given bounded piecewise continuous matrix weighting functions satisfying the following condition. There exists a constant $\delta > 0$ such that $Q(t) \geq \delta I$, $R(t) \geq \delta I$ for all t. Then for a given time interval $[0, \ \ s]$, $s \leq NT$, we will consider the uncertainty input $w(\cdot)$ and the measurement noise $v(\cdot)$ and initial condition $x(0)$ such that

$$\begin{aligned} &(x(0) - x_0)'Y_0(x(0) - x_0) + \int_0^s (w(t)'Q(t)w(t) + \\ &v(t)'R(t)v(t))dt \leq d + \int_0^s \|z(t)\|^2 dt. \end{aligned} \tag{3}$$

Notation 2.3: Let $y(t) = y_0(t)$ be a fixed measured output of the uncertain system (1) and let the finite-time interval $[0, s]$ be given. Furthermore, let $\mathcal{F}$ and $\mathcal{G}$ be given coders and decoder-estimator, respectively. Then, $\mathcal{X}_s[x_0, y_0(\cdot)|_0^s, d, \mathcal{F}, \mathcal{G}]$ denotes the set produces by the coders/decoder-estimator pair $(\mathcal{F}, \mathcal{G})$ that captures all possible state $x(s)$ at time s for the uncertain system (1) with uncertainty input $w(t)$ and measurement noise $v(t)$ satisfying the constraint (3).

The problem of decentralized robust state estimation via a sensor networks with data-rate constrained communication channels considered in this paper is the problem of constructing the coders/decoder-estimator pair $(\mathcal{F}, \mathcal{G})$ and the set $\mathcal{X}_s[x_0, y_0(\cdot)|_0^s, d, \mathcal{F}, \mathcal{G}]$.

Definition 2.1: The coders/decoder-estimator pair $(\mathcal{F}, \mathcal{G})$ is said to *detect* the state of system (1) via a data-rate constrained sensor network if for any vector $x_0 \in \mathbb{R}^n$, any time $s \in [0, NT]$, any constant $d > 0$, and any sampling period $T > 0$, and any fixed output $y(t) = y_0(t)$, the set $\mathcal{X}_s[x_0, y_0(\cdot)|_0^s, d, \mathcal{F}, \mathcal{G}]$ is bounded.

III. Preliminary Results

This section presents two useful preliminary results that are important for the development of the main results in this paper. The first result concerns with robust prediction that will be applied to determining the size of a quantization region for the design of coders $\mathcal{F}$ and decoder-estimator $\mathcal{G}$. The second result concerns with the set-valued state estimation of uncertain continuous systems with discrete measurements.

A. Robustly Predictable Systems

We consider system (1) satisfying the following IQC condition. Let $\hat{d} > 0$ be a given constant, $S_0 = S_0' > 0$ be a given matrix and $Q(\cdot) = Q(\cdot)'$ be given bounded piecewise continuous matrix weighting function satisfying the following condition. There exists a constant $\delta > 0$ such that $Q(t) \geq \delta I$ for all t. Then for a given time interval $[0,\ s]$, $s \leq NT$, we will consider the uncertainty input $w(\cdot)$ and initial condition $x(0)$ such that

$$x(0)'S_0x(0) + \int_0^s w(t)'Q(t)w(t) \leq \hat{d} + \int_0^s \|z(t)\|^2 dt. \tag{4}$$

Next, consider the following Riccati differential equation:

$$\begin{aligned} -\dot{S}(t) = S(t)A(t) + A(t)'S(t) - S(t)B(t)Q(t)^{-1}B(t)'S(t) \\ - K(t)'K(t), \quad S(0) = S_0, t \in [0,\ NT]. \end{aligned} \tag{5}$$

Definition 3.1: Uncertain system (1), (4) is said to be *robustly predictable* on $[0, NT]$ if any time $s \in [0, NT]$ and any constant $\hat{d} > 0$, the set $\mathcal{X}_s[\hat{d}]$ is bounded where $\mathcal{X}_s[\hat{d}]$ denotes the set of all possible state $x(s)$ at time s for the uncertain system (1) with uncertainty input $w(t)$ and initial condition $x(0)$ satisfying the constraint (4).

Theorem 3.1: Consider system (1). Let $S_0 = S_0' > 0$ be a given matrix, and $Q(\cdot) = Q(\cdot)' > 0$ be given matrix function such that condition (4) holds over time interval $[0, NT]$. Then, for a given constant $\hat{d} > 0$ and any time $s \in [0, NT]$, the system (1), (4) is robustly predictable on $[0, NT]$ if and only if the Riccati equation (5) has a solution over $[0, NT]$ such that $S(\cdot) = S(\cdot)' > 0$. Furthermore, the set $\mathcal{X}_s[\hat{d}]$ is given by $\mathcal{X}_s[\hat{d}] = \{x_s \in \mathbb{R}^n : x_s'S(s)x_s \leq \hat{d}\}$.

Proof: The proof of Theorem 3.1 will be given in the full version of the paper. □

Using Theorem 3.1, the following corollary can be obtained immediately.

Corollary 3.1: Suppose that Assumption 2.1 holds and the Riccati equation (5) has a solution $S(\cdot) = S(\cdot)' > 0$ over $[0, NT]$ with initial condition $S(0) = Y_0$. Then system (1) is robustly predictable on $[0, NT]$.

B. Robust State Estimation with Discrete Measurements

Again, we consider the continuous system (1), but assume that the measurements of the system can only be observed at discrete times by the remote sensors. In other words, the augmented measurement equation is in the form:

$$\bar{y}(kT) = C(kT)x(kT) + \bar{v}(kT) \tag{6}$$

where $\bar{y}(\cdot) = [\bar{y}_1(\cdot), \bar{y}_2(\cdot), \ldots, \bar{y}_l(\cdot)] \in \mathbb{R}^{\bar{m}}$ is the measurement vector, $\bar{v}(\cdot) = [\bar{v}_1(\cdot), \bar{v}_2(\cdot), \ldots, \bar{v}_l(\cdot)] \in \mathbb{R}^{\bar{m}}$ is some measurement noise vector.

We also assume that the uncertainty $w(\cdot)$ and the measurement noise $\bar{v}(\cdot)$ satisfy a Sum Integral Quadratic Constraint (SIQC) such that

$$\begin{aligned} (x(0) - x_0)'P_0^{-1}(x(0) - x_0) + \int_0^s w(t)'\bar{Q}(t)w(t)dt + \\ \sum_{kT \leq s} \bar{v}(kT)'\bar{R}\bar{v}(kT) \leq \bar{d} + \int_0^s \|z(t)\|^2 dt, \end{aligned} \tag{7}$$

where P_0, $\bar{Q}(\cdot)$ and $\bar{R}(\cdot)$ are given symmetric positive definite weighting matrices of suitable dimensions, and $\bar{d} > 0$ is a given constant. Note that $\bar{y}(\cdot)$, $\bar{v}(\cdot)$, $\bar{Q}(\cdot)$, $\bar{R}(\cdot)$ and $\bar{d}$ can be different from $y(\cdot)$, $v(\cdot)$, $Q(\cdot)$, $R(\cdot)$ and d in (2) and (3).

Notation 3.1: Let $\bar{y}(kT) = \bar{y}_0(kT)$ be a given fixed discrete measurement of system (1). The set $\bar{\mathcal{X}}_s[x_0, \bar{y}_0(\cdot)|_0^s, \bar{d}]$ denotes a set containing all the possible states of system (1) at time $s \in [0, NT]$ with uncertainty and measurement noise satisfying SIQC (7).

The following set-valued state estimation of an uncertain continuous system with discrete measurements is a special case of Theorem 6.3.1 in [15]. Before we state this result, we introduce a notation: the term $\nu(t^-)$ denotes the limit of the function $\nu(\cdot)$ at the point t from the left, i.e., $\nu(t^-) := \lim_{\epsilon > 0, \epsilon \to 0} \nu(t - \epsilon)$.

Theorem 3.2: Let $P_0 = P_0' > 0$ be a given matrix, $\bar{Q}(\cdot) = \bar{Q}(\cdot)' > 0$ and $\bar{R}(\cdot) = \bar{R}(\cdot)' > 0$ be given matrix functions. Consider uncertain system (1) and constraint (7) with discrete measurement (6). Then, the set $\bar{\mathcal{X}}_s[x_0, \bar{y}_0(\cdot)|_0^s, \bar{d}]$ is bounded over $[0, NT]$ if and only if the following jump Riccati equation

$$\begin{aligned} \dot{P}(t) &= A(t)P(t) + P(t)A(t)' + B(t)Q^{-1}(t)B(t)' \\ &\quad + P(t)K(t)'K(t)P(t), \qquad \text{for } t \neq kT \\ P(kT) &= [P^{-1}(kT^-) + C(kT)'\bar{R}(kT)C(kT)]^{-1}, \\ &\qquad \text{for } k = 1, 2, \ldots, N, \end{aligned} \tag{8}$$

has a solution over $[0, NT]$ such that $P(\cdot) = P(\cdot)' > 0$ and $P(0) = P_0$. Furthermore, the set $\bar{\mathcal{X}}_s[x_0, \bar{y}_0(\cdot)|_0^s, \bar{d}] = \{x_s \in \mathbb{R}^n : (x_s - \hat{x}(s))'P(s)^{-1}(x_s - \hat{x}(s)) \leq \bar{d} + \bar{\rho}(s)\}$ for any

$s \in [0, NT]$, where $\hat{x}(\cdot)$ is the solution to the following jump state equation:

$$\begin{aligned}\dot{\hat{x}}(t) &= [A(t) + P(t)K(t)'K(t)]\hat{x}(t), \quad \text{for } t \neq kT\\ \hat{x}(kT) &= \hat{x}(kT^-) - P(kT^-)C(kT)'\bar{R}(kT)C(kT)\hat{x}(kT^-)\\ &\quad + P(kT^-)C(kT)'\bar{R}(kT)\bar{y}(kT), \text{ for } k = 1, 2, \ldots, N,\end{aligned} \tag{9}$$

with initial condition $\hat{x}(0) = x_0$, and the function $\bar{\rho}(s)$ is defined as $\bar{\rho}(s) := \int_0^s \|K(t)\hat{x}(t)\|^2 dt - \sum_{kT \le s} \|\bar{R}(kT)^{1/2}(C(kT)\hat{x}(kT) - \bar{y}_0(kT))\|^2$.

Proof: See Theorem 6.3.1 in [15]. □

IV. CODERS AND DECODER-ESTIMATOR

In this section, we design coders $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_l\}$ and a decoder-estimator $\mathcal{G}$ to solve the proposed state estimation problem via a sensor network with data-rate constrained communication channels. For each sensor Ω_i, its coder $\mathcal{F}_i$ uses uniform quantization of the measurement $y_i(\cdot)$ of the uncertain system (1), (2). The coder $\mathcal{F}_i$ is static and does not have any memory, and therefore, computations at the coders can be kept at minimal. It measures $y_i(t)$ and converts it into a finite-length codeword $h_i(kT)$ through sampling and quantization.

To construct each $\mathcal{F}_i$, we first need to know the possible range of $y_i(\cdot)$ for quantization *a priori*. By using Theorem 3.1(Robust predictability), if system (1) with uncertainty $w(\cdot)$ and initial condition $x(0)$ satisfying the IQC (4), and the Riccati equation (5) has a solution over $[0, NT]$ such that $S(\cdot) = S(\cdot)' > 0$, then for all $s \in [0, NT]$,

$$\|x(s)\|_\infty \le \beta\sqrt{\hat{d}}, \tag{10}$$

where the scalar $\beta > 0$ is defined as

$$\beta := \max_{k=0,1,2,\ldots,N}\left(\max_{j=1,2,\ldots,n}\sqrt{[S(kT)^{-1}]_{j,j}}\right), \tag{11}$$

where $[S(kT)^{-1}]_{j,j}$ denotes the (j, j) element of the matrix $S(kT)^{-1}$.

Since each measurement matrix $C_i(\cdot)$ in (2) is a bounded piecewise continuous matrix function, there exists a constant $\gamma_i > 0$ such that

$$\max_{k=0,1,2,\ldots,N} \|C_i(kT)\|_\infty = \gamma_i, \tag{12}$$

where $\|C_i(\cdot)\|_\infty$ denotes the maximum row sum matrix norm of the matrix $C_i(\cdot)$, i.e., $\|C_i(\cdot)\|_\infty := \max_i \sum_{j=1}^n |[C_i(\cdot)]_{i,j}|$. To get a bound for $y_i(\cdot)$, we impose the following assumption on the measurement noise (2).

Assumption 4.1: The measurement noise $v_i(\cdot)$ in (2) from each sensor Ω_i is bounded and there exists a known bound $\alpha_i > 0$ such that

$$\|v_i(s)\| \le \alpha_i, \text{ for all } s \le NT. \tag{13}$$

Then using Eqns (2), (13), (10) and (12), a bound L_i for the measurement $y_i(\cdot)$ over the time interval $[0, NT]$ can be defined as follows:

$$L_i := \gamma_i\beta\sqrt{\hat{d}} + \alpha_i \ge \|y_i(s)\|_\infty \tag{14}$$

for all $s \in [0, NT]$. The bound L_i (14) can be pre-computed without the knowledge of the actual output $y_i(\cdot)$. This bound is then used to define a quantization region for the output measurement $y_i(kT)$, for $k = 0, 1, 2, \ldots, N$.

In our proposed scheme, each coder $\mathcal{F}_i$, $i = 1, 2, \ldots, l$, uses uniform quantization of the measurement $y_i(\cdot)$. Let the set $\mathcal{B}_{L_i} := \{y_i \in \mathbb{R}^{m_i} : \|y_i\|_\infty \le L_i\}$ be the quantization region. We quantize the measurement $y_i(\cdot)$ by dividing the quantization region $\mathcal{B}_{L_i}$ uniformly into $q_i^{m_i}$ hypercubes where q_i is a specified integer.

For each $j \in \{1, 2, \ldots, m_i\}$, we divide the corresponding component of the vector $y_i = [y_{i,1}\ \ y_{i,2} \ldots\ \ y_{i,m_i}]'$ into q_i intervals as follows:

$$\begin{aligned} I_1^j(L_i) &:= \left\{y_{i,j} : y_{i,j} \in [-L_i,\ -L_i + \frac{2L_i}{q_i})\right\};\\ I_2^j(L_i) &:= \left\{y_{i,j} : y_{i,j} \in [-L_i + \frac{2L_i}{q_i},\ -L_i + \frac{4L_i}{q_i})\right\}; \cdots\\ I_{q_i}^j(L_i) &:= \left\{y_{i,j} : y_{i,j} \in [L_i - \frac{2L_i}{q_i}, L_i]\right\}.\end{aligned}$$

Then for any $y_i \in \mathcal{B}_{L_i}$, y_i belongs to one of the hypercubes in $\mathcal{B}_{L_i}$. In other words, there exist unique m_i integers $\theta_{i,1}, \theta_{i,2}, \ldots, \theta_{i,m_i} \in \{1, 2, \ldots, q_i\}$ such that $y_i \in I^1_{\theta_{i,1}}(L_i) \times I^2_{\theta_{i,2}}(L_i) \times \ldots \times I^{m_i}_{\theta_{i,m_i}}(L_i)$, where $I^1_{\theta_{i,1}}(L_i) \times I^2_{\theta_{i,2}}(L_i) \times \ldots \times I^{m_i}_{\theta_{i,m_i}}(L_i)$ is one of the $q_i^{m_i}$ hypercubes containing y_i. Also, corresponding to the integers $\theta_{i,1}, \theta_{i,2}, \ldots, \theta_{i,m_i}$, we define the vector η_i as follows:

$$\eta_i(\Theta_i) := -L_i + \left[\frac{L_i(2\theta_{i,1} - 1)}{q_i}\ \ \frac{L_i(2\theta_{i,2} - 1)}{q_i}\ \cdots\ \frac{L_i(2\theta_{i,m_i} - 1)}{q_i}\right]'.$$

where $\Theta_i := [\theta_{i,1}\ \theta_{i,2}\ \ldots\ \theta_{i,m_i}]'$. The vector $\eta_i(\cdot)$ is the center of the hypercube $I^1_{\theta_{i,1}}(L_i) \times I^2_{\theta_{i,2}}(L_i) \times \ldots \times I^{m_i}_{\theta_{i,m_i}}(L_i)$ containing the original point y_i.

In our proposed coders/decoder-estimator, each one of the hypercubes in the quantization region $\mathcal{B}_{L_i}$ will be assigned a codeword

$$h_i(kT) = \Theta_i \tag{15}$$

and the coder $\mathcal{F}_i$ will transmit the codeword $h_i(kT)$ corresponding to the current measurement vector $y_i(kT)$. By defining $\bar{y}_i(kT) := \eta_i(\Theta_i)$, for a given $\epsilon > 0$, we can choose $q_i > 0$ such that

$$\|y_i(kT) - \bar{y}_i(kT)\|_\infty \le L_i/q_i \le \epsilon, \tag{16}$$

for all $k = 0, 1, 2, \ldots, N$. In other words, ϵ gives the quantization error and it can be controlled by varying the parameter q_i. However, the allowable quantization parameter q_i is limited by the capacity of the communication channel between the sensor Ω_i and the fusion center. If q_i is unbounded, it means the measurement $y_i(kT)$ can be transmitted with an infinite precision.

Now we are in position to introduce our proposed coders and decoder-estimator:

Coder $\mathcal{F}_i$ $(i = 1, 2, \dots, l)$:

$$h_i(kT) = \Theta_i, \text{ for } k = 0, 1, 2, \dots, N \\ \text{and } y_i \in I^1_{\theta_{i,1}}(L_i) \times I^2_{\theta_{i,2}}(L_i) \times \dots \times I^{m_i}_{\theta_{i,m_i}}(L_i). \tag{17}$$

Decoder-estimator $\mathcal{G}$:

Consists of jump state Eqn (9) and Riccati Eqn. (8) with

$$\hat{x}(0) = x_0, \quad P(0) = Y_0^{-1}, \text{ and} \\ \bar{y}(kT) = [\eta_1(\Theta_1)\ \ \eta_2(\Theta_2)\ \ \dots\ \ \eta_l(\Theta_l)]', \quad \text{for} \\ h(kT) = [\Theta_1\ \ \Theta_2\ \ \dots\ \Theta_l]'. \tag{18}$$

The main result of this paper is stated as follows:

Theorem 4.1: Consider the uncertain system (1), (2). Let $\bar{R} = \operatorname{diag}\{r_1^{-1}, r_2^{-1}, \dots, r_{\bar{m}}^{-1}\}$ be a given diagonal constant matrix with $r_i > 0$, and let $T > 0$ and $\epsilon > 0$ be given constants, and $s \in (0, NT]$ be given. Suppose that Assumptions 2.1 and 4.1 hold and also that the solution $S(\cdot)$ to the Riccati equation (5) with initial condition $S(0) = Y_0$ and the solution $P(\cdot)$ to the jump Riccati equation (8) with initial condition $P(0) = Y_0^{-1}$ are both defined and positive-definite on the interval $[0, NT]$. Furthermore, suppose that the quantization parameter q_i satisfies

$$q_i \geq L_i/\epsilon, \qquad i = 1, 2, \dots, l, \tag{19}$$

where L_i is defined in (14). Then the coders/decoder-estimator pair $(\mathcal{F}, \mathcal{G})$ (17), (18) detects the state of system (1), (2) via a data-rate constrained sensor network and the set $\mathcal{X}_s[x_0, y_0(\cdot)|_0^s, d, \mathcal{F}, \mathcal{G}]$ is given by

$$\mathcal{X}_s[x_0, y_0(\cdot)|_0^s, d, \mathcal{F}, \mathcal{G}] = \{x_s \in \mathbb{R}^n \\ : (x_s - \hat{x}(s))' P(s)^{-1} (x_s - \hat{x}(s)) \leq d + \rho(s)\} \tag{20}$$

where $\rho(s) := \int_0^s \|K(t)\hat{x}(t)\|^2 dt + N(\|\alpha\| + \epsilon\sqrt{\bar{m}})^2/r - \sum_{kT \leq s} \|\bar{R}^{1/2}(C(kT)\hat{x}(kT) - \bar{y}_0(kT))\|^2$, the state $\hat{x}(\cdot)$ is defined by (18) with initial condition x_0, $\bar{y}_0(\cdot)$ is the sampled and quantized signal of the fixed measurement vector $y_0(\cdot)$, the vector $\alpha := [\alpha_1\ \alpha_2\ \dots \alpha_l]'$ and $r := \min_{i \leq \bar{m}}\{r_i\}$.

Proof: The proof of Theorem 4.1 will be given in the full version of the paper. □

V. Illustrative Example

We consider a state estimation problem of an uncertain two-mass-spring system via a data-rate constrained sensor network. The system to be estimated is used in a well-known benchmark example in robust control (see, e.g. [17]). It consists of two masses connected by a spring as shown in Fig. 2. The masses are assumed to be $m_1 = 1$ and $m_2 = 1$, whereas the spring constant k of the spring is uncertain. The spring constant k has a nominal value of $k_0 = 1.25$, but can vary up to 15% of its nominal value. Based on these parameters, a model of the dynamics of the masss-spring system can be obtained from [17] and is described by the equation

$$\dot{x}(t) = Ax(t) + Bw(t), \qquad z(t) = Kx(t) \tag{21}$$

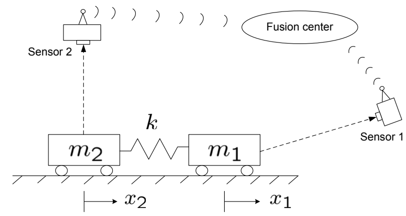

Fig. 2. Estimation of a two-mass-spring system via a data-rate constrained sensor network.

where $x := [x_1\ x_2\ \dot{x}_1\ \dot{x}_2]' \in \mathbb{R}^4$,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1.25 & 1.25 & 0 & 0 \\ 1.25 & -1.25 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ -0.1875 \\ 0.1875 \end{bmatrix}, \\ K = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix},$$

and the uncertain input $w(t)$ is given by

$$w(t) = \Delta(t) z(t), \qquad |\Delta(t)| \leq 1. \tag{22}$$

In this example, we employ a sensor network to estimate the state of the system. The sensor network in this example consists of two remote sensors as shown in Fig. 2. The first sensor (Sensor 1) measures the position of the first mass, whereas the second sensor (Sensor 2) measures the position of the second mass. Both encoded measurements from Sensors 1 and 2 are sent via a data-rate constrained communication network. The fusion center collects and decodes the measurements, and then carries out state estimation using its embedded state estimator. The sensors and the fusion center are located far away, the only way of communication from the sensors to the fusion center is through data-rate limited communication channels. A schematic of the state estimation of the uncertain mass-spring system via the two sensors is shown in Fig. 2.

We let $y_1(t)$ and $y_2(t)$ be the measurements taken by Sensors 1 and 2, respectively. Then the measurement equation is given by

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t). \tag{23}$$

In this example, using Sensors 1 and 2, we are interested in estimating the state of system (21) over the time interval $[0,\ 15]$ with a sampling period of $T = 0.3$ seconds in the coders. Assuming that the initial condition of system (21) is $x(0) = [1\ \ -1\ \ 1\ \ -0.5]'$ and there are no measurement noises from the sensors, i.e. $v_1(\cdot) \equiv v_2(\cdot) \equiv 0$. Also, the time-varying uncertain function $\Delta(t)$ in (22) is given as $\Delta(t) = \sin 0.2\pi t$.

We choose the vector $x_0 = [0\ 0\ 0\ 0]'$ and the matrices Y_0, S_0, $\bar{R}$ and scaler Q as follows: $Y_0 = S_0 = \operatorname{diag}\{1.5, 1.5, 1.5, 1.5\}$, $\bar{R} = \operatorname{diag}\{1, 1\}$ and $Q = 1$. Then the parameters d and $\hat{d}$ can be defined as $d = \hat{d} = 8.78$ so that both conditions (3) and (4) hold. Using the initial conditions $P(0) = Y_0^{-1}$, $S(0) = Y_0$, the solutions of both the Riccati differential equation (5) and the jump Riccati differential equation (8) are defined and positive definite over the time interval $[0,\ 15]$.

The solution $S(t)$ of the Riccati differential equation (5) allows us to estimate the bounds L_1 and L_2 of the measurements $y_1(t)$ and $y_2(t)$, respectively, for all $t \in [0,\ 15]$ by using Eqns (10)–(14). The bounds are found to be $L_1 = L_2 = 25.7$. Therefore, given a quantization error bound as $\epsilon = 0.1$, we pick the quantization parameters q_1 and q_2 as $q_1 = q_2 = 518$ so that condition (19) holds.

Finally, using the proposed coders $\mathcal{F}_1$ and $\mathcal{F}_2$ (17) for Sensors 1 and 2, and the decoder-estimator $\mathcal{G}$ (18) for the fusion center together with the above-mentioned design parameters, we obtain the simulation results for the state estimation of system (21) and they are shown in Fig. 3 and Fig. 4. The upper and lower bounds of the state estimates are evaluated from (20).

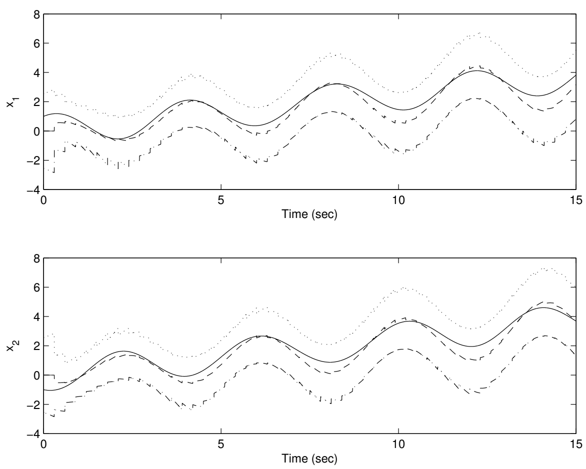

Fig. 3. Estimation of states x_1 (Top) and x_2 (Bottom). True value $x(t)$ $(-)$, estimate $\hat{x}(t)(--)$, upper bound $(\cdots)$, lower bound $(\cdot -)$

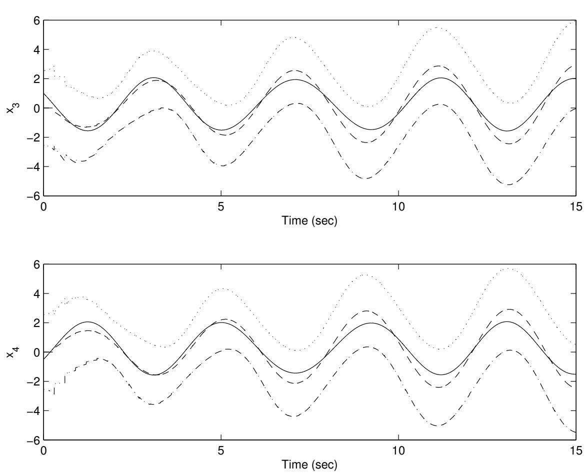

Fig. 4. Estimation of states x_3 (Top) and x_4 (Bottom). True value $x(t)$ $(-)$, estimate $\hat{x}(t)$ $(--)$, upper bound $(\cdots)$, lower bound $(\cdot -)$

References

[1] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Automat. Contr.*, vol. 45, no. 7, pp. 1279–1289, 2000.

[2] T. M. Cheng and A. V. Savkin, "Output feedback stabilisation of nonlinear networked control systems with non-decreasing nonlinearities: A matrix inequalities approach," *International Journal of Robust and Nonlinear Control*, vol. 17, no. 5, pp. 387–404, 2007.

[3] C. De Persis, "On stabilization of nonlinear systems under data rate constraints using output measurements," *International Journal of Robust and Nonlinear Control*, vol. 16, no. 6, pp. 315–332, 2006.

[4] C. De Persis and A. Isidori, "Stabilizability by state feedback implies stabilizability by encoded state feedback," *Systems & Control Letters*, vol. 53, pp. 249–258, 2004.

[5] D. L. Hall and J. Llinas, "An introduction to multisensor data fusion," *Proceedings of the IEEE*, vol. 85, no. 1, pp. 6–23, 1997.

[6] D. Liberzon and J. P. Hespanha, "Stabilization of nonlinear systems with limited information feedback," *IEEE Trans. Automat. Contr.*, vol. 50, no. 6, pp. 910–915, 2005.

[7] V. Malyavej, I. R. Manchester, and A. V. Savkin, "Precision missile guidance using radar/multiple-video sensor fusion via communication channels with bit-rate constraints," *Automatica*, vol. 42, pp. 763–769, 2006.

[8] V. Malyavej and A. V. Savkin, "The problem of optimal robust Kalman state estimation via limited capacity digital communication channels," *Systems & Control Letters*, vol. 54, pp. 283–292, 2005.

[9] A. S. Matveev and A. V. Savkin, "Multi-rate stabilization of linear multiple sensor systems via limited capacity communication channels," *SIAM Journal on Control and Optimization*, vol. 44, no. 2, pp. 584–617, 2005.

[10] ——, "An analogue of shannon information theory for detection and stabilization of via noisy discrete comunication channels," *SIAM Journal on Control and Optimization*, vol. 46, no. 4, pp. 1323–1367, 2007.

[11] ——, *Estimation and Control over Communication Networks*. Boston: Birkhäuser, 2008.

[12] R. R. Murphy, "Sensor and information fusion for improved vision-based vehicle guidance," *IEEE Intelligent Systems*, vol. 13, no. 6, pp. 49–56, 1998.

[13] G. N. Nair and R. J. Evans, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM Journal on Control and Optimization*, vol. 43, no. 2, pp. 413–436, 2004.

[14] P. N. Pathirana, A. E. K. Lim, A. V. Savkin, and P. D. Hodgson, "Robust video/ultrasonic fusion-based estimation for automotive applications," *IEEE Transactions on Vehicular Technology*, vol. 56, no. 4, pp. 1631–1639, 2007.

[15] I. R. Petersen and A. V. Savkin, *Robust Kalman Filtering for Signals and Systems with Large Uncertainties*. Boston: Birkhäuser, 1999.

[16] ——, "Multi-rate stabilization of multivariable discrete-time linear systems via a limited capacity communication channel," in *Proc. 40th IEEE Conf. Decision and Control*, Orlando, Florida, USA, 2001, pp. 304–309.

[17] I. R. Petersen, V. A. Ugrinovskii, and A. V. Savkin, *Robust Control Design Using H^∞ Methods*. London: Springer-Verlag, 2000.

[18] A. V. Savkin, "Analysis and synthesis of networked control systems: Topological entropy, observability, robustness, and optimal control," *Automatica*, vol. 42, no. 1, pp. 51–62, 2006.

[19] A. V. Savkin and T. M. Cheng, "Detectability and output feedback stabilizability of nonlinear networked control systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 4, pp. 730–735, 2007.

[20] A. V. Savkin and I. R. Petersen, "Set-valued state estimation via a limited capacity communication channel," *IEEE Transactions on Automatic Control*, vol. 48, no. 4, pp. 676–680, 2003.